Example: A box contains 4 white , 6 red , 5 blue and 3 orange balls. Three balls are drawn at random from the box ; what is the probability that :

a_ Two of them are white ,

b_ one is red and two are orange ,

c_ none of them are blue ,

d_ all of them are either white or blue .

Sol/

$$1 - \boldsymbol{P(A)} = \frac{\boldsymbol{C_2^4 C_1^{14}}}{\boldsymbol{C_3^{18}}} = \frac{\boldsymbol{6*14}}{\boldsymbol{816}} = \frac{\boldsymbol{84}}{\boldsymbol{816}} = \boldsymbol{0.103}$$

$$2 - \boldsymbol{P(B)} = \frac{\boldsymbol{C_1^6 C_2^3}}{\boldsymbol{C_3^{18}}} = \boldsymbol{0.0221}$$

$$3 - \boldsymbol{P(C)} = \frac{\boldsymbol{C_3^{13}}}{\boldsymbol{C_3^{18}}} = \boldsymbol{0.351}$$

$$4 - \boldsymbol{P(B)} = \frac{\boldsymbol{C_3^4 + C_3^5}}{\boldsymbol{C_3^{18}}} = \boldsymbol{0.0172}$$

Example: Two cards are drawn one after the other without replacement from a well-shuffled deck of 52 cards ; Find the probability that they are :

A_ of spades

B_ one is heart and the other is diamond

C_ number 10 of spade and Jack of club

D_ red cards

E_ numbered cards

F_ picture cards

Sol/

ملاحظه: عندما يذكر بالسؤال without replacement بدون ارجاع أي عملية السحب والاختيار تكون بدون ارجاع فإن العدد يتناقص، وإذا لم يذكر في السؤال (السحب او الاختيار بدون ارجاع) فإن العدد يبقى كما هو او يحل شيء محل اخر. وسيتم توضيح ذلك في حل السؤال التالي:

Without replacement	with replacement
$P(A) = \frac{C_1^{13}}{C_1^{52}} * \frac{C_1^{12}}{C_1^{51}} = \frac{13}{52} * \frac{12}{51} =$	$P(A) = \frac{C_2^{13}}{C_2^{52}}$
$P(B) = \frac{C_1^{13}}{C_1^{52}} * \frac{C_1^{13}}{C_1^{51}} = \frac{13}{52} * \frac{13}{51} =$	$P(B) = \frac{C_1^{13} * C_1^{13}}{C_2^{52}}$
$P(C) = \frac{C_1^{1}}{C_1^{52}} * \frac{C_1^{1}}{C_1^{51}} = \frac{1}{52} * \frac{1}{51} =$	$P(C) = \frac{C_1^{1} * C_1^{1}}{C_2^{52}}$
$P(D) = \frac{C_1^{26}}{C_1^{52}} * \frac{C_1^{25}}{C_1^{51}} = \frac{26}{52} * \frac{25}{51} =$	$P(D) = \frac{C_2^{26}}{C_2^{52}}$
$P(E) = \frac{C_1^{40}}{C_1^{52}} * \frac{C_1^{39}}{C_1^{51}} = \frac{40}{52} * \frac{39}{51} =$	$P(E) = \frac{C_2^{40}}{C_2^{52}}$
$P(F) = \frac{C_1^{12}}{C_1^{52}} * \frac{C_1^{11}}{C_1^{51}} = \frac{12}{52} * \frac{11}{51} =$	$P(F) = \frac{C_2^{12}}{C_2^{52}}$

Example: A card is selected at random from an ordinary deck of 52 cards , let

A= {the card is spade}

B= {the card is a picture card; i.e., a Jack, Queen and king}

Find $P(A)$, $P(B)$ & $P(A \cup B)$

Sol/

$$P(A) = \frac{C_1^{13}}{C_1^{52}} = \frac{13}{52} \quad ; \quad P(B) = \frac{C_1^{12}}{C_1^{51}} = \frac{12}{51}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cap B) = \{\, the\ card\ is\ picture\ card\ of\ spade\}$$

$$\therefore P(A \cap B) = \frac{C_1^{3}}{C_1^{52}} = \frac{3}{52}$$

$$\therefore P(A \cup B) = \frac{13}{52} + \frac{12}{51} - \frac{3}{52} = \frac{22}{52}$$

Example: Two prizes are to be given to two students who are chosen at random from the students [Ayad, Bashir, Khalid, Mazin and Sami];

1- Find elements of the sample space

2- Find the probability of the events

E_1 : Ayad receives a prize

E_2 : either Bashir or Sami receives a prize

E_3 : Khalid dose not receives a prize

E_4 : either Khalid or Mazin receives a prize but not both

Sol/

$$\boldsymbol{C_2^5} = \frac{\mathbf{5!}}{\mathbf{2! * 3!}} = \mathbf{10} \quad ways$$

1- $\therefore \boldsymbol{S} = \{ \boldsymbol{AB}, \boldsymbol{AK}, \boldsymbol{AM}, \boldsymbol{AS}, \boldsymbol{BK}, \boldsymbol{BM}, \boldsymbol{BS}, \boldsymbol{KM}, \boldsymbol{KS}, \boldsymbol{MS}\}$

2- $E_1 = \{ \boldsymbol{AB}, \boldsymbol{AK}, \boldsymbol{AM}, \boldsymbol{AS}\}$

$$\therefore P(E_1) = \frac{\boldsymbol{n}(E_1)}{\boldsymbol{n}} = \frac{\mathbf{4}}{\mathbf{10}} \quad or \quad \frac{\boldsymbol{C_1^4}}{\boldsymbol{C_2^5}}$$

$E_2 = \{ \boldsymbol{B}, \boldsymbol{S}\}$ &

$\boldsymbol{B} = \{ \boldsymbol{AB}, \boldsymbol{BK}, \boldsymbol{BM}, \boldsymbol{BS}\} \quad S = \{ \boldsymbol{AS}, \boldsymbol{BS}, \boldsymbol{KS}, \boldsymbol{MS}\}$

$$\therefore P(B) = \frac{\boldsymbol{n}(B)}{\boldsymbol{n}} = \frac{\mathbf{4}}{\mathbf{10}} \quad ; \quad \therefore P(S) = \frac{\boldsymbol{n}(S)}{\boldsymbol{n}} = \frac{\mathbf{4}}{\mathbf{10}}$$

$$\therefore (B \cap S) = \{BS\} \quad \rightarrow P(B \cap S) = \frac{\mathbf{1}}{\mathbf{10}}$$

$$\therefore P(B \cup S) = P(B) + P(S) - P(B \cap S) = \frac{\mathbf{4}}{\mathbf{10}} + \frac{\mathbf{4}}{\mathbf{10}} - \frac{\mathbf{1}}{\mathbf{10}} = \frac{\mathbf{7}}{\mathbf{10}}$$

$E_3 = \{ AB, AM, AS, BM, BS, MS\}$

$$\therefore P(E_3) = \frac{\boldsymbol{n}(E_3)}{\boldsymbol{n}} = \frac{\mathbf{6}}{\mathbf{10}}$$

$E_4 = \{ K, M\}$ & $\quad K = \{AK, BK, KS\} \quad M = \{AM, BM, MS\}$

$$\therefore P(K \cup M) = P(K) + P(M) = \frac{\mathbf{3}}{\mathbf{10}} + \frac{\mathbf{3}}{\mathbf{10}} = \frac{\mathbf{6}}{\mathbf{10}}$$

4-3 : Independent Events الحوادث المستقلة

Two events A & B are independent iff :-

$$P(A \cap B) = P(A) * P(B)$$

Then :

Any two events A & B the probability :

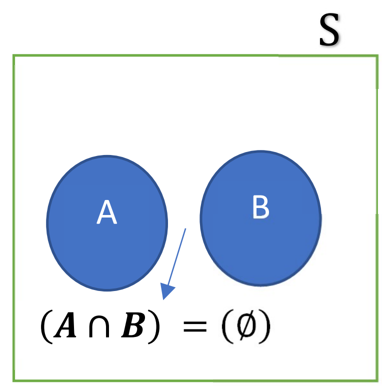
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If : - هما الحدثان اللذان لا يمكن ان يحدثا في آن واحد

Mutually exclusive : $P(A \cup B) = P(A) + P(B) - 0$

حوادث متنافية

$\therefore P(A \cap B) = P(\emptyset) = 0$



If :

Independent events

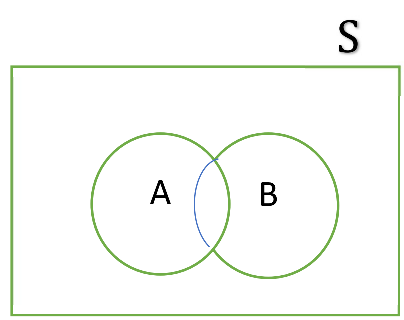
يقال لحدثان انهما مستقلان اذا كان احتمال ظهور احدهما لا يتأثر بحدوث الحدث الاخر : -

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$\because P(A \cap B) = P(A) * P(B)$

في حال تحقق الشرط أعلاه يقال ان الحدثان مستقلان وان

احتمال الاتحاد كما يلي :-



$$P(A \cup B) = P(A) + P(B) - P(A) * P(B)$$

Example: Suppose that two machines 1 & 2 are operated in a factory independently of each other. Let $\boldsymbol{A}$ be the event that machine 1 will become inoperative during a given 8- hour period, let $\boldsymbol{B}$ be the event that machine 2 will become inoperative during the same period, and suppose that $\boldsymbol{P}(\boldsymbol{A}) = \frac{\mathbf{1}}{\mathbf{3}}$ and $\boldsymbol{P}(\boldsymbol{B}) = \frac{\mathbf{1}}{\mathbf{4}}$. we shall determine the probability that at least one of the machines will become inoperative during a given period.

The probability $\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$; that both machines will become inoperative during the period is :

$$\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) * \boldsymbol{P}(\boldsymbol{B}) = \frac{\mathbf{1}}{\mathbf{3}} * \frac{\mathbf{1}}{\mathbf{4}} = \frac{\mathbf{1}}{\mathbf{12}}$$

Therefore, the probability $\boldsymbol{P}(\boldsymbol{A} \cup \boldsymbol{B})$ that at least one of machines will become inoperative during the period is :

$$\boldsymbol{P}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) + \boldsymbol{P}(\boldsymbol{B}) - \boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$$

$$= \frac{\mathbf{1}}{\mathbf{3}} + \frac{\mathbf{1}}{\mathbf{4}} - \frac{\mathbf{1}}{\mathbf{12}} = \frac{\mathbf{6}}{\mathbf{12}} = \frac{\mathbf{1}}{\mathbf{2}}$$

-.-

A & B are independent

$$\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A}) * \boldsymbol{P}(\boldsymbol{B})$$

$$\boldsymbol{P}(\boldsymbol{A}) * \boldsymbol{P}(\boldsymbol{B}) = \boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B})$$

A & B are not independent

$$\boldsymbol{P}(\boldsymbol{A} \cap \boldsymbol{B}) \neq \boldsymbol{P}(\boldsymbol{A}) * \boldsymbol{P}(\boldsymbol{B})$$

Example: $\boldsymbol{S} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\}$

$\boldsymbol{A} = \{\mathbf{2}, \mathbf{4}, \mathbf{6}\}$ & $\boldsymbol{B} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$

Are $\boldsymbol{A}$ & $\boldsymbol{B}$ independent ?

Sol/

$$\boldsymbol{P(A) = \frac{3}{6}} \quad ; \quad \boldsymbol{P(B) = \frac{4}{6}}$$

$$\boldsymbol{(A \cap B) = \{2, 4\}} \quad ; \quad \boldsymbol{P(A \cap B) = \frac{2}{6} = \frac{1}{3}}$$

$$\boldsymbol{P(A \cap B) = P(A) * P(B)}$$

$$\boldsymbol{\frac{1}{3} = \frac{3}{6} * \frac{4}{6}} \quad \rightarrow \quad \boldsymbol{\frac{1}{3} = \frac{1}{3}}$$

$\therefore$ A & B are independent events .

- Theorem :-

If A & B are two independent events then :-

1- A^c & B are also independent.

2- A & B^c are also independent.

3- A^c & B^c are also independent.

Proof/

1- $\boldsymbol{P(A^c \cap B) = P(A^c) * P(B)}$

$$\because \boldsymbol{P(A^c \cap B) = P(B/A) = P(B) - P(A \cap B)}$$
$$\boldsymbol{= P(B) - [P(A) * P(B)]}$$
$$\boldsymbol{= P(B)[1 - P(A)]}$$
$$\boldsymbol{= P(B) * P(A^c)}$$

2- $\boldsymbol{P(A \cap B^c) = P(A) * P(B^c)}$

$$\because \boldsymbol{P(A \cap B^c) = P(A/B) = P(A) - P(A \cap B)}$$
$$\boldsymbol{= P(A) - [P(A) * P(B)]}$$
$$\boldsymbol{= P(A)[1 - P(B)]}$$
$$\boldsymbol{= P(A) * P(B^c)}$$

3- $P(A^c \cap B^c) = P(A^c) * P(B^c)$

$P(A^c \cap B^c) = P(A \cup B)^c$ حسب قانون دي موركان

$= 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [P(A) + P(B) - P(A) * P(B)]$

$= [1 - P(A)] - P(B) + P(A) * P(B)$

$= P(A^c) - P(B)[1 - P(A)]$

$= P(A^c) - P(B)P(A^c)$

$= P(A^c)[1 - P(B)]$

$\therefore P(A^c \cap B^c) = P(A^c) * P(B^c)$

-.-

If A, B & C are three events , we said that A, B & C are independent events iff: -

1- $P(A \cap B) = P(A) * P(B)$

1- $P(A \cap C) = P(A) * P(C)$

1- $P(B \cap C) = P(B) * P(C)$

1- $P(A \cap B \cap C) = P(A) * P(B) * P(C)$

If only three satisfy then we said A, B & C are pairwise-independent.

بمعنى مستقلة مثنى - مثنى (زوج - زوج)

Example: Let $S = \{123, 132, 321, 312, 231, 213, 111, 222, 333\}$ and each of the nine elementary events in S occurs with probability 1/9 . Let A_k be the event that the k-th digit is 1 , $k = 1,2,3$ then :-

Sol/

$A_1 = \{111, 123, 132\}$

$\rightarrow P(A_1) = \frac{3}{9} = \frac{1}{3}$