Chapter Four

_ Axiomatic Approach of Probability _

4.1_ Probability Defined on Event: -

In this chapter, we develop a mathematical model of an experiment which is defined mathematically by three things.

1_ Specifying the sample space S on which probability statements are made;

تحديد فضاء العينة التي يتم فيها وضع بيانات الاحتمالية؛

2_ defining the events of interest; and

تحديد الأحداث ذات الأهمية؛ و

3_ defining a numerical measure for probability statements, which is assigned to the events.

تحديد مقياس رقمي للبيانات الاحتمالية، والذي يتم تعيينه للأحداث

To each event A defined on a sample space S we shall assign a non-negative real number which is called the probability of A , denoted by $P(A)$.

4.1_ Axioms of Probability: بديهيات الاحتمال

Let S be sample space, let σ be the class of events, and let P be a real value function defined on σ .

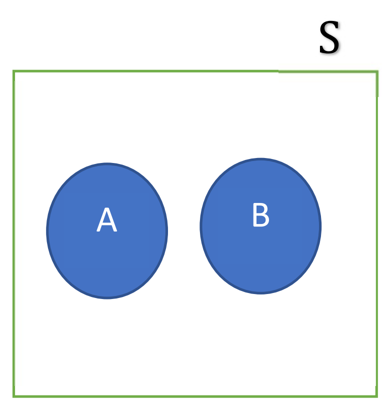
Then P is called a probability function, and $P(A)$ is called probability of the event A if the following axioms hold

1_ For every event A , $P(A) \geq 0$; $0 \leq P(A) \leq 1$

2_ $P(S) = 1$; $P(S) = \frac{h}{n} = \frac{n(S)}{n(S)} = 1$

3_ If A and B are mutually exclusive events, then:

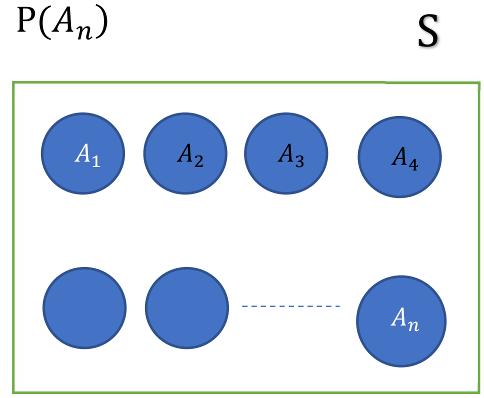
$$P(A \cup B) = P(A) + P(B)$$



4_ If $A_1, A_2, A_3, \ldots, A_n$ is a sequence of mutually exclusive events, then:

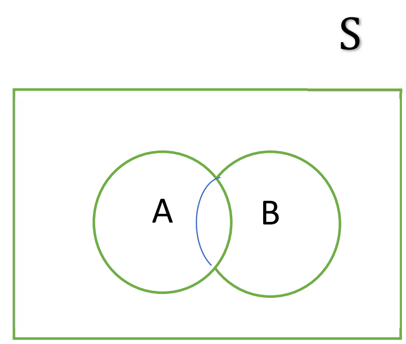
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) = P(\cup_{i\in I} A_i) = \sum_{i=1}^{n} P(A_i)$$

$$= P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$



5_ If A *and* B are not mutually exclusive events, then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Theorem (1) :

$P(\emptyset) = 0$ for any sample space.

Proof:

Let A be any subset , then A and $\emptyset$ are disjoint , and

$A = A \cup \emptyset \quad \rightarrow \quad P(A) = P(A \cup \emptyset)$

Since A and $\emptyset$ are **m.e.** events by axiom (3)

$\rightarrow P(A \cup \emptyset) = P(A) + P(\emptyset)$

$\rightarrow P(A) = P(A) + P(\emptyset)$

$\rightarrow P(\emptyset) = 0$

Or

$$P(\emptyset) = \frac{n(\emptyset)}{n(S)} = \frac{0}{n} = 0$$

Theorem (2) :

For any event $A \subset S$, $P(A^c) = 1 - P(A)$

Proof

Let $S = A \cup A^c \rightarrow P(S) = P(A \cup A^c)$

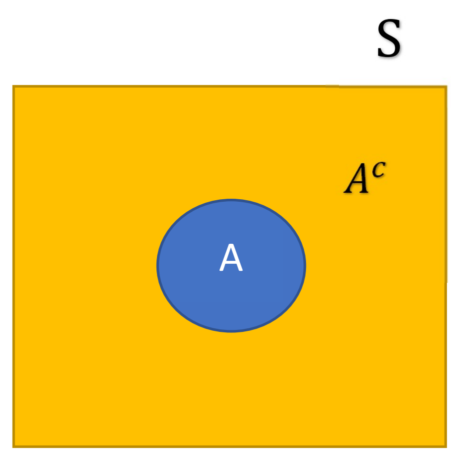
Since A & A^c are m.e. events, then:

$P(A \cup A^c) = P(A) + P(A^c)$

$\therefore P(S) = 1$

$\therefore 1 = P(A) + P(A^c)$

$\therefore P(A) = 1 - P(A^c)$



Or

$P(A^c) = 1 - P(A)$

$\therefore A \cap A^c = \emptyset$

&

$P(A \cap A^c) = P(\emptyset) = 0$

Theorem (3) :

$$0 \leq P(A) \leq 1$$

Proof

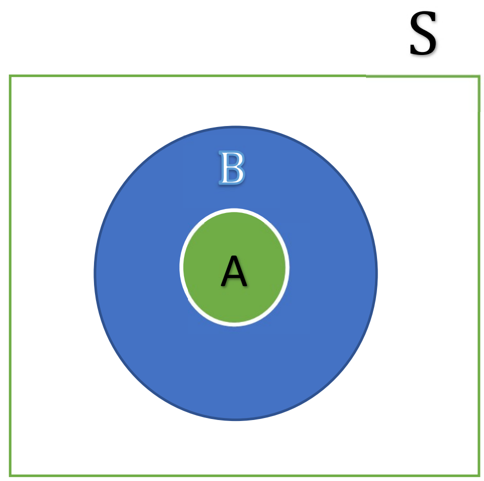
$\because P(\emptyset) = 0$

& $P(S) = 1$, $P(\emptyset) \leq P(A) \leq P(S)$

$\therefore 0 \leq P(A) \leq 1$

Theorem (4) :

If $\quad A \subset B \;, \quad then \quad P(A) \le P(B)$



Proof

m.e.

We have $\quad B = A \cup (B \cap A^c)$

$\therefore \quad P(B) = P[A \cup (B \cap A^c)]$

$\rightarrow \quad P(B) = P(A) + P(B \cap A^c)]$

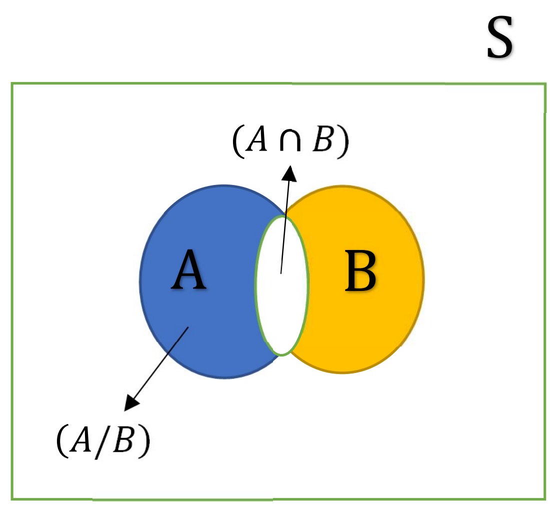
كمية موجبة

$\therefore \quad P(B) \ge P(A)$

Theorem (5) :

For any two events $A \; and \; B$

$P(A/B) = P(A) - P(A \cap B)$



Proof

Let $A = (A/B) \cup (A \cap B)$

$\rightarrow P(A) = P[(A/B) \cup (A \cap B)]$

m.e

$= P(A/B) + P(A \cap B)$

$\rightarrow P(A/B) = P(A) - P(A \cap B)$

ويمكن كتابة A/B بالشكل التالي

$A/B = A \cap B^c$

وكذلك يمكن كتابة B/A بالشكل التالي

$B/A = B \cap A^c$

$\& \ P(B/A) = P(B) - P(B \cap A)$

Proof

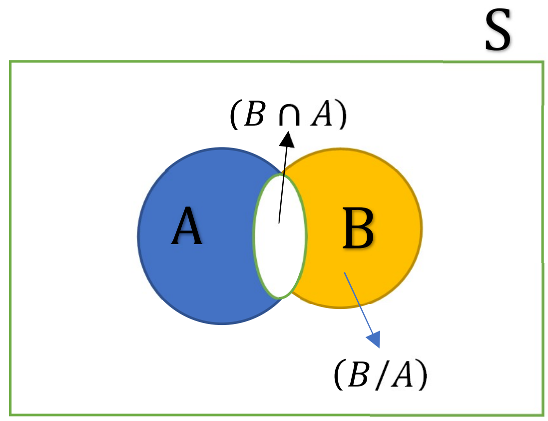
Let $B = (B/A) \cup (B \cap A)$

$\rightarrow P(B) = P[(B/A) \cup (B \cap A)]$

m.e

$= P(B/A) + P(B \cap A)$

$\therefore \ P(B/A) = P(B) - P(B \cap A)$



Theorem (6) :

For any two events $A \ and \ B$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof

Let $A \cup B = A \cup (B/A)$

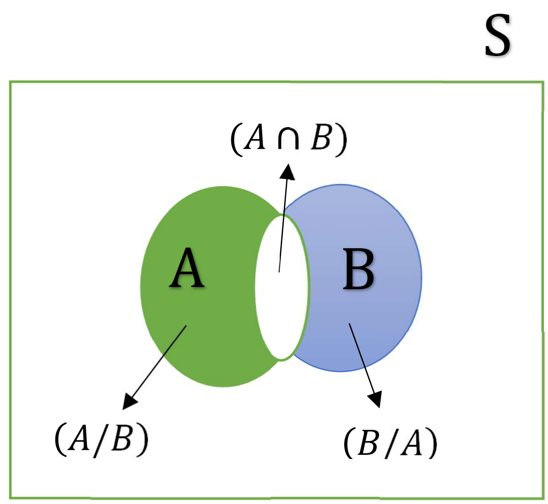
$\rightarrow P(A \cup B) = P[(A) \cup (B/A)]$

m.e

$= P(A) + P(B/A)$

$\because \ P(B/A) = P(B) - P(B \cap A)$

$\therefore \ P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Proof with another way:

$(A \cup B) = (A/B) \cup (A \cap B) \cup (B/A)$

m.e

$\therefore P(A \cup B) = P[(A/B) \cup (A \cap B) \cup (B/A)]$

$= P(A/B) + P(A \cap B) + P(B/A)$

$\because\ P(A/B) = P(A) - P(A \cap B)$

$\&\ P(B/A) = P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) - P(A \cancel{\cap} B) + P(A \cancel{\cap} B) + P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) + P(B) - P(A \cap B)$

Proof again: -

$(A \cup B) = (A \cap B^c)\ \cup (A \cap B) \cup (B \cap A^c)$

m.e

$\therefore P(A \cup B) = P[(A \cap B^c) \cup (A \cap B) \cup (B \cap A^c)]$

$= P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$

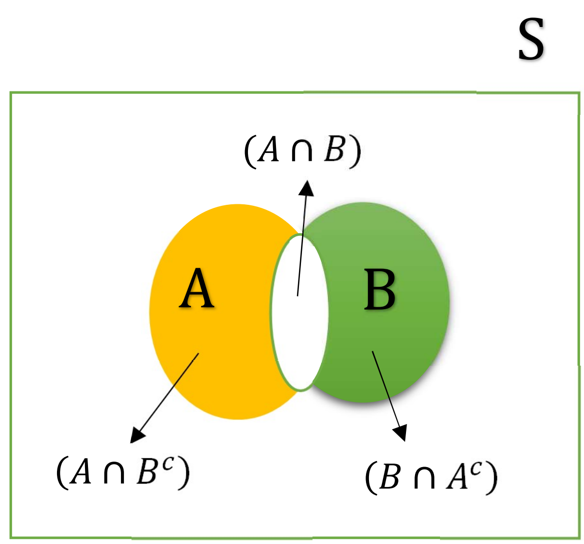
$\because\ (A/B)\ =\ (A \cap B^c)$

$\therefore\ P(A \cap B^c) = P(A) - P(A \cap B)$

$\&\ P(B \cap A^c) = P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) - P(A \cancel{\cap} B) + P(A \cancel{\cap} B) + P(B) - P(A \cap B)$

$\therefore P(A \cup B)\ = P(A) + P(B) - P(A \cap B)$

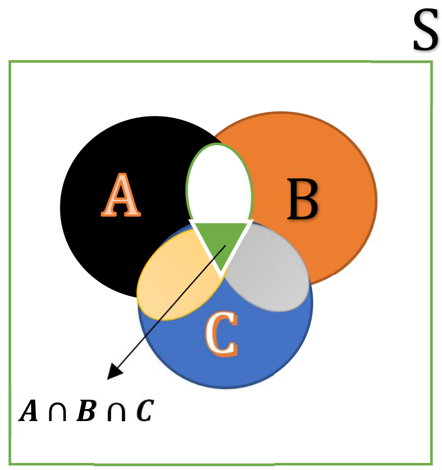


Corollary: -

If A , B & C are any three events; then:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof:



$$P(A \cup B \cup C) = P[A \cup (B \cup C)]$$

$$\therefore P[A \cup (B \cup C)] = P(A) + P(B \cup C) - P[A \cap (B \cup C)]$$

And we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\therefore P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

And $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

4_2_1_ Examples of probability laws:

EX/

Two coins are tossed, what is the probability that at least one head appears?

SOL/

$$S = \{HH, HT, TH, TT\}$$

$$A = \{at\ least\ one\ head\ appears\} = \{HH, HT, TH\} = 3$$

$$\therefore P(A) = \frac{h}{n} = \frac{n(A)}{n} = \frac{3}{4} \quad or \quad \frac{C_1^3}{C_1^4}$$

$$B = \{exactly\ one\ head\ appear\} = \{HT, TH\} = 2$$