Estimating parameters using the least squares method

The basis for using the least squares method is to make the sum of the squares of the residuals or the error as small as possible:

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_m X_{im} + e_i$$
$$e_i = y_i - (\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots + \beta_m X_{im})$$

Reaching the natural equations.

Thus, we have a vector of estimated regression parameters that takes the following form:

$$\hat{\beta} = (X'X)^{-1} X'y$$

whereas:

$$X = \begin{bmatrix} 1 & x_{11} & x_{12} & x_{1m} \\ 1 & x_{21} & x_{22} & x_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} & x_{nm} \end{bmatrix}, \qquad X'y = \begin{bmatrix} \sum y_i \\ \sum x_{i1} y_i \\ \sum x_{i2} y_i \\ \vdots \\ \sum x_{im} y_i \end{bmatrix}$$

$$\therefore X'X = \begin{bmatrix} n & \sum X_{i1} & \sum X_{i2} & \cdots & \sum X_{im} \\ \sum X_{i1} & \sum X_{i1}^2 & \sum X_{i1} X_{i2} & \cdots & \sum X_{i1} X_{im} \\ \sum X_{i2} & \sum X_{i2} X_{i1} & \sum X_{i2}^2 & \cdots & \sum X_{i2} X_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum X_{im} & \sum X_{im} X_{i1} & \sum X_{im} X_{i2} & \cdots & \sum X_{im}^2 \end{bmatrix}$$

If we have one independent variable, the natural equations can be written in matrix form as follows:

$$X'y = X'X\,b$$

That is:

$$X'X\,b = X'y$$

$$X'X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{11} & X_{21} & \cdots & X_{n1} \end{bmatrix} \begin{bmatrix} 1 & X_{11} \\ 1 & X_{21} \\ \vdots & \vdots \\ 1 & X_{n1} \end{bmatrix} = \begin{bmatrix} n & \sum X_{i1} \\ \sum X_{i1} & \sum X_{i1}^2 \end{bmatrix}$$

and that:

$$X'y = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{11} & X_{21} & \cdots & X_{n1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum X_{i1} y_i \end{bmatrix}$$

Thus, natural equations can be written using matrices and m independent variables, as follows:

$$X'X\,b = X'y$$

$$\therefore \quad \begin{bmatrix} n & \sum X_{i1} & \sum X_{i2} & \cdots & \sum X_{im} \\ \sum X_{i1} & \sum X_{i1}^2 & \sum X_{i1}X_{i2} & \cdots & \sum X_{i1}X_{im} \\ \sum X_{i2} & \sum X_{i2}X_{i1} & \sum X_{i2}^2 & \cdots & \sum X_{i2}X_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum X_{im} & \sum X_{im}X_{i1} & \sum X_{im}X_{i2} & \cdots & \sum X_{im}^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_m \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum X_{i1}y_i \\ \sum X_{i2}y_i \\ \vdots \\ \sum X_{im}y_i \end{bmatrix}$$

Multiply both sides by the matrix $(X'X)^{-1}$ We get:

$$\hat{\beta} = (X'X)^{-1}X'y$$

Where the matrix $(X'X)^{-1}$ is represented by the symbol C:

$$(X'X)^{-1} = C$$

$$C = \begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0m} \\ C_{10} & C_{11} & \cdots & C_{1m} \\ C_{20} & C_{21} & \cdots & C_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ C_{m0} & C_{m1} & \cdots & C_{mm} \end{bmatrix}_{(m+1)(m+1)}$$

That is:

$$\hat{\beta} = \begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0m} \\ C_{10} & C_{11} & \cdots & C_{1m} \\ C_{20} & C_{21} & \cdots & C_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ C_{m0} & C_{m1} & \cdots & C_{mm} \end{bmatrix} \begin{bmatrix} \sum y_i \\ \sum X_{i1} y_i \\ \sum X_{i2} y_i \\ \vdots \\ \sum X_{im} y_i \end{bmatrix}$$

Sum of the corrected product

Another way to write natural equations is using the sum of squares and the sum of the corrected product. In this way:

$$X_i = \begin{bmatrix} (x_{11} - \bar{X}_1) & (x_{12} - \bar{X}_2) & \cdots & (x_{1m} - \bar{X}_m) \\ (x_{21} - \bar{X}_1) & (x_{22} - \bar{X}_2) & \cdots & (x_{2m} - \bar{X}_m) \\ \vdots & \vdots & \vdots & \vdots \\ (x_{n1} - \bar{X}_1) & (x_{n2} - \bar{X}_2) & \cdots & (x_{nm} - \bar{X}_m) \end{bmatrix}$$

Let us denote the corrected sum of squares $S_{X_iX_i}$by S_{ii} for ease, and for the sum of the corrected product $S_{X_iX_j}$by S_{ij} ,and the sum of the corrected product S_{X_iy} by S_{iy}:

$$S_{ii} = \sum X_i^2 - \frac{(\sum X_i)^2}{n}$$

$$S_{ij} = \sum X_i X_j - \frac{(\sum X_i)(\sum X_j)}{n}$$

$$S_{iy} = \sum X_i y_i - \frac{(\sum X_i)(\sum y_i)}{n}$$

So, the matrix $X_c'X_c$ (corrected) will be:

$$X_c'X_c = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1m} \\ S_{21} & S_{22} & \cdots & S_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ S_{m1} & S_{m2} & \cdots & S_{mm} \end{bmatrix}$$

$$y_c = \begin{bmatrix} (y_1 - \bar{y}) \\ (y_2 - \bar{y}) \\ \vdots \\ (y_n - \bar{y}) \end{bmatrix}, \quad X_c'y_c = \begin{bmatrix} S_{1y} \\ S_{2y} \\ \vdots \\ S_{my} \end{bmatrix}$$

So, the estimated corrected vector of parameters is:

$$\tilde{\beta} = \begin{bmatrix} \tilde{\beta}_1 \\ \tilde{\beta}_2 \\ \vdots \\ \tilde{\beta}_m \end{bmatrix} = (X_c'X_c)^{-1} X_c'y_c$$

Whereas

$$(X_c'X_c)^{-1} = \tilde{C}$$

where

$$\tilde{C} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ C_{m1} & C_{m2} & \cdots & C_{mm} \end{bmatrix}$$

Comparison between the two methods:

1- The size of the matrix is (m+1)(m+1), while the size of the matrix $X_c'X_c$ is (m×m), so the process of finding the inverse is easier.
2- The elements of the matrix $\tilde{C}$ are the same C (after deleting the first row and first column of C).
3- The values $\hat{\beta}$ are the same as the values $\tilde{\beta}$ (after removing β_0 from the vector $\hat{\beta}$).

The method that will be followed in our lectures is the uncorrected method.