Exp) Prove the following :

1_ $C_r^n = C_{n-r}^n$

2- $C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$

3_ $C_r^{n+1} = C_{r-1}^n + C_r^n$

Proof /

1_ $C_r^n = C_{n-r}^n = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!(\cancel{n}-\cancel{n}+r)!} = \frac{n!}{r!(n-r)!}$

2_ $C_r^n = C_r^{n-1} + C_{r-1}^{n-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)![n-1-(r-1)]!}$

$= \frac{(n-1)!}{r!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)!} = \frac{(n-1)!}{r(r-1)!(n-r-1)!} + \frac{(n-1)!}{(r-1)!(n-r)(n-r-1)!}$

$= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{1}{r} + \frac{1}{n-r}\right] = \frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{n-r+r}{r(n-r)}\right] = \frac{(n-1)!}{(r-1)!(n-r-1)!} * \frac{n}{r(n-r)}$

$= \frac{n(n-1)!}{r(r-1)!(n-r)(n-r-1)!} = \frac{n!}{r!(n-r)!} = C_r^n$

3_ $C_r^{n+1} = C_{r-1}^n + C_r^n = \frac{n!}{(r-1)![n-(r-1)]!} + \frac{n!}{r!(n-r)!}$

$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r+1)(n-r)!} + \frac{n!}{r(r-1)!(n-r)!}$

$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$

$= \frac{n!}{(r-1)!(n-r)!} \cdot \left[\frac{r+n-r+1}{r(n-r+1)}\right] = \frac{n!}{(r-1)!(n-r)!} * \frac{n+1}{r(n-r+1)}$

$= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!} = \frac{(n+1)!}{r!(n-r+1)!}$

$= \frac{(n+1)!}{r![(n+1)-r]!} = C_r^{n+1}$

2-4) Combinations and Binomial theorem

_ التوافيق ونظرية ثنائي الحدين _

If (n) is positive integer then :

$$(a+b)^n = \sum_{r=0}^{n} C_r^n \, a^r \, b^{n-r}$$

$$(a+b)^n = C_0^n a^0 b^n + C_1^n a^1 b^{n-1} + C_2^n a^2 b^{n-2} + \cdots + C_n^n a^n b^{n-n}$$

$$= b^n + nab^{n-1} + \frac{n(n-1)}{2!} a^2 b^{n-2} + \cdots + a^n$$

$\rightarrow n = 0 \;:\; (a+b)^0 = 1$

$n = 1 \;:\; (a+b)^1 = a + b$

$n = 2 \;:\; (a+b)^2 = a^2 + 2ab + b^2$

$n = 3 \;:\; (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$n = 4 \;:\; (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$n = 5 \;:\; (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$n = 6 \;:\; (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

مما تقدم يمكن ملاحظة الخواص التالية من مفكوك $(a+b)^n$

1_ يوجد في المفكوك $(n+1)$ من الحدود

2_ مجموع اسي كل من a و b يساوي n

3_ يتناقص اس a حدا بعد من n الى الصفر ويتزايد اس b من الصفر الى n حداً بعد حد .

4_ معامل كل حد هو C_r^n حيث ان r هو اس أي من a أو b .

5_ تتساوى معاملات الحدود التي تبتعد عن بداية المفكوك ونهايته بنفس المقدار .

6_ نلاحظ أيضاً ان معاملات القوى $(a+b)$ المتتالية يمكن ترتيبها في مثلث من الاعداد يسمى مثلث باسكال

n=0	1
n=1	1 1
n=2	1 2 1
n=3	1 3 3 1
n=4	1 4 6 4 1
n=5	1 5 10 10 5 1
n=6	1 6 15 20 15 6 1
n=7	1 7 21 35 35 21 7 1
n=8	1 8 28 56 70 56 28 8 1

$\therefore\ (a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

Lemma :-

For any positive integer n , we have :

$(1+x)^n = \sum_{r=0}^{n} C_r^n\, x^r$

Proof /

We have $(a+b)^n = \sum_{r=0}^{n} C_r^n\, a^r b^{n-r}$

Let $a = x$, $b = 1$

$$\rightarrow (1+x)^n = \sum_{r=0}^{n} C_r^n\, x^r 1^{n-r} = \sum_{r=0}^{n} C_r^n\, x^r$$

Prove that

$2^n = \sum_{r=0}^{n} C_r^n$

Proof / we have $(a+b)^n = \sum_{r=0}^{n} C_r^n\, a^r b^{n-r}$

Let $a = 1$ and $b = 1$

Then$\rightarrow (1+1)^n = \sum_{r=0}^{n} C_r^n\, 1^r 1^{n-r} \rightarrow 2^n = \sum_{r=0}^{n} C_r^n$

$= C_0^n + C_1^n + C_2^n + \cdots + C_n^n$

Exp) How many different subject can be made for a set A which contain 3 elements .

$$C_0^3 + C_1^3 + C_2^3 + C_3^3 = 1 + 3 + 3 + 1 = 8$$

Or

$$\rightarrow \ 2^n = 2^3 = 8$$

Exp) Prove that $(3x^2 + y)^5$

Proof/

We have $(a+b)^n = \sum_{r=0}^{n} C_r^n \, a^r b^{n-r}$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Let $a = 3x^2$ *and* $b = y$

$$(3x^2 + y)^5 = (3x^2)^5 + 5(3x^2)^4y + 10(3x^2)^3y^2 + 10(3x^2)^2y^3 + 5(3x^2)y^4 + y^5$$

$$= 342x^{10} + 405x^8y + 270x^6y^2 + 90x^4y^3 + 15x^2y^4 + y^5$$

Exp) Prove that $(x + y)^5$

Proof/

We have $(a+b)^n = \sum_{r=0}^{n} C_r^n \, a^r b^{n-r}$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Let $a = x$ *and* $b = y$

$$\therefore (x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

2-5) Problems

1_ A coin is thrown and a dice is cast ; determine the number of sample points in the sample space of this experiment , If we tossing the coin first and if we tossing the dice first . (H.W)

2_ A dice is tossing three time , Determine the number of sample points in the sample space of this experiment . (H.W)

استخدم مخطط الشجرة لحل السؤال الأول والثاني .

3_ How many 3_digit numbers can be formed from the digits 2,4,6,7,9 ? (H.W)

4_ How many four number committee can be formed from a group of 7 person.(H.W)

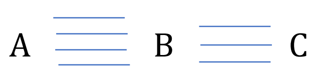
5_A student is to answer 10 out of 13 questions on an exam.

I. How many choices has he? (H.W)
II. How many choices has if he must answer the first two questions?
III. How many choices has if he must answer the first or second questions?
IV. How many choices has if he must answer exactly 3 of the first 5 questions?
V. How many choices has if he must answer at least 3 of the first 5 questions?

6_ There are 5 roads between city A and city B ; and 4 roads between B and C .

I. In how many ways can one drive from city A to city C by way of B .
II. In how many ways can one drive from A to C and back to A passing through B on both trips without the same road more than once ?

Sol/



I. 5 * 4 = 20
II. 5 * 4 * 3 * 4 = 240

7_a_ In how many ways can a student arrange his 8 textbooks on a shelf ?

b_ In how many ways can he do this if three specific books must be together ?

Sol/

a_ $n! = 8! = 8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 = 40320 \quad ways$

b_ [ABCDEFGH] على فرض ان الكتب تأخذ الاحرف التالية

3! * 6! = 6 * 120 = 720 ways

8_ Find n if i) $P_3^n = 336$ (H.W)

ii) $P_3^n = 5P_2^n$ (H.W)

iii) $2P_2^n + 50 = P_2^{2n}$

Sol/ $2P_2^n + 50 = P_2^{2n}$; we have $P_r^n = \frac{n!}{(n-r)!}$

$$\therefore 2\frac{n!}{(n-2)!} + 50 = \frac{2n!}{(2n-2)!} \rightarrow 2\frac{n(n-1)(n-2)!}{(n-2)!} + 50 = \frac{2n(2n-1)(2n-2)!}{(2n-2)!}$$

$$\rightarrow 2n(n-1) + 50 = 2n(2n-1)$$

$$\rightarrow 2n^2 - 2n + 50 = 4n^2 - 2n$$

$$\rightarrow 50 = 2n^2 \rightarrow n^2 = \frac{50}{2} = 25$$

$$\therefore n = \sqrt{25} = 5$$

9_ Mr. Ahmed has 5 suits , 7 ties and 4 pairs of shoes ; In how many ways can he chouse 3 suits , 4 ties and 2 pairs of shoes to take along on a business trip . (H.W)

Hint : $C_3^5 * C_4^7 * C_2^4$

10_ A student is to answer 7 questions out of 10 in an exam . How many choices has he :

I. If he must answer the first 3 questions
II. If he must not answer the last question ?

Hint : I. C_4^7 II. C_7^9

11_ If 4 Iraqis , 3 Egyptians and 2 Jordanians are to be seated in a row . In how many ways can this be done if the people of the same nationality must sit together ?

Sol/

$$n! = (4! * 3! * 2!) * 3! = 1728 \ ways$$

12_ Solve problem 11 if the sit at around table .

Sol/

$$(n-1)! = (4! * 3! * 2!) * 2! = 576 \ ways$$

13_ A class contains 10 boys and 5 girls : (H.W)

I. In how many ways can a teacher choose a committee of 4 ?
II. How many ways of these committee will contain 2 girls ?
III. How many of these committee will contain at least one boy ?

14_ A women has 11 class friends :

I. In how many ways can she invite 5 of them to dinner ?
II. In how many ways if two of the friends are married and will not attend separately ?
III. In how many ways if two of them are not on speaking terms and will not attend together ?

Sol/

I. C_5^{11}
II. $C_3^9 + C_5^9$
C_3^9 حضورهم معا + C_5^9 في حالة عدم حضورهم
III. $C_5^{11} + C_4^9 + C_4^9$
C_5^{11} عدم حضورهم معا + C_4^9 حضور الاول + C_4^9 حضور الثاني