$$A_2 = \{111, 213, 312\}$$

$$\rightarrow P(A_2) = \frac{3}{9} = \frac{1}{3}$$

$$A_3 = \{111, 321, 231\}$$

$$\rightarrow P(A_3) = \frac{3}{9} = \frac{1}{3}$$

$$A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \{111\}$$

$$P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = 1/9$$

Then :

$$P(A_1 \cap A_2) = P(A_1) * P(A_2) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$$

And

$$P(A_1 \cap A_3) = P(A_1) * P(A_3) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$$

And

$$P(A_2 \cap A_3) = P(A_2) * P(A_3) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$$

$\therefore$ A , B & C are pairwise-independent.

$$A_1 \cap A_2 \cap A_3 = \{111\}$$

$$\therefore P(A_1 \cap A_2 \cap A_3) = \frac{n(A_1 \cap A_2 \cap A_3)}{n} = 1/9$$

And

$$\rightarrow P(A_1 \cap A_2 \cap A_3) = P(A_1) * P(A_2) * P(A_3)$$

$$\rightarrow \frac{1}{9} = \frac{1}{3} * \frac{1}{3} * \frac{1}{3}$$

$$\rightarrow \frac{1}{9} \neq \frac{1}{27}$$

$\therefore$ A , B & C are not independent.

4-4 : Conditional probability and Bay's theorem الاحتمال الشرطي ونظرية بيز

Definition:

If A & B are two events of a probability space S . then the conditional probability of event A on the occurrence (حدوث) of the event B, denoted by $P(A|B)$ is: -

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided that} \quad P(B) > 0$$

This symbol may be read " probability of A given B "

the conditional probability of B given A is given as:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided that} \quad P(A) > 0$$

- Notes:
- If A & B are dependent events, then:
  a- $P(A \cap B) = P(A|B) * P(B)$
  b- $P(A \cap B) = P(B|A) * P(A)$

- If A & B are independent events, then:
  a- $P(A|B) = P(A)$

  Where $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) * P(B)}{P(B)} = P(A)$, $P(B) > 0$

  b- $P(B|A) = P(B)$

  Where $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A) * P(B)}{P(A)} = P(B)$, $P(A) > 0$

-.-

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for any two events: -

For mutually exclusive $A \cap B = \emptyset$; $P(A \cap B) = 0$

$\therefore$ $P(A \cup B) = P(A) + P(B)$; m.e.

For independent events $P(A \cap B) = P(A) * P(B)$

$$\therefore\ P(A \cup B) = P(A) + P(B) - [\,P(A) * P(B)]$$

For conditional probability

$$\therefore\ P(A \cup B) = P(A) + P(B) - [\,P(A|B) * P(B)]$$

Or $P(A \cup B) = P(A) + P(B) - [\,P(B|A) * P(A)]$

Example: A box contains Red and White balls. Each ball is labeled either X or Z . the composition of the box is show below:

	R	W	
X	5	3	8
Z	1	2	3
	6	5	11

Let us now assume that a ball is to be selected at random from this box; What is the probability of getting a Red ball if it was labeled X .

Sol/ A = a ball is labeled X

B = a Red ball

$$\therefore P(A) = \frac{8}{11} \quad ; \quad P(B) = \frac{6}{11} \quad ; \quad P(A \cap B) = \frac{5}{11}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{5/11}{8/11} = \frac{5}{8}$$

- If a ball is drawn and found it white, what is the probability that it was labeled Z ? $P(Z|W)$

$$P(A) = \frac{8}{11} \quad ; \quad P(B) = \frac{6}{11} \quad ; \quad P(A \cap B) = \frac{5}{11} \ ; \quad P(B \cap Z) = \frac{1}{11}$$

$$P(W) = \frac{5}{11} \quad ; \quad P(W \cap A) = \frac{3}{11} \quad ; \quad P(Z) = \frac{3}{11} \quad ; \quad P(W \cap Z) = \frac{2}{11}$$

$$P(Z|W) = \frac{P(Z \cap W)}{P(W)} = \frac{2/11}{5/11} = \frac{2}{5}$$

- What is the probability that it is labeled X if it was Red ?

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{5/11}{6/11} = \frac{5}{6}$$

- What is the probability that it is labeled X if it was White ?

$$P(A|W) = \frac{P(A\cap W)}{P(W)} = \frac{3/11}{5/11} = \frac{3}{5}$$

4-4-1 : Three Conditional events or more:

If we have 3 events A,B and C ; then :

$$P(A \cap B \cap C) = P(A) * P(B|A) * P(C|A \cap B)$$

$$\therefore \ P(A \cap B) = P(A) * P(B|A)$$

Where

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

And

$$P(A \cap B \cap C) = P(A \cap B) * P(C|A \cap B)$$

$$\therefore \ P(C|A \cap B) = \frac{P(A\cap B\cap C)}{P(A\cap B)}$$

In general; If we have n events then: -

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) * P(A_2|A_1) * P(A_3|A_1 \cap A_2) * \ldots * P(A_n|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{n-1})$$

Example: Four cards are to be drawn successively (على التوالي) at random and without replacement from an ordinary deck of playing cards; What is the probability of receiving a spade , a heart , a diamond , and a club.

Sol/ without replacement (العدد يتناقص) السحب بدون ارجاع

Let Spade , Heart , diamond , Club

A B C D

$$P(A \cap B \cap C \cap D) = P(A) * P(B|A) * P(C|A \cap B) * P(D|A \cap B \cap C)$$

$$= \frac{C_1^{13}}{C_1^{52}} * \frac{C_1^{13}}{C_1^{51}} * \frac{C_1^{13}}{C_1^{50}} * \frac{C_1^{13}}{C_1^{49}}$$

$$= \frac{13}{52} * \frac{13}{51} * \frac{13}{50} * \frac{13}{49} = \frac{2197}{519800}$$

- في هذا السؤال نلاحظ ان السحب كان من كل نوع من أنواع الأوراق، اما اذا كانت جميع الأوراق المسحوبة هي (4 اوراق) من نوع واحد، على سبيل المثال من النوع Spade سيكون الحل :

$$P(A_1 \cap A_2 \cap A_3 \cap A_4)$$
$$= P(A_1) * P(A_2|A_1) * P(A_3|A_1 \cap A_2) * P(A_4|A_1 \cap A_2 \cap A_3)$$
$$= \frac{C_1^{13}}{C_1^{52}} * \frac{C_1^{12}}{C_1^{51}} * \frac{C_1^{11}}{C_1^{50}} * \frac{C_1^{10}}{C_1^{49}}$$
$$= \frac{13}{52} * \frac{12}{51} * \frac{11}{50} * \frac{10}{49}$$

Example: Let A & B be events with $P(A) = \frac{1}{2}$; $P(B) = \frac{1}{3}$ & $P(A \cap B) = \frac{1}{4}$
Find $P(A|B)$; $P(B|A)$; $P(A \cup B)$; $P(A^c|B^c)$; $P(B^c|A^c)$

Sol/

- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$
- $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{2}{4}$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}$
- $P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)}$
- $P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(A^c \cap B^c)$ by De Morgan's $(A^c \cap B^c) = (A \cup B)^c$

$\therefore$ $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}$

$\therefore$ $P(A^c|B^c) = \frac{5/12}{2/3} = \frac{5}{8}$

- $P(B^c|A^c) = \frac{P(A^c \cap B^c)}{P(A^c)} = \frac{P(A^c \cap B^c)}{1 - P(A)} = \frac{5/12}{1/2} = \frac{5}{6}$

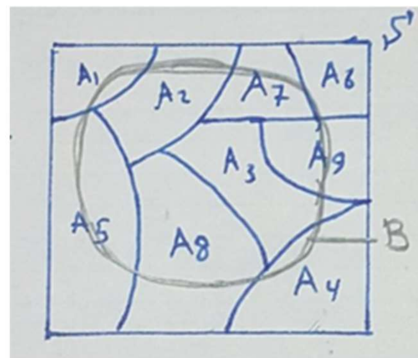
4-4-2 : Baye's Low

General theorem for conditional probability :-

Suppose that $A_1, A_2, A_3, \dots, A_n$ are mutually exclusive events. Such that $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n = S$ and $P(A_1) > 0$ where $i = 1,2,3, \dots, n$ then for any event B.

$$P(B) = P(B|A_1)\,P(A_1) + P(B|A_2)\,P(A_2) + \cdots + P(B|A_n)\,P(A_n)$$

Proof/



$B = B \cap S$

$A_1, A_2, A_3, \dots, A_n$ partitions of the S then:-

1- $A_\mathrm{i} \cap A_\mathrm{j} = \emptyset$ mutually exclusive
2- $\bigcup_{i=1}^{n} A_i = S$

$$\therefore B = B \cap \left(\bigcup_{i=1}^{n} A_i\right) = B \cap [A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n]$$

$$= [(B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup \dots \cup (B \cap A_n)]$$

$$\Longrightarrow P(B) = P[(B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup \dots \cup (B \cap A_n)]$$

$$= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + \cdots + P(B \cap A_n)$$

$$\therefore P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Longrightarrow P(B \cap A_1) = P(B|A_1)\,P(A_1) \;;\; P(B \cap A_2) = P(B|A_2)\,P(A_2) \dots$$

$$\Longrightarrow P(B) = P(B|A_1)\,P(A_1) + P(B|A_2)\,P(A_2) + \cdots + P(B|A_n)\,P(A_n)$$

$$\therefore\; P(B) = \sum_{i=1}^{n} P(B|A_i)\,P(A_i)$$

$$\therefore\; P(A) = \sum_{i=1}^{n} P(A|B_i)\,P(B_i)$$

4-4-3 : Baye's Theorem

Let the events $A_1, A_2, A_3, \dots, A_k$ for a partition of the sample space S such that $P(A_i) > 0$, for $i = 1,2,3, \dots, k$ and let B be any event in S such that $P(B) > 0$

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} \quad ; P(B) > 0$$

$\therefore$ from **4-4-2** $\Rightarrow P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$

$$\therefore P(A_i|B) = \frac{P(A_i \cap B)}{\sum_{i=1}^{n} P(B|A_i)\, P(A_i)} = \frac{P(B|A_i)\, P(A_i)}{\sum_{i=1}^{n} P(B|A_i)\, P(A_i)}$$

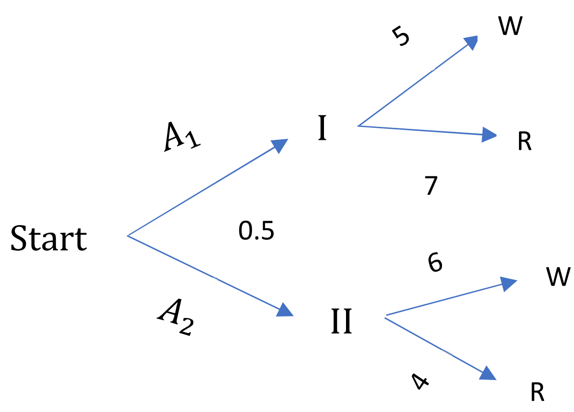
Example: consider two urns ; urnI contains 5 white and 7 red balls , urnII contains 6 white and 4 red balls; one of the urns is selected at random and a ball is drawn from it :

Find the probability that the ball drawn will be white ?

Sol/ A_1 = urnI is chosen

A_2 = urnII is chosen

B = white ball is drawn



$$P(B) = P(B|A_1)\, P(A_1) + P(B|A_2)\, P(A_2)$$

$$P(B|A_1) = \frac{P(B \cap A_1)}{P(A_1)} = \frac{C_1^5}{C_1^{12}} = \frac{5}{\mathbf{12}}$$

$$P(B|A_2) = \frac{P(B \cap A_2)}{P(A_2)} = \frac{C_1^6}{C_1^{10}} = \frac{6}{\mathbf{10}}$$