$F'(X) = (x^2)' = 2x = f(x)$

$$\therefore f(x) = \frac{\partial F(X)}{\partial X}$$

Ex) Suppose that X is a continuous r.v. whose probability density function (p.d.f.) is given by :

$$f(X) = \begin{cases} c(4x - 2x^2) & 0 < X < 2 \\ 0 & o.w. \end{cases}$$

Find :

1- What is the value of the constant c ?

2- Find the probability that $X > 1$ ?

3- Find the probability that $X > 0$ ?

4- The distribution function $F(X)$ ?

Sol)

1- $\int_{-\infty}^{\infty} f(x)dx = 1 \rightarrow \int_0^2 c(4x - 2x^2)\, dx = 1$

$\rightarrow c[\int_0^2 4x\, dx - \int_0^2 2x^2\, dx] = 1$

$\rightarrow c\left[\frac{4\,x^2}{2}|_0^2 - \frac{2\,x^3}{3}|_0^2\right] = 1$

$\rightarrow c\left[(8-0) - (\frac{16}{3} - 0)\right] = 1 \rightarrow c\left[\frac{8}{3}\right] = 1 \rightarrow \therefore c = \frac{3}{8}$

$\therefore p.d.f.$

$$f(X) = \begin{cases} \left(\frac{12x}{8} - \frac{6x^2}{8}\right) & 0 < X < 2 \\ 0 & o.w. \end{cases}$$

ويجب التأكد من انها دالة كثافة احتمالية : -

$\rightarrow \int_0^2 \frac{12x}{8} - \frac{6x^2}{8}\, dx \rightarrow \frac{12}{8}\int_0^2 x\, dx - \frac{6}{8}\int_0^2 x^2\, dx$

$\rightarrow \frac{12}{8}\left(\frac{x^2}{2}|_0^2 - \frac{6}{8}\frac{x^3}{3}|_0^2\right) \Rightarrow \frac{12}{8}\left(\frac{4}{2}\right) - \frac{6}{8}\left(\frac{8}{3}\right) \Rightarrow 3 - 2 = 1$ تحقق الشرط

2- $P(X > 1) = \int_1^2 f(x)dx = \int_1^2 \frac{12}{8}x\, dx - \int_1^2 \frac{6}{8}x^2 dx =$

$\rightarrow \frac{12}{8}\left(\frac{x^2}{2}|_1^2 - \frac{6}{8}\frac{x^3}{3}|_1^2\right) = \frac{12}{8}\left(\frac{4}{2} - \frac{1}{2}\right) - \frac{6}{8}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{12}{8}\left(\frac{3}{2}\right) - \frac{6}{8}\left(\frac{7}{3}\right) = \frac{9}{4} - \frac{7}{4} = \frac{1}{2}$

Or $\quad 1 - F(1) = \frac{1}{2}$

3- $P(X > 0) = \int_0^2 f(x)dx = \frac{12}{8}\int_0^2 x\,dx - \frac{6}{8}\int_0^2 x^2 dx = 1$

4- $P(-1 < x < 1) = P(-1 < x < 0) + P(0 < x < 1)$

$$= 0 + \int_0^1 f(x)dx = \frac{1}{2}$$

Or $\quad F(1) - F(0) = \frac{1}{2}$

5- Distribution Function or $F(X)$ .

$$F(X) = P(X \le x) = \int_0^x f(t)dt = \frac{12}{8}\int_0^x t\,dt - \frac{6}{8}\int_0^x t^2\,dt$$

$$\therefore F(X) = \frac{12}{8}\left(\frac{t^2}{2}\Big|_0^x - \frac{6}{8}\,\frac{t^3}{3}\Big|_0^x\right) = \frac{12}{8}\left(\frac{x^2}{2}\right) - \frac{6}{8}\left(\frac{x^3}{3}\right) = \frac{6\,x^2}{8} - \frac{2\,x^3}{8}$$

$$\therefore F(X) = \begin{cases} 0 & X < 0 \\ \frac{6\,x^2}{8} - \frac{2\,x^3}{8} & 0 \le X \le 2 \\ 1 & X \ge 2 \end{cases}$$

| ***X*** | 0 | 1 | 2 |
|---|---|---|---|
| ***F(X)*** | 0 | ½ | 1 |

Ex) let the random variable X have the probability density function .

$$f(X) = \begin{cases} cx^2 & -1 < X < 1 \\ 0 & o.w. \end{cases}$$

1- Determine the value of the constant c and $F(X)$

2- Find $P(0 < X < 1)$ , $P(0 < X \le 3)$ , $P\left(X = \frac{1}{2}\right)$ , $P\left(-1 < X < \frac{1}{2}\right)$

Sol)

1- $\int_{-1}^1 f(x)\,dx = 1 \Rightarrow c\int_{-1}^1 x^2\,dx = 1 \Rightarrow c(\frac{x^3}{3}\Big|_{-1}^1 = 1 \Rightarrow c\left[\frac{1}{3} + \frac{1}{3}\right] = 1$

$$\Rightarrow c\left[\frac{2}{3}\right] = 1 \Rightarrow c = \frac{3}{2}$$

$$\therefore f(X) = \begin{cases} \frac{3}{2}x^2 & -1 < X < 1 \\ 0 & o.w. \end{cases}$$

And

$$F(X) = P(X \le x) = \int_{-1}^{x} f(t)dt = \int_{-1}^{x} \frac{3}{2}t^2\, dt = \frac{3}{2}(\frac{t^3}{3}|_{-1}^{x} = \frac{3}{2}\left(\frac{x^3}{3} + \frac{1}{3}\right) = \frac{1}{2}(x^3 + 1)$$

$$\therefore F(X) = \begin{cases} 0 & X < -1 \\ \frac{1}{2}(x^3 + 1) & -1 \le X \le 1 \\ 1 & X \ge 1 \end{cases}$$

2- i- $P(0 < X < 1) = \int_0^1 f(x)dx = \frac{3}{2}\int_0^1 x^2 dx = \frac{3}{2}\ (\frac{x^3}{3}|_0^1 = \frac{1}{2}$

Or $\quad F(b) - F(a) = F(1) - F(0) = 1 - \frac{1}{2} = \frac{1}{2}$

ii- $P(0 < X \le 3) = P(0 < X < 1) + P(1 < X \le 3) = \frac{1}{2} + 0 = \frac{1}{2}$

iii- $P\left(X = \frac{1}{2}\right) = zero \quad or \quad F\left(\frac{1}{2}\right) - F\left(\frac{1}{2}\right) = zero$

iv- $P\left(-1 < X < \frac{1}{2}\right) = \int_{-1}^{\frac{1}{2}} f(x)dx = \frac{3}{2}\int_{-1}^{\frac{1}{2}} x^2 dx = \frac{3}{2}\ (\frac{x^3}{3}|_{-1}^{\frac{1}{2}} = \frac{3}{2}(\frac{\left(\frac{1}{2}\right)^3}{3} - \frac{(-1)^3}{3}) = \frac{9}{16}$

Or $\quad F(b) - F(a) = F\left(\frac{1}{2}\right) - F(-1) = \frac{9}{16}$

Ex) Consider the distribution function

$$F(X) = \begin{cases} 0 & X = 0 \\ X & 0 \le X < 1 \\ 1 & X \ge 1 \end{cases}$$

Find : $P\left(\frac{1}{4} < X < \frac{3}{4}\right)$ ; $P\left(-1 < X < \frac{1}{2}\right)$

Sol)

$$\frac{\partial F(X)}{\partial X} = f(x) = \begin{cases} 0 & for \quad x < 0 \quad and \quad x \ge 1 \\ 1 & for \quad 0 < x < 1 \end{cases}$$

i- $P\left(\frac{1}{4} < X < \frac{3}{4}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} 1\ dx = x\,|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

Or $\quad F\left(\frac{3}{4}\right) - F\left(\frac{1}{4}\right) = \frac{1}{2}$

ii- $P\left(-1 < X < \frac{1}{2}\right) = P(-1 < X < 0) + P\left(0 < X < \frac{1}{2}\right)$

$$= 0 + \int_0^{\frac{1}{2}} 1 \, dx = x \,\Big|_0^{\frac{1}{2}} = \frac{1}{2} - 0 = \frac{1}{2}$$

Or $F\left(\frac{1}{2}\right) - F(0) = \frac{1}{2}$