3.4 Higher Transition probabilities

In order to determine the absolute probabilities at any stage, we shall need the idea of n-step transition probabilities. Let us in fact write:

$$P_{ij}^{n} = P_r\{X_{n+m} = j|X_m = i\}$$

With

$$P_{ij}^{1} = P_{ij}$$

Here P_{ij}^{n} denotes the probability that the process goes from state i to state j in n transitions.

Consider a fixed value of k then :

$$P_r\{X_{n+2} = j, X_{n+1} = k|X_n = i\}$$

$$=P_r\{X_{n+2} = j|X_{n+1} = k, X_n = i\}. P_r\{X_{n+1} = k|X_n = i\}$$

$$=P_r\{X_{n+2} = j|X_{n+1} = k\}. P_r\{X_{n+1} = k|X_n = i\}$$

$$=P_{kj}^{1}P_{ik}^{1}$$

$$=P_{ik}P_{kj}$$

$$\therefore P_{ij}^{2} = P_r\{X_{n+2} = j|X_n = i\} = \sum_k P_{ik}P_{kj}$$

This is the transition from state i to state j in 2 steps.

Theorem(2): The n-step transition probabilities of a Markov chain satisfy

$$P_{ij}^{n} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{n-1}$$

Where we define

$$P_{ij}^{0} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

<u>Proof:</u>

$$P_{ij}^{n} = P_r\{X_n = j | X_0 = i\}$$

The event of going from state *i* to state *j* in *n* transitions can be realized in the mutually exclusive ways of going to some intermediate state k, we have:

$$P_{ij}^{n} = \sum_{k=0}^{\infty} P_r\{X_n = j, X_1 = k | X_0 = i\}$$

By the definition of the conditional Prob. ,we have :

$$P_r(C \cap B | A) = P_r(C | B \cap A).P_r(B | A)$$

$$P_{ij}^{n} = \sum_{k=0}^{\infty} P_r\{X_1 = k | X_0 = i\}. P_r\{X_n = j | X_1 = k, X_0 = i\}$$

Now by the Markovian property:

$$P_{ij}^{n} = \sum_{k=0}^{\infty} P_r\{X_1 = k | X_0 = i\}. P_r\{X_n = j | X_1 = k\}$$