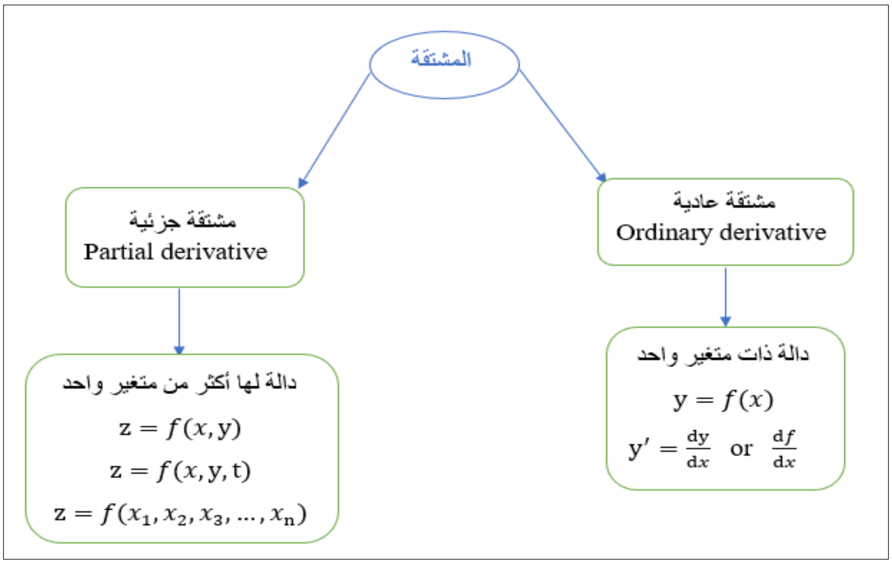
المشتقات الجزئية Partial Derivatives



تعريف: لتكن $f(x, y)$ دالة معرفة في المنطقة $A \subseteq \mathbb{R}^2$. تعرف المشتقتان الجزئيتان كالاتي:

$$\frac{\partial f}{\partial x} = f_x = \lim_{\triangle x \to 0} \frac{f(x + \triangle x, y) - f(x, y)}{\triangle x}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{\triangle y \to 0} \frac{f(x, y + \triangle y) - f(x, y)}{\triangle y}$$

وتعرف مشتقات الدالة في ثلاثة متغيرات أو أكثر بطريقة مشابهة.

أما المشتقات الجزئية من الرتبة الثانية للدالة الاصلية (أو المشتقات الثانية) تعرف كالاتي:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \lim_{\triangle x \to 0} \frac{f_x(x + \triangle x, y) - f_x(x, y)}{\triangle x}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \lim_{\triangle y \to 0} \frac{f_y(x, y + \triangle y) - f_y(x, y)}{\triangle y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = \lim_{\triangle y \to 0} \frac{f_x(x, y + \triangle y) - f_x(x, y)}{\triangle y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = \lim_{\triangle x \to 0} \frac{f_y(x + \triangle x, y) - f_y(x, y)}{\triangle x}$$

المشتقات الجزئية f_{xy} و f_{yx} تسمى مشتقات جزئية مختلطة.

مثال: اذا كانت $f(x,\mathrm{y}) = x\mathrm{y}^2 + x^2\mathrm{y} + 3$ ، أوجد f_x و f_y بأستعمال التعريف.

الحل: لأيجاد $\frac{\partial f}{\partial x}$

$$\frac{\partial f}{\partial x} = f_x = \lim_{\triangle x \to 0} \frac{f(x + \triangle x, \mathrm{y}) - f(x,\mathrm{y})}{\triangle x}$$

$$f(x + \triangle x, \mathrm{y}) = (x + \triangle x)\mathrm{y}^2 + (x + \triangle x)^2\, \mathrm{y} + 3$$

$$= x\mathrm{y}^2 + \triangle x\mathrm{y}^2 + x^2\mathrm{y} + 2x \triangle x\mathrm{y} + (\triangle x)^2\, \mathrm{y} + 3$$

$$f(x + \triangle x, \mathrm{y}) - f(x,\mathrm{y})$$

$$= x\mathrm{y}^2 + \triangle x\mathrm{y}^2 + x^2\mathrm{y} + 2x \triangle x\mathrm{y} + (\triangle x)^2\, \mathrm{y} + 3 - x\mathrm{y}^2 - x^2\mathrm{y} - 3$$

$$= \triangle x\, \mathrm{y}^2 + 2x \triangle x\, \mathrm{y} + (\triangle x)^2\, \mathrm{y}$$

$$= \triangle x\, (\mathrm{y}^2 + 2x\mathrm{y} + \triangle x\, \mathrm{y})$$

$$\frac{\partial f}{\partial x} = f_x = \lim_{\triangle x \to 0} \frac{\triangle x\, (\mathrm{y}^2 + 2x\mathrm{y} + \triangle x\mathrm{y})}{\triangle x}$$

$$= \lim_{\triangle x \to 0} (\mathrm{y}^2 + 2x\mathrm{y} + \triangle x\mathrm{y})$$

$$= \mathrm{y}^2 + 2x\mathrm{y}$$

ولأيجاد $\frac{\partial f}{\partial \mathrm{y}}$

$$\frac{\partial f}{\partial \mathrm{y}} = f_\mathrm{y} = \lim_{\triangle \mathrm{y} \to 0} \frac{f(x, \mathrm{y} + \triangle \mathrm{y}) - f(x,\mathrm{y})}{\triangle \mathrm{y}}$$

$$f(x, \mathrm{y} + \triangle \mathrm{y}) = x(\mathrm{y} + \triangle \mathrm{y})^2 + x^2(\mathrm{y} + \triangle \mathrm{y}) + 3$$

$$= x(\mathrm{y}^2 + 2\mathrm{y}\, \triangle \mathrm{y} + (\triangle \mathrm{y})^2) + x^2(\mathrm{y} + \triangle \mathrm{y}) + 3$$

$$= x\mathrm{y}^2 + 2x\mathrm{y}\, \triangle \mathrm{y} + x(\triangle \mathrm{y})^2 + x^2\mathrm{y} + x^2 \triangle \mathrm{y} + 3$$

$$f(x, \mathrm{y} + \triangle \mathrm{y}) - f(x,\mathrm{y})$$

$$= x\mathrm{y}^2 + 2x\mathrm{y}\, \triangle \mathrm{y} + x(\triangle \mathrm{y})^2 + x^2\mathrm{y} + x^2 \triangle \mathrm{y} + 3 - x\mathrm{y}^2 - x^2\mathrm{y} - 3$$

$$= 2x\mathrm{y}\, \triangle \mathrm{y} + x(\triangle \mathrm{y})^2 + x^2 \triangle \mathrm{y}$$

$$= \triangle \mathrm{y}\, (2x\mathrm{y} + x \triangle \mathrm{y} + x^2)$$

$$\frac{\partial f}{\partial \mathrm{y}} = f_\mathrm{y} = \lim_{\triangle \mathrm{y} \to 0} \frac{\triangle \mathrm{y}\, (2x\mathrm{y} + x \triangle \mathrm{y} + x^2)}{\triangle \mathrm{y}} = \lim_{\triangle \mathrm{y} \to 0} (2x\mathrm{y} + x \triangle \mathrm{y} + x^2)$$

$$= 2x\mathrm{y} + x^2$$

ملاحظة:

- قواعد تفاضل حاصل ضرب دالتين، دالة الدالة، والدالة الكسرية تعتبر صحيحة في المشتقات الجزئية.
- المشتقة الجزئية للدالة $f(x, \mathrm{y})$ بالنسبة للمتغير x هي المشتقة الاعتيادية للدالة $f(x, \mathrm{y})$ بالنسبة للمتغير x وذلك بأعتبار y ثابت. والمشتقة الجزئية للدالة $f(x, \mathrm{y})$ بالنسبة للمتغير y هي المشتقة الاعتيادية للدالة $f(x, \mathrm{y})$ بالنسبة للمتغير y وذلك بأعتبار x ثابت.

As is true for ordinary derivatives, it is possible to take second, third, and higher-order partial derivatives of a function of several variables, provided such derivatives exist. Higher-order derivatives are denoted by the order in which the differentiation occurs. For instant, the function $\mathrm{z} = f(x, \mathrm{y})$ has the following second partial derivatives.

1	differentiate twice with respect to x	$f_{xx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$
2	differentiate twice with respect to y	$f_{\mathrm{yy}} = \dfrac{\partial}{\partial \mathrm{y}}\left(\dfrac{\partial f}{\partial \mathrm{y}}\right) = \dfrac{\partial^2 f}{\partial \mathrm{y}^2}$
3	differentiate first with respect to x and then with respect to y	$f_{x\mathrm{y}} = \dfrac{\partial}{\partial \mathrm{y}}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial \mathrm{y} \partial x}$
4	differentiate first with respect to y and then with respect to x	$f_{\mathrm{y}x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \mathrm{y}}\right) = \dfrac{\partial^2 f}{\partial x \partial \mathrm{y}}$

The third and fourth cases are called mixed partial derivatives.

مثال: اذا كانت $f(x, \mathrm{y}) = x\,\mathrm{y}\,\mathrm{e}^{x^2+\mathrm{y}^2}$ ، جد f_x و f_{y} .

الحل:

$$f_x = x\,\mathrm{y} \cdot \left(\mathrm{e}^{x^2+\mathrm{y}^2} \cdot (2x)\right) + \mathrm{e}^{x^2+\mathrm{y}^2} \cdot (\mathrm{y}) = 2\,x^2\,\mathrm{y}\,\mathrm{e}^{x^2+\mathrm{y}^2} + \mathrm{y}\,\mathrm{e}^{x^2+\mathrm{y}^2}$$

$$= (2x^2 + 1)\,\mathrm{y}\,\mathrm{e}^{x^2+\mathrm{y}^2}$$

$$f_{\mathrm{y}} = x\,\mathrm{y} \cdot \left(\mathrm{e}^{x^2+\mathrm{y}^2} \cdot (2\mathrm{y})\right) + \mathrm{e}^{x^2+\mathrm{y}^2} \cdot (x) = 2x\mathrm{y}^2\,\mathrm{e}^{x^2+\mathrm{y}^2} + x\,\mathrm{e}^{x^2+\mathrm{y}^2}$$

$$= (2\mathrm{y}^2 + 1)x\,\mathrm{e}^{x^2+\mathrm{y}^2}$$

مثال: اذا كانت $f(x,\mathrm{y}) = \mathrm{e}^{\sin(\mathrm{y}/x)}$ ، جد f_x و f_{y} حيث $x \neq 0$.

الحل:

$$f_x = \mathrm{e}^{\sin(\mathrm{y}/x)} \cdot \cos(\mathrm{y}/x) \cdot \left(\frac{-\mathrm{y}}{x^2}\right) = \frac{-\mathrm{y}}{x^2} \cdot \cos(\mathrm{y}/x) \cdot \mathrm{e}^{\sin(\mathrm{y}/x)}$$

$$f_{\mathrm{y}} = \mathrm{e}^{\sin(\mathrm{y}/x)} \cdot \cos(\mathrm{y}/x) \cdot \left(\frac{1}{x}\right) = \frac{1}{x} \cdot \cos(\mathrm{y}/x) \cdot \mathrm{e}^{\sin(\mathrm{y}/x)}$$

مثال: اذا كانت $f(x,\mathrm{y}) = x\mathrm{y}^4 - 2x^2\mathrm{y}^5 + 4x^6 - 3\mathrm{y}^2$. جد كلاً من $\frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial f}{\partial x}\right)$ و $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \mathrm{y}}\right)$. ماذا تلاحظ في هذا المثال؟.

الحل: المطلوب أيجاد $f_{x\mathrm{y}}$ و $f_{\mathrm{y}x}$. نوجد أولاً $\frac{\partial f}{\partial x}$ و $\frac{\partial f}{\partial \mathrm{y}}$

$$\frac{\partial f}{\partial x} = f_x = \mathrm{y}^4 - 4x\mathrm{y}^5 + 24x^5$$

$$\frac{\partial f}{\partial \mathrm{y}} = f_{\mathrm{y}} = 4x\mathrm{y}^3 - 10x^2\mathrm{y}^4 - 6\mathrm{y}$$

الان

$$\frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial f}{\partial x}\right) = f_{x\mathrm{y}} = 4\mathrm{y}^3 - 20x\mathrm{y}^4 = 4\mathrm{y}^3(1 - 5\,x\,\mathrm{y})$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \mathrm{y}}\right) = f_{\mathrm{y}x} = 4\mathrm{y}^3 - 20x\mathrm{y}^4 = 4\mathrm{y}^3(1 - 5\,x\,\mathrm{y})$$

نلاحظ في هذا المثال أن

$$\frac{\partial}{\partial \mathrm{y}}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \mathrm{y}}\right)$$

مثال: اذا كانت $f(x,\mathrm{y}) = \frac{x^3\mathrm{y} - x\mathrm{y}^3}{x^2 + \mathrm{y}^2}$. جد f_x و f_{y} وبين مجال كل منهما.

الحل:

$$f_x = \frac{(x^2 + \mathrm{y}^2) \cdot (3x^2\mathrm{y} - \mathrm{y}^3) - (x^3\mathrm{y} - x\mathrm{y}^3) \cdot (2x)}{(x^2 + y^2)^2}$$

$$= \frac{3x^4\mathrm{y} - x^2\mathrm{y}^3 + 3x^2\mathrm{y}^3 - \mathrm{y}^5 - 2x^4\mathrm{y} + 2x^2\mathrm{y}^3}{(x^2 + \mathrm{y}^2)^2} = \frac{x^4\mathrm{y} + 4x^2\mathrm{y}^3 - \mathrm{y}^5}{(x^2 + \mathrm{y}^2)^2}$$

واضح أن مجال f_x هو

$$\mathrm{D}_{f_x} = \mathbb{R}^2 \setminus \{(0,0)\}$$

وبطريقة مشابهة نجد أن

$$f_{\mathrm{y}} = \frac{x^5 - 4x^3\mathrm{y}^2 - x\mathrm{y}^4}{(x^2 + \mathrm{y}^2)^2}$$

كذلك مجال f_{y} هو $\mathrm{D}_{f_{\mathrm{y}}} = \mathbb{R}^2 \setminus \{(0,0)\}$

مثال: اذا كانت $f(x,y)=\sqrt{x^2+y^2}$ ، جد f_x و f_y .

الحل:

$$f_x=\frac{1}{2}(x^2+y^2)^{\frac{-1}{2}}(2x)=\frac{x}{\sqrt{x^2+y^2}}$$

$$f_y=\frac{1}{2}(x^2+y^2)^{\frac{-1}{2}}(2y)=\frac{y}{\sqrt{x^2+y^2}}$$

Example: If $f(x,y)=x^3+x^2y^3-2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

Solution:

Holding y constant and differentiate with respect to x, we get

$$f_x(x,y)=3x^2+2xy^3$$

and so

$$f_x(2,1)=3.(2^2)+2.(2).1^3=12+4=16$$

Holding x constant and differentiate with respect to y, we get

$$f_y(x,y)=3x^2y^2-4y$$

$$f_y(2,1)=3.(2^2).1^2-4.(1)=12-4=8$$

Example: If $f(x,y)=4-x^2-2y^2$, find $f_x(1,1)$ and $f_y(1,1)$.

Solution: we have

$$f_x(x,y)=-2x \qquad , \qquad f_y(x,y)=-4y$$

$$f_x(1,1)=-2.(1)=-2 \qquad , \qquad f_y(1,1)=-4.(1)=-4$$

Example: If $f(x,y)=\sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution:

$$\frac{\partial f}{\partial x}=\cos\left(\frac{x}{1+y}\right)\cdot\left(\frac{(1+y)\cdot 1-x.(0)}{(1+y)^2}\right)=\frac{(1+y)}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$$
$$=\frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$$

$$\frac{\partial f}{\partial y}=\cos\left(\frac{x}{1+y}\right)\cdot\left(\frac{(1+y)\cdot 0-x.(1)}{(1+y)^2}\right)=\frac{-x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation $x^3 + y^3 + z^3 + 6xyz = 1$

Solution:

To find $\frac{\partial z}{\partial x}$ we differentiate implicitly with respect to x

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + \left(6xy\frac{\partial z}{\partial x} + 6yz\right) = 0$$

Solving this equation for $\frac{\partial z}{\partial x}$, we obtain

$$3x^2 + (3z^2 + 6xy)\frac{\partial z}{\partial x} + 6yz = 0$$

$$(3z^2 + 6xy)\frac{\partial z}{\partial x} = -3x^2 - 6yz$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy}$$

Similarly, implicit differentiation with respect to y gives

$$\frac{\partial z}{\partial y} = \frac{-y^2 - 2xz}{z^2 + 2xy}$$

Example: Find f_x , f_y and f_z if $f(x, y, z) = e^{xy}\ln z$

Solution:

Holding y and z constant and differentiating with respect to x , we have

$$f_x = ye^{xy}\ln z$$

Similarly,

$$f_y = xe^{xy}\ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$

Example: Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution:

$$f_x = 3x^2 + 2xy^3$$

$$f_y = 3x^2y^2 - 4y$$

$$f_{xx} = 6x + 2y^3$$

$$f_{yy} = 6x^2y - 4$$

$$f_{xy} = 6xy^2$$

$$f_{yx} = 6xy^2$$

Example: Calculate f_{xxyz} if $f(x,\mathrm{y},\mathrm{z}) = \sin(3x+\mathrm{yz})$

Solution:

$$f_x = 3\cos(3x+\mathrm{yz})$$

$$f_{xx} = -9\sin(3x+\mathrm{yz})$$

$$f_{xx\mathrm{y}} = -9\,\mathrm{z}\ \cos(3x+\mathrm{yz})$$

$$f_{xx\mathrm{yz}} = -9\,\mathrm{z}\,(-\sin(3x+\mathrm{yz}))\cdot\mathrm{y} + \cos(3x+\mathrm{yz})\cdot(-9)$$

$$= 9\,\mathrm{zy}\ \sin(3x+\mathrm{yz}) - 9\cos(3x+\mathrm{yz})$$

Example: Find $f_x(1,0)$ and $f_{\mathrm{y}}(1,0)$ if $f(x,\mathrm{y},\mathrm{z}) = x\mathrm{e}^{x\mathrm{y}}$

Solution:

$$f_x(x,\mathrm{y}) = x\,(\mathrm{e}^{x\mathrm{y}}\cdot\mathrm{y}) + \mathrm{e}^{x\mathrm{y}}\cdot(1) = x\mathrm{y}\mathrm{e}^{x\mathrm{y}} + \mathrm{e}^{x\mathrm{y}}$$

$$f_{\mathrm{y}}(x,\mathrm{y}) = x^2\mathrm{e}^{x\mathrm{y}}$$

$$f_x(1,0) = 1\cdot(0)\cdot\mathrm{e}^{1\cdot(0)} + \mathrm{e}^{1\cdot(0)} = 0+\mathrm{e}^0 = 0+1 = 1$$

$$f_{\mathrm{y}}(1,0) = 1^2\cdot\mathrm{e}^{1\cdot(0)} = \mathrm{e}^0 = 1$$

Example: For $f(x,\mathrm{y}) = x\mathrm{e}^{x^2\mathrm{y}}$, find f_x and f_{y} , evaluate each at the point $(1,\ln 2)$

Solution:

$$f_x(x,\mathrm{y}) = x\left(\mathrm{e}^{x^2\mathrm{y}}\cdot 2x\mathrm{y}\right) + \mathrm{e}^{x^2\mathrm{y}}\cdot(1) = 2x^2\mathrm{y}\mathrm{e}^{x^2\mathrm{y}} + \mathrm{e}^{x^2\mathrm{y}}$$

The partial derivative of f with respect to x at $(1,\ln 2)$ is

$$f_x(1,\ln 2) = 2\cdot(1^2)\cdot(\ln 2)\cdot\mathrm{e}^{1^2\ln 2} + \mathrm{e}^{1^2\ln 2}$$

$$= 2\ln 2\cdot\mathrm{e}^{\ln 2} + \mathrm{e}^{\ln 2} = 2\ln 2\cdot(2) + 2 = 4\ln 2 + 2.$$

$$f_{\mathrm{y}}(x,\mathrm{y}) = x\ \mathrm{e}^{x^2\mathrm{y}}(x^2) = x^3\mathrm{e}^{x^2\mathrm{y}}$$

The partial derivative of f with respect to y at $(1,\ln 2)$ is

$$f_{\mathrm{y}}(1,\ln 2) = 1^3\cdot\mathrm{e}^{1^2\ln 2} = \mathrm{e}^{\ln 2} = 2$$

Example: Find the second partial derivatives of

$f(x,\mathrm{y}) = 3x\mathrm{y}^2 - 2\mathrm{y} + 5x^2\mathrm{y}^2$, and determine the value of $f_{x\mathrm{y}}(-1,2)$

Solution:

$$f_x(x,\mathrm{y}) = 3\mathrm{y}^2 + 10x\mathrm{y}^2$$

$$f_{xx}(x,\mathrm{y}) = 10\mathrm{y}^2$$

$$f_{x\mathrm{y}}(x,\mathrm{y}) = 6\mathrm{y} + 20xy$$

$$f_y(x,y) = 6xy - 2 + 10x^2y$$

$$f_{yy}(x,y) = 6x + 10x^2$$

$$f_{yx}(x,y) = 6y + 20xy$$

At $(-1,2)$ the value of f_{xy} is

$f_{xy}(-1,2) = 6(2) + 20(-1)(2) = 12 - 40 = -28$

Example: Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function given by $f(x,y,z) = ye^x + x \ln z$.

Solution:

First partials: $$f_x(x,y,z) = ye^x + \ln z$$

$$f_z(x,y,z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x,y,z) = \frac{1}{z}$$

$$f_{zx}(x,y,z) = \frac{1}{z}$$

$$f_{zz}(x,y,z) = \frac{-x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x,y,z) = \frac{-1}{z^2}$$

$$f_{zxz}(x,y,z) = \frac{-1}{z^2}$$

$$f_{zzx}(x,y,z) = \frac{-1}{z^2}$$

مثال: اذا كانت $f(x,y) = \sin^{-1}(2x + 3y)$ ، جد f_x و f_y .

الحل: $f_x = \frac{2}{\sqrt{1-(2x+3y)^2}}$ ، $f_y = \frac{3}{\sqrt{1-(2x+3y)^2}}$

مثال: جد $f_x(1,-1)$ و $f_y(1,-1)$ للدالة $f(x,y) = \ln(1 + xy^2)$

الحل: $f_x = \frac{y^2}{1+xy^2}$, $f_x(1,-1) = \frac{(-1)^2}{1+1\times(-1)^2} = \frac{1}{1+1} = \frac{1}{2}$

$f_y = \frac{2xy}{1 + xy^2}$, $f_y(1,-1) = \frac{2(1)(-1)}{1 + 1 \times (-1)^2} = \frac{-2}{1+1} = -1$