$$E(x) = \sum_{\forall x} x\, p(x) \qquad E(x^2) = \sum_{\forall x} x^2\, p(x)$$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx \qquad E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

We know that $\sigma_x^2 \geq 0$

$\therefore\ E(X^2) \geq (E(X))^2$

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$$\sigma_x^2 = E(X^2) - (E(X))^2$$

$$A = E(X^2) - (E(X))^2$$

$$\therefore\ A + (E(X))^2 = E(X^2)$$

$$\therefore\ E(X^2) \geq (E(X))^2$$

And we can write

$$E(X^2) = E[X(X-1) + X]$$

$$= E\big(X(X-1)\big) - E(X)$$

**Ex)**

$$P(X) = \begin{cases} \frac{1}{6} & X = 1,2,3 \\ 0 & o.w. \end{cases}$$

Find: $V(x)$ , $E(x)$

Sol)

$$E(x) = \sum_{\forall x} X\, P(X) = 1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} = \frac{1}{6}(1+2+3) = \frac{1}{6} * 6 = 1$$

$$V(x) = E(X^2) - (E(X))^2$$

$$\therefore E(X^2) = \sum_{\forall x} X^2\, P(X) = 1^2 * \frac{1}{6} + 2^2 * \frac{1}{6} + 3^2 * \frac{1}{6} = \frac{1}{6}(1+4+9) = \frac{14}{6}$$

$$\therefore V(x) = \frac{14}{6} - (1)^2 = \frac{14}{6} - 1 = \frac{8}{6}$$

Ex)

$$f(X) = \begin{cases} \frac{1}{2} & 0 < X < 1 \\ 0 & o.w. \end{cases}$$

Find: $V(x)$

Sol)

$$E(X) = \int_{-\infty}^{\infty} X\, f(x)dx = \int_{0}^{1} X \frac{1}{2}\, dx = \frac{1}{2}(\frac{X^2}{2}|_0^1 = \frac{1}{2}\left(\frac{1}{2} - 0\right) = \frac{1}{4}$$

$$E(X^2) = \int_{-\infty}^{\infty} X^2\, f(x)dx = \int_{0}^{1} X^2 \frac{1}{2}\, dx = \frac{1}{2}(\frac{X^3}{3}|_0^1 = \frac{1}{2}\left(\frac{1}{3} - 0\right) = \frac{1}{6}$$

$$\therefore V(x) = E(X^2) - (E(X))^2 = \frac{1}{6} - (\frac{1}{4})^2 = \frac{1}{6} - \frac{1}{16} = \frac{10}{96}$$

Properties of variance

1- $V(c) = 0$ where c = constant

2- $V(cx) = c^2 V(x)$

3- $V(x \mp c) = V(x)$

4- $V(cx \mp b) = c^2 V(x)$

5- $V(z) = 1$ where

$$z = \frac{x-\mu}{\sigma} \quad \rightarrow \quad V(z) = V(\frac{x-\mu}{\sigma})$$

$$= \frac{1}{\sigma^2} V(x-\mu) = \frac{1}{\sigma^2} V(x)$$

$$= \frac{1}{\sigma^2} * \sigma^2 = 1$$

**4.3) Moments:**

The r-th moment about the origin is denoted $m_r$ , if it exists it is defined as:

Discrete:

$$m_r = E(X^r) = \sum_{\forall x} X^r P(X) ; \qquad r = 1,2,3, \ldots$$

Or

Continuous:

$$m_r = E(X^r) = \int_{-\infty}^{\infty} X^r f(x) dx$$

The r-th central moment about the mean $m = E(X)$ , is denoted by $\mu_r$ , if it exists it is defined as:

Discrete:

$$\mu_r = E(X-m)^r = \sum_{\forall x} (X-m)^r P(X) ; \qquad r = 1,2,3, \ldots$$

Or

Continuous:

$$\mu_r = E(X-m)^r = \int_{-\infty}^{\infty} (X-m)^r f(x) dx$$

Note that:

If $r = 2, \; then \; \mu_2 = E(X-m)^2 = E(X-E(X))^2 = Var(X)$

The standard deviation of X is denoted by $\sigma$ , and is defined as: $\sigma = \sqrt{Var(X)}$

Ex) Let X be a r.v. with a *p.m.f.* given by:

| $\boldsymbol{x}$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\boldsymbol{P(X = x)}$ | 1/10 | 2/10 | 3/10 | 4/10 |

Find: $m_1 , m_2 , m_4 , 4 - th \; central \; moment \; \mu_4 = E(X-m)^4$

Sol)

$$r=1\ \ ,\ m_1=E(X)=\sum_{x=0}^{3}XP(X)=\left(0*\frac{1}{10}\right)+\left(1*\frac{2}{10}\right)+\left(2*\frac{3}{10}\right)+\left(3*\frac{4}{10}\right)$$

$$=\frac{20}{10}=2$$

$$r=2\ \ ,\ m_2=E(X^2)$$

$$=\sum_{x=0}^{3}X^2P(X)=\left(0^2*\frac{1}{10}\right)+\left(1^2*\frac{2}{10}\right)+\left(2^2*\frac{3}{10}\right)+\left(3^2*\frac{4}{10}\right)$$

$$=\frac{50}{10}=5$$

$$r=4\ \ ,\ m_4=E(X^4)$$

$$=\sum_{x=0}^{3}X^4P(X)=\left(0^4*\frac{1}{10}\right)+\left(1^4*\frac{2}{10}\right)+\left(2^4*\frac{3}{10}\right)+\left(3^4*\frac{4}{10}\right)$$

$$=\frac{374}{10}=37.4$$

$$\therefore\ the\ 4-th\ central\ moment\ \ about\ the\ mean\ m_1=2=m$$

$$\therefore\ \mu_4=E(X-m)^4=\ E(X-2)^4=\sum_{x=0}^{3}(X-2)^4P(X)$$

$$=(0-2)^4\left(\frac{1}{10}\right)+(1-2)^4\left(\frac{2}{10}\right)+(2-2)^4\left(\frac{3}{10}\right)+(3-2)^4\left(\frac{4}{10}\right)=\frac{22}{10}=2.2$$

Ex) Let X be a r.v. with a density function:

$$f(X)=\begin{cases}\frac{1}{b} & 0<X<b\\ 0 & o.w.\end{cases}$$

Find: $m_1$ , $m_2$ , $Var(X)$ , the 3-rd central moment and the 3-rd moment.

$$r=1\ \ ,\ m_1=E(X)=\int_{-\infty}^{\infty}Xf(x)dx=\int_{0}^{b}X*\frac{1}{b}\ dx=\frac{1}{b}(\frac{X^2}{2}\mid_0^b$$

$$=\frac{1}{b}(\frac{b^2}{2}-0)=\frac{b}{2}$$

$$r = 2 \ , \ m_2 = E(X^2) = \int_{-\infty}^{\infty} X^2 f(x)dx = \int_0^b X^2 * \frac{1}{b}\ dx = \frac{1}{b}(\frac{X^3}{3}|_0^b$$

$$= \frac{1}{b}(\frac{b^3}{3} - 0) = \frac{b^2}{3}$$

$$Var(X) = E\big(X - E(X)\big)^2 = E(X^2) - (E(X))^2 = m_2 - m_1^2$$

$$= \frac{b^2}{3} - (\frac{b}{2})^2 = \frac{b^2}{3} - \frac{b^2}{4} = \frac{4b^2 - 3b^2}{12} = \frac{b^2}{12}$$

$$\mu_3 = E(X - m)^3 = E\left(X - \frac{b}{2}\right)^3 = \int_0^b \left(X - \frac{b}{2}\right)^3 * \frac{1}{b}\ dx = \frac{\left(X - \frac{b}{2}\right)^4}{4b}|_0^b$$

$$= \frac{1}{4b}\left[\frac{b^4}{16} - \frac{b^4}{16}\right] = 0$$

$$r = 3 \ , \ m_3 = E(X^3) = \int_{-\infty}^{\infty} X^3 f(x)dx = \int_0^b X^3 * \frac{1}{b}\ dx = \frac{1}{b}(\frac{X^4}{4}|_0^b$$

$$= \frac{1}{b}(\frac{b^4}{4} - 0) = \frac{b^3}{4}$$

## 4.4) Moment Generating function: الدالة المولدة للعزوم

The moment generating function (m.g.f.) denoted by $m_x(t)$ for a r.v. X is defined to be the expectation of the exponential function $e^{tx}$ , where $t$ = all real numbers.

Where

$$m_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} P(X)$$

If X is discrete r.v. with *p.m.f.* $P(X)$

Or

$$m_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

If X is continuous r.v. with *p.d.f.* $f(X)$

We call $m_x(t)$ the moment generating function because all of the moments of X can be obtained by successively على التوالي differentiating اشتقاق جزئي $m_x(t)$ and evaluating the result at $(t = 0)$.

**Theorem:**

If $m_x(t)$ is (*m.g.f.*) of X and if $m_k$ is will be defined:

$$m_x^k(0) = \frac{d^k}{d_t^k} m_k(t)|_{t=0} = m_k$$

Or the relation between ( *m.g.f* ) and moment , we know that

$m_x(t) = E(e^{tx})$

$m_x^{'}(t) = E(xe^{tx})$

$m_x^{'}(0) = E(xe^{0})\,, \quad t = 0$

Hence

$$m_x^{'}(0) = E(x * 1) = E(x) = m_1 = mean$$

- $m_x^{''}(t) = E(x^2 e^{tx})$

  $m_x^{''}(0) = E(x^2 e^0) = E(x^2) = m_2$

- $m_x^{'''}(t) = E(x^3 e^{tx})$

  $m_x^{'''}(0) = E(x^3 e^0) = E(x^3) = m_3$

In general, the k-th derivative of $m_x(t)$ is given by:

$m_x^k(t) = E(x^k e^{tx})$

$m_x^k(0) = E(x^k) = m_k$ , k-th moment.

Another way to see the derivative of $m_x(t)$ is by taylor series expansion of $e^{tx}$:

$$m_x(t) = E(e^{tx}) = E\left(\sum_{i=0}^{\infty} \frac{(tx)^i}{i!}\right)$$

$$= E\left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \frac{(tx)^4}{4!} + \cdots\right)$$

$$= \left[1 + tE(x) + t^2\,\frac{E(x^2)}{2!} + t^3\,\frac{E(x^3)}{3!} + t^4\,\frac{E(x^4)}{4!} + \cdots\right]$$

$$= \left[1 + tm_1 + t^2\,\frac{m_2}{2!} + t^3\,\frac{m_3}{3!} + t^4\,\frac{m_4}{4!} + \cdots\right]$$

$$\therefore m_x'(t) = m_1 + \frac{2t\,m_2}{2!} + \frac{3t^2\,m_3}{3!} + \frac{4t^3 m_4}{4!} + \cdots$$

$$m_x'(0) = m_1 + 0 + 0 + 0 + \cdots$$

$\therefore\ m_x'(0) = m_1$ first moment.

$$m_x''(t) = m_2 + \frac{6t\,m_3}{3!} + \frac{12t^2\,m_4}{4!} + \frac{20t^3 m_5}{5!} + \cdots$$

$$m_x''(0) = m_2 + 0 + 0 + 0 + \cdots$$

$\therefore m_x''(0) = m_2$ Second moment

It is easy to see that:

$m_x^k(0) = m_k$ , k-th moment.

Ex) Find the *m.g.f.* of X when

$$f(X) = \begin{cases} \frac{1}{\theta} & 0 < X < \theta \\ 0 & o.w. \end{cases}$$

Sol)

$$m_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_0^{\theta} e^{tx}\frac{1}{\theta}dx = \frac{1}{\theta t}\int_0^{\theta} te^{tx}dx = \frac{1}{\theta t}(\,e^{tx}|_0^{\theta}$$

$$= \frac{1}{\theta t}\left(e^{t\theta} - e^0\right) = \frac{1}{\theta t}\left(e^{t\theta} - 1\right) \quad \textit{m.g.f.}$$

$$m_x'(t) = m_1 = E(x) \quad ; \; m_x''(t) = m_2 = E(x^2)$$

$\therefore Var(X) = E\big(X - E(X)\big)^2 = E(X^2) - (E(X))^2 = m_2 - m_1^2$

Ex) A fair coin is flipped twice, let X be the number of head that occur, then:

$S = \{HH, HT, TH, TT\}$ and:

| $\boldsymbol{x}$ | 0 | 1 | 2 |
|---|---|---|---|
| $\boldsymbol{P(X = x)}$ | $1/4$ | $2/4$ | $1/4$ |

1-find the moment generating function of X .

2- find the $\mu_x$ $and\ \sigma_x^2$

Sol)

1- $m_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} P(X)$

$$= e^{t0} * \frac{1}{4} + e^{t1} * \frac{2}{4} + e^{t2} * \frac{1}{4}$$

$$\therefore m_x(t) = \frac{1}{4} + \frac{1}{2}e^t + \frac{1}{4}e^{2t}$$

2- $m_x'(t) = 0 + \frac{1}{2}e^t + \frac{2}{4}e^{2t} = \frac{1}{2}e^t + \frac{1}{2}e^{2t}$

$$m_x'(0) = \frac{1}{2}e^0 + \frac{1}{2}e^{2*0} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\therefore m_x'(0) = 1 = m_1 = E(x) = \mu_x$$

$$m_x''(t) = \frac{1}{2}e^t + \frac{2}{2}e^{2t} = \frac{1}{2}e^t + e^{2t}$$

$$\therefore m_x''(0) = \frac{1}{2}e^0 + e^{2*0} = \frac{1}{2} + 1 = \frac{3}{2}$$

$$\therefore m_x''(0) = \frac{3}{2} = m_2 = E(x^2)$$

$$\therefore \sigma_x^2 = E(x^2) - (E(x))^2 = m_2 - m_1^2$$

$$\therefore \sigma_x^2 = \frac{3}{2} - 1^2 = \frac{3}{2} - 1 = \frac{1}{2}$$