5- Weigh the errors with the corresponding predictive values.

That is,

$$\sum \hat{y}_i e_i = 0$$

Proof:

$$\hat{y}_i = (\bar{y} + \hat{\beta}_1(x_i - \bar{X}))$$

$$e_i = (y_i - (\bar{y} + \hat{\beta}_1(x_i - \bar{X}))$$

So

$$\sum \hat{y}_i e_i = \sum (\bar{y} + \hat{\beta}_1(x_i - \bar{X}))(y_i - (\bar{y} + \hat{\beta}_1(x_i - \bar{X}))$$

Through the brackets, the second bracket is arranged as follows:

$$\sum \hat{y}_i e_i = \sum (\bar{y} + \hat{\beta}_1(x_i - \bar{X}))((y_i - \bar{y}) - \hat{\beta}_1(x_i - \bar{X}))$$

The brackets are multiplied together as follows:

$$\sum \hat{y}_i e_i = \sum (\bar{y}(y_i - \bar{y}) - \bar{y}\hat{\beta}_1(x_i - \bar{X}) + \hat{\beta}_1(x_i - \bar{X})(y_i - \bar{y}) - (\hat{\beta}_1(x_i - \bar{X}))^2)$$

By inserting the sum into the parentheses, we get:

$$\sum \hat{y}_i e_i = \bar{y}\underline{\sum (y_i - \bar{y})} - \bar{y}\hat{\beta}_1 \underline{\sum (x_i - \bar{X})} + \hat{\beta}_1 \sum (x_i - \bar{X})(y_i - \bar{y}) - \hat{\beta}_1^2 \sum (x_i - \bar{X})^2$$

So, the result will be:

$$\sum \hat{y}_i e_i = \hat{\beta}_1 \sum (x_i - \bar{X})(y_i - \bar{y}) - \hat{\beta}_1^2 \sum (x_i - \bar{X})^2$$

Since

$$\because S_{Xy} = \sum y_i (x_i - \bar{X})$$

$$\because S_{XX} = \sum (x_i - \bar{X})^2$$

So

$$\sum \hat{y}_i e_i = \hat{\beta}_1 S_{Xy} - \hat{\beta}_1^2 S_{XX}$$

The above equation can be abbreviated as follows:

$$\sum \hat{y}_i e_i = \hat{\beta}_1 S_{Xy} - \hat{\beta}_1 \frac{S_{Xy}}{S_{XX}} S_{XX}$$

$$\sum \hat{y}_i e_i = \hat{\beta}_1 S_{Xy} - \hat{\beta}_1 S_{Xy}$$

$$\sum \hat{y}_i e_i = 0$$

6− The estimated parameter $\hat{\beta}_1$ is an unbiased estimate of the parameter β_1.

So:

$$E(\hat{\beta}_1) = \beta_1$$

Proof:

$$E(\hat{\beta}_1) = E\left[\frac{S_{Xy}}{S_{XX}}\right] = E\left[\frac{\sum (x_i - \bar{X}) y_i}{\sum (x_i - \bar{X})^2}\right]$$

By entering the expectation into both sides, we get:

$$= \frac{\sum (x_i - \bar{X}) y_i}{\sum (x_i - \bar{X})^2} = \frac{\sum (x_i - \bar{X})\,(\beta_0 + \beta_1 x_i + e_i)}{\sum (x_i - \bar{X})^2}$$

$$= \frac{\beta_0 \sum (x_i - \bar{X}) + \beta_1 \sum (x_i - \bar{X})\, x_i + \sum (x_i - \bar{X})\, e_i)}{\sum (x_i - \bar{X})^2}$$

$$= \frac{\beta_1 \sum (x_i - \bar{X})\, x_i + \sum (x_i - \bar{X})\, e_i)}{\sum (x_i - \bar{X})^2}$$

Where $S_{XX} = \sum (x_i - \bar{X})\, x_i$

Entering the sum and multiplying the e_i over the bracket by the second term, we get:

$$= \frac{\beta_1 S_{XX} + \sum x_i e_i - \bar{X} \sum e_i)}{S_{XX}}$$

We have $\sum x_i e_i = 0$and $\sum e_i = 0$, so:

We have $\sum x_i e_i = 0$, and $\sum e_i = 0$So:

$$= \frac{\beta_1 S_{XX}}{S_{XX}}$$

$$\therefore E(\hat{\beta}_1) = \beta_1$$

7-The estimated parameter $\hat{\beta}_0$ is an unbiased estimate of the parameter β_0.

That mean:
$E(\hat{\beta}_0) = \beta_0$

Proof:

$$E(\hat{\beta}_0) = E(\bar{y} - \hat{\beta}_1 \bar{X})$$
$$= \bar{y} - E(\hat{\beta}_1)\bar{X}$$
$$= \left(\frac{\sum y_i}{n}\right) - \bar{X}\beta_1$$
$$= \left(\frac{\sum(\beta_0 + \beta_1 x_i + e_i)}{n}\right) - \bar{X}\beta_1$$

By inserting the sum into the parentheses, we get:

$$= \left(\frac{n\beta_0 + \beta_1 \sum x_i + \sum e_i)}{n}\right) - \bar{X}\beta_1$$
$$= \frac{n\beta_0}{n} + \beta_1 \frac{\sum X_i}{n} + \frac{\sum e_i}{n} - \bar{X}\beta_1$$
$$= \beta_0 + \beta_1 \bar{X} + 0 - \beta_1 \bar{X}$$
$$\therefore E(\hat{\beta}_0) = \beta_0$$

Estimate population variance σ^2:

SSe =Error Sum of Squares
=Residual Sum of squares

The population variance can be estimated by the sum of squared errors as follows:

$$SSe = \sum e_i^2 = \sum (y_i - \hat{y}_i)^2$$

We have $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{X}$ so:

$$SSe = \sum (y_i - ((\bar{y} - \hat{\beta}_1 \bar{X}) + \hat{\beta}_1 x_i))^2$$

Simplifying the above equation we get:

$$SSe = \sum ((y_i - \bar{y}) + \hat{\beta}_1 \bar{X} - \hat{\beta}_1 x_i))^2$$

By factoring out $\hat{\beta}_1$ we get:

$$SSe = \sum ((y_i - \bar{y}) - \hat{\beta}_1 (x_i - \bar{X}))^2$$

Open the bracket with the square:

$$SSe = \sum ((y_i - \bar{y})^2 - 2\hat{\beta}_1 (x_i - \bar{X})(y_i - \bar{y}) + \hat{\beta}_1^2 (x_i - \bar{X})^2)$$

By inserting the sum into the parentheses, we get:

$$SSe = \sum (y_i - \bar{y})^2 - 2\hat{\beta}_1 \sum (x_i - \bar{X})(y_i - \bar{y}) + \hat{\beta}_1^2 \sum (x_i - \bar{X})^2$$

We have: $S_{XX} = \sum (x_i - \bar{X})^2, S_{yy} = \sum (y_i - \bar{y})^2, S_{Xy} = \sum (x_i - \bar{X})(y_i - \bar{y})$

$$SSe = S_{yy} - 2\hat{\beta}_1 S_{Xy} + \hat{\beta}_1^2 S_{XX}$$

So:

$$SSe = S_{yy} - 2\hat{\beta}_1 S_{Xy} + \hat{\beta}_1 \hat{\beta}_1 S_{XX}$$

We have $\hat{\beta}_1 = \frac{S_{Xy}}{S_{XX}}$:

$$SSe = S_{yy} - 2\hat{\beta}_1 S_{Xy} + \hat{\beta}_1 \frac{S_{Xy}}{\cancel{S_{XX}}} \cancel{S_{XX}}$$

$$SSe = S_{yy} - 2\hat{\beta}_1 S_{Xy} + \hat{\beta}_1 S_{Xy}$$

That is:

$$SSe = S_{yy} - \hat{\beta}_1 S_{Xy}$$

The sum of squared errors (SSe) has (n-2) degrees of freedom because it requires two degrees of estimation β_0, β_1 to obtain $\hat{y}_i$. Thus:

$$\hat{\sigma}^2 = S_{y/x}^2 = MSe = V(y_i) = \frac{SSe}{n-2} = \frac{S_{yy} - \hat{\beta}_1 S_{Xy}}{n-2}$$

9- Estimating the variances of the regression coefficient β_1.

$V(\hat{\beta}_1)$

Proof:

$$\hat{\beta}_1 = \frac{S_{Xy}}{S_{XX}} = \frac{\sum (x_i - \bar{X}) y_i}{\sum (x_i - \bar{X})^2} = \sum \left(\frac{(x_i - \bar{X})}{\sum (x_i - \bar{X})^2} \right) y_i$$

If we have

$$a = a_1 y_1 + a_2 y_2 + \ldots + a_n y_n$$

Taking the variance of both sides, we get:

$$V(a) = a_1^2 V(y_1) + a_2^2 V(y_2) + \ldots + a_n^2 V(y_n)$$

We have $V(y_1) = V(y_2) = \ldots = V(y_n) = V(y_i)$, so

We have $V(y_1) = V(y_2) = \ldots = V(y_n) = V(y_i)$ so:

$$V(a) = V(y_i)(a_1^2 + a_2^2 + \ldots + a_n^2)$$

$$V(a) = \sum_{i=1}^{n} a_i^2 V(y_i)$$

Thus:

$$V(\hat{\beta}_1) = V\left(\sum \left(\frac{(x_i - \bar{X})}{\sum (x_i - \bar{X})^2} \right) y_i \right)$$

$$V(\hat{\beta}_1) = \sum \left(\frac{(x_i - \bar{X})}{\sum (x_i - \bar{X})^2} \right)^2 V(y_i)$$

$$V(\hat{\beta}_1) = \sum \left(\frac{(x_i - \bar{X})}{S_{XX}} \right)^2 V(y_i)$$

$$V(\hat{\beta}_1) = \frac{\sum (x_i - \bar{X})^2}{S_{XX}^2} V(y_i)$$

$$V(\hat{\beta}_1) = \frac{S_{XX}}{S_{XX}^2} \hat{\sigma}^2$$

$$\therefore V(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{S_{XX}}$$

10-Estimating the variances of the regression coefficient β_0

So: $V(\hat{\beta}_0)$