Properties of least squares estimators

1- $\hat{\beta}$ is an unbiased estimate of β.

the proof:

$$\hat{\beta} = (X'X)^{-1}X'y$$
$$= (X'X)^{-1}X'(X\beta + e)$$
$$= (X'X)^{-1}X'X\beta + (X'X)^{-1}X'e$$
$$E(\hat{\beta}) = E(\beta) + (X'X)^{-1}X'E(e)$$
$$E(\hat{\beta}) = \beta$$

2- The variance and covariance matrix of $\hat{\beta}$ is $V(\hat{\beta})$ and is equal to:

$$V(\hat{\beta}) = \hat{\sigma}^2(X'X)^{-1} = \hat{\sigma}^2 C$$

Whereas

$$C = \begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0m} \\ C_{10} & C_{11} & \cdots & C_{1m} \\ C_{20} & C_{21} & \cdots & C_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ C_{m0} & C_{m1} & \cdots & C_{mm} \end{bmatrix}_{(m+1)(m+1)}$$

The variance of the estimated vector of parameters can be demonstrated as follows:

$$V(\hat{\beta}) = E\left[(\hat{\beta} - E(\hat{\beta}))\ (\hat{\beta} - E(\hat{\beta}))' \right]$$

We have:

$$\hat{\beta} = (X'X)^{-1}X'y$$

and

$$E(\hat{\beta}) = (X'X)^{-1}X'E(y)$$

So that:

$$V(\hat{\beta}) = E\left[(\hat{\beta} - E(\hat{\beta}))\,(\hat{\beta} - E(\hat{\beta}))'\right]$$

$$= E((X'X)^{-1}X'y - (X'X)^{-1}X'E(y))\,((X'X)^{-1}X'y - (X'X)^{-1}X'E(y))'$$

Extracting $(X'X)^{-1}X'$ as a common factor for each arc we get:

$$= E((X'X)^{-1}X'(y - E(y))\,((X'X)^{-1}X'(y - E(y))'$$

Since $e = (y - \hat{y})$ is the same as $e = (y - E(y))$, then:

Since $e = (y - \hat{y})$ and $e = (y - E(y))$, are the same, then:

$$= E((X'X)^{-1}X'e)\,((X'X)^{-1}X'e)'$$

Taking the transpose of the second parenthesis, we get:

$$= E((X'X)^{-1}X'e)\,((e'X(X'X)^{-1})$$

By introducing the expectation after multiplying the parentheses, we get:

$$= (X'X)^{-1}X'E(ee')X(X'X)^{-1})$$

We have $E(ee') = \sigma^2$, so we get

$$= \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$$

We have $(X'X)^{-1}X'X = I$, so we get:

$$V(\hat{\beta}) = \hat{\sigma}^2(X'X)^{-1}$$

3- Estimating population variance

Proof:

as:

$$SSe = e'e$$

And we have $e = (y - X\beta)$, Thus:

$$SSe = (y - X\beta)'(y - X\beta)$$

Taking the transpose of the first parenthesis, we get:

$$\begin{aligned} SSe &= (y - \beta' X')(y - X\beta) \\ &= (y'y - \beta' X' y - y' X\beta + \beta' X' X\beta) \end{aligned}$$

We have $\beta' X' y = y' X\beta$, Thus:

$$SSe = (y'y - 2\beta' X' y + \beta' X' X\beta)$$

We have $X' X\beta = X' y$, Thus:

$$SSe = (y'y - 2\beta' X' y + \beta' X' y)$$

Subtracting the second term from the third term we get:

$$SSe = y'y - \beta'X'y$$

Where SSe is the sum of squares of error.

The degrees of freedom for the sum of squares of error are (n-m-1), where m is the number of independent variables in the model, and one is the parameter β_0.

The mean square error is equal to:

$$Mse = \hat{\sigma}^2 = \frac{SSe}{n-m-1} = \frac{y'y - \beta' X' y}{n-m-1}$$

Example: Using the data below, which consists of two independent variables and a dependent variable, find the following:

1- Equation of the regression line for the data below.
2- Find the variance of the estimated parameters $V(\hat{\beta})$.

X_2	X_1	y
1	3	6
2	5	5
5	4	8
7	9	7
9	2	6
8	1	2

the solution:

1- Estimate the parameters of the regress formula:

$$\hat{\beta} = (X'X)^{-1}X'y$$

The two independent variables are arranged in a matrix as follows:

$$X = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 5 & 2 \\ 1 & 4 & 5 \\ 1 & 9 & 7 \\ 1 & 2 & 9 \\ 1 & 1 & 8 \end{bmatrix}, \; y = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \\ 6 \\ 2 \end{bmatrix}$$

We first find the matrix $(X'X)$ and vector $X'y$ as follows:

$$(X'X) = \begin{bmatrix} n & \sum X_{i1} & \sum X_{i2} \\ \sum X_{i1} & \sum X_{i1}^2 & \sum X_{i1}X_{i2} \\ \sum X_{i2} & \sum X_{i2}X_{i1} & \sum X_{i2}^2 \end{bmatrix}, X'y = \begin{bmatrix} \sum y_i \\ \sum x_{i1}y_i \\ \sum x_{i2}y_i \end{bmatrix}$$

We find the product of multiplying the variables together as follows: