Calculus

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Chapter One

Sets

Sets

Sets are defined as the collection of well-defined data. In Math, Set is that tool that helps to classify and collect data belonging to the same category, even though the elements used in sets are all different from each other, but they all are similar as they belong to one group. For instance, a set of different outdoor games, say set A= {Football, basketball, volleyball, cricket, badminton} all the games mentioned are different, but they all are similar in one way as they belong to the same group (outdoor games).

Set is denoted as a capital letter, for example, set A, set B, etc., and the elements belonging to the set are denoted as a small letter, and they are kept in curly brackets {}, for example, set A= {a, b, c, d}, as it is clear that a, b, c, d belong to set A, it can be written a ∈ A, does p belong to set A? No. Therefore, it will be written as, p∉ A.

Representation of Sets

Sets can be represented in two ways, one is known as the Roster form and the other is famous as the Set-Builder form, these two forms can be used to represent the same data, just the style varies in both cases.

Roster Form

In Roster Form, the elements are inside {}⇢ Curly brackets. All the elements are mentioned inside and are separated by commas. Roster form is the easiest way to represent the data in groups. For example, the set for the table of 5 will be, A= {5, 10, 15, 20, 25, 30, 35…..}.

Properties of Roster form of Sets:

- *The arrangement in the Roster form does not necessarily to be in the same order every time. For example, A= {a, b, c, d, e} is equal to A= {e, d, a, c, b}.*
- *The elements are not repeated in the set in Roster form, for example, the word "apple" will be written as, A= {a, p, l, e}*
- *The Finite sets are represented either with all the elements or if the elements are too much, they are represented as dots in the middle. The infinite sets are represented with dots in the end.*

Set-Builder Form

In Set-builder form, elements are shown or represented in statements expressing relation among elements. The standard form for Set-builder, A= {a: statement}. For example, $A = \{x: x = a^3, a \in N, a < 9\}$

Properties of Set-builder form:

- *In order to write the set in Set- builder form, the data should follow a certain pattern.*
- *Colons (:) are necessary in Set-builder form.*
- *After colon, the statement is to be written.*

Order of the Set

The order of the Set is determined by the number of elements present in the Set. For example, if there are 10 elements in the set, the order of the set becomes 10. For finite sets, the order of the set is finite and for infinite sets, the order of the set is infinite.

Sample Problems

Question 1: Determine which of the following are considered as sets and which are not.

1. **All even numbers on the number line.**
2. **All the good basketball players from class 9th.**
3. **The bad performers from the batch of dancers.**
4. **All prime numbers from 1 to 100.**
5. **Numbers that are greater than 5 and less than 15.**

Answer:

Sets are not those bunches or groups where some quality or characteristic comes in the picture. Therefore,

1. *"All even numbers on the number line" is a set.*
2. *"All the good basketball players from class 9th" is not a Set as "good" is a quality which is involved.*
3. *"The bad performers from the batch of dancers" cannot be a Set since "bad" is a characteristic.*
4. *"All prime numbers from 1 to 100" is a Set.*
5. *"Numbers that are greater than 5 and less than 15" is a Set.*

Question 2: Represent the following information into the Roster form.

1. **All Natural numbers.**
2. **Numbers greater than 6 and less than 3.**
3. **All even numbers from 10 to 25.**

Answer:

The Roster form for the above information,

1. *Set A= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11......}*
2. *Set B= {} ⇢ Null set, since there are no numbers greater than 6 and less than 3.*
3. *Set C= {10, 12, 14, 16, 18, 20, 22, 24}*

Question 3: Express the given information in the Set-Builder form.

1. **Numbers that are greater than 10 and less than 20.**
2. **All Natural numbers greater than 25.**
3. **Vowels in English Alphabets.**

Answer:

The Set-Builder form for the above information,

1. *A= {a: a$\in$ N and 10< a > 20}*
2. *B= {b: b$\in$ N and b > 25}*
3. *C= {c: c is the vowel of English Alphabet}*

Question 4: Convert the following Sets given in Roster form into Set -Builder form.

1. **A= {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}**
2. **B= {2, 4, 6, 8, 10}**
3. **C= {5, 7, 9, 11,13, 15, 17, 19}**

Answer:

The Set- builder form for the above Sets,

1. *A= {a: a is a consonant of the English Alphabet}*
2. *B= {b: b is an Even number and 2≤ b ≤10}*
3. *C= {c: c is an odd number and 5≤ c ≤ 19}*

Question 5: Give an example of the following types of Sets in both Roster form and Set-builder form.

1. **Singular Set.**
2. **Finite Set.**
3. **Infinite Set.**

Solution:

The Examples can be taken as per choice since there can be a infinite number of examples for any of the above Sets,

- *Singular Set*

Roster Form: A= {2}

Set- builder form: A= {a: a$\in$N and 1<a<3}

- *Finite Set*

Roster Form: B= {0,1, 2, 3, 4, 5}

Set-builder form: B= {b: b is a whole number and b<6}

- *Infinite Set*

Roster Form: C= {2, 4, 6, 8, 10, 12, 14, 16.....}

Set- builder form: C= {c: c is a Natural and Even number}

Question 6: What is the order of the given sets,

1. **A= {7, 14, 21, 28, 35}**
2. **B= {a, b, c, d, e, f, e....x, y, z}**
3. **C= {2, 4, 6, 8, 10, 12, 14......}**

Answer:

The order of the set tells the number of element present in the Set.

1. *The order of Set A is 5 as it has 5 elements.*
2. *The order of set B is 26 as the English Alphabets have 26 letters.*
3. *The order of set C is infinite as the set has the infinite number of elements.*

Question 7: Express the given Sets in Roster form,

1. **A = {a: a = n/2, n ∈ N, n < 10}**
2. **B = {b: b = n^2, n ∈ N, n ≤ 5}**

Answer:

Representing the above Set-builder sets in Roster form,

1. *A= {1/2, 1, 3/2, 2, 5/2, 3, 7/2, 4, 9/2}*
2. *B= {1, 4, 9, 16, 25}*

Types of Sets in Mathematics

Sets are the collection of different elements belonging to the same category and there can be different types of sets seen. A set may have an infinite number of elements, may have no elements at all, may have some elements, may have just one element, and so on. Based on all these different ways, sets are classified into different types.

The different types of sets are:

Singleton Set

Singleton Sets are those sets that have only 1 element present in them.

Example:

- *Set A= {1} is a singleton set as it has only one element, that is, 1.*
- *Set P = {a : a is an even prime number} is a singleton set as it has only one element 2.*

Similarly, all the sets that contain only one element are known as Singleton sets.

Empty Set

Empty sets are also known as Null sets or Void sets. They are the sets with no element/elements in them. They are denoted as φ.

Example:

- *Set A= {a: a is a number greater than 5 and less than 3}*
- *Set B= {p: p are the students studying in class 7 and class 8}*

Finite Set

Finite Sets are those which have a finite number of elements present, no matter how much they're increasing number, as long as they are finite in nature, They will be called a Finite set.

Example:

- *Set A= {a: a is the whole number less than 20}*
- *Set B = {a, b, c, d, e}*

Infinite Set

Infinite Sets are those that have an infinite number of elements present, cases in which the number of elements is hard to determine are known as infinite sets.

Example:

- *Set A= {a: a is an odd number}*
- *Set B = {2,4,6,8,10,12,14,.....}*

Equal Set

Two sets having the same elements and an equal number of elements are called equal sets. The elements in the set may be rearranged, or they may be repeated, but they will still be equal sets.

Example:

- *Set A = {1, 2, 6, 5}*
- *Set B = {2, 1, 5, 6}*

In the above example, the elements are 1, 2, 5, 6. Therefore, A= B.

Equivalent Set

Equivalent Sets are those which have the same number of elements present in them. It is important to note that the elements may be different in both sets but the number of elements present is equal. For Instance, if a set has 6 elements in it, and the other set also has 6 elements present, they are equivalent sets.

Example:

Set A= {2, 3, 5, 7, 11}

Set B = {p, q, r, s, t}

Set A and Set B both have 5 elements hence, both are equivalent sets.

Subset

Set A will be called the Subset of Set B if all the elements present in Set A already belong to Set B. The symbol used for the subset is **⊆**
If A is a Subset of B, It will be written as A ⊆ B

Example:
Set A= {33, 66, 99}

Set B = {22, 11, 33, 99, 66}

Then, Set A ⊆ Set B

Power Set

Power set of any set A is defined as the set containing all the subsets of set A. It is denoted by the symbol **P(A)** and read as Power set of A.
For any set A containing n elements, the total number of subsets formed is 2^n. Thus, the power set of A, P(A) has 2^n elements.
Example: For any set A = {a,b,c}, the power set of A is?
Solution:
Power Set P(A) is,

P(A) = {ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {c, a}, {a, b, c}}

Universal Set

A universal set is a set that contains all the elements of the rest of the sets. It can be said that all the sets are the subsets of Universal sets. The universal set is denoted as U.

Example: **For Set A = {a, b, c, d} and Set B = {1,2} find the universal set containing both sets.**
Solution:
Universal Set U is,

U = {a, b, c, d, e, 1, 2}

Disjoint Sets

For any two sets A and B which do have no common elements are called Disjoint Sets. The intersection of the Disjoint set is φ, now for set A and set B A∩B = φ.

Example: Check whether Set A ={a, b, c, d} and Set B= {1,2} are disjoint or not.
Solution:
Set A ={a, b, c, d}
Set B= {1,2}

Here, A∩B = ϕ *Thus, Set A and Set B are disjoint sets.*

Example 2: Which of the given below sets are equal and which are equivalent in nature?

- **Set A= {2, 4, 6, 8, 10}**
- **Set B= {a, b, c, d, e}**
- **Set C= {c: c ∈ N, c is an even number, c ≤ 10}**
- **Set D = {1, 2, 5, 10}**
- **Set E= {x, y, z}**

Solution:

Equivalent sets are those which have the equal number of elements, whereas, Equal sets are those which have the equal number of elements present as well as the elements are same in the set.

Equivalent Sets = Set A, Set B, Set C.

Equal Sets = Set A, Set C.

Example 3: Determine the types of the below-given sets,

- **Set A= {a: a is the number divisible by 10}**
- **Set B = {2, 4, 6}**
- **Set C = {p}**
- **Set D= {n, m, o, p}**
- **Set E= φ**

Solution:

From the knowledge gained above in the article, the above-mentioned sets can easily be identified.

- *Set A is an Infinite set.*
- *Set B is a Finite set*
- *Set C is a singleton set*
- *Set D is a Finite set*
- *Set E is a Null set*

Example 4: Explain which of the following sets are subsets of Set P,
Set P = {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

- **Set A = {a, 1, 0, 2}**
- **Set B ={0, 2, 4}**
- **Set C = {1, 4, 6, 10}**
- **Set D = {2, 20}**
- **Set E ={18, 16, 2, 10}**

Solution:

- *Set A has elements a, 1, which are not present in the Set P. Therefore, set A is not a Subset.*
- *Set B has elements which are present in set P, Therefore, Set B ⊆ Set P*
- *Set C has 1 as an extra element. Hence, not a subset of P*

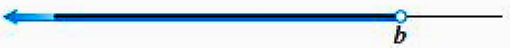
- *Set D has 2, 20 as element. Therefore, Set D ⊆ Set P*
- *Set E has all its elements matching the elements of set P. Hence, Set E ⊆ Set P.*

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and also contains all real numbers between any two of its elements.

Types of intervals

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Inequalities

Definition *The expression,* $x < y$*; in words, (*x *is less than* y*) means* y *lies to the right of* x *on the line number. The expression* $x > y$*; in words (*x *is greater than* y*) means is to the right of* y *on the number line.* $x \leq y$ *if either* $x = y$ *or* $x < y$. $x \geq y$ *if either* $y < x$ *or* $x = y$.

A number, x *is positive if* $x > 0$:

Rules for Inequalities

If x, y and z are reals, then

1) $x < y \Longrightarrow x + z < y + z$
2) $x < y \Longrightarrow x - z < y - z$
3) $x < y \text{ and } z > 0 \Longrightarrow xz < yz$
4) $x < y \text{ and } z < 0 \Longrightarrow yz < xz$
5) $x > 0 \Longrightarrow \frac{1}{x} > 0$
6) If x and y are both positive or both negative ,then

$x < y \Longrightarrow \frac{1}{y} < \frac{1}{x}$

EXAMPLE 2 Solve the following inequalities. Express the solution sets in terms of intervals and graph them.

(a) $2x - 1 > x + 3$ (b) $-\frac{x}{3} \geq 2x - 1$ (c) $\frac{2}{x-1} \geq 5$

Solution

(a) $2x - 1 > x + 3$ Add 1 to both sides.

$2x > x + 4$ Subtract x from both sides.

$x > 4$ The solution set is the interval $(4, \infty)$.

(b) $-\frac{x}{3} \geq 2x - 1$ Multiply both sides by -3.

$x \leq -6x + 3$ Add $6x$ to both sides.

$7x \leq 3$ Divide both sides by 7.

$x \leq \frac{3}{7}$ The solution set is the interval $(-\infty, 3/7]$.

(c) We transpose the 5 to the left side and simplify to rewrite the given inequality in an equivalent form:

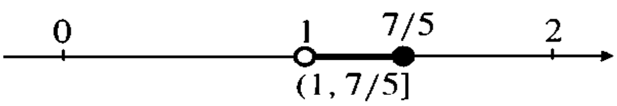
$$\frac{2}{x-1} - 5 \geq 0 \iff \frac{2 - 5(x-1)}{x-1} \geq 0 \iff \frac{7-5x}{x-1} \geq 0.$$

The fraction $\frac{7-5x}{x-1}$ is undefined at $x = 1$ and is 0 at $x = 7/5$. Between these numbers it is positive if the numerator and denominator have the same sign, and negative if they have opposite sign. It is easiest to organize this sign information in a chart:

x		1		7/5	
$7-5x$	$+$	$+$	$+$	0	$-$
$x-1$	$-$	0	$+$	$+$	$+$
$(7-5x)/(x-1)$	$-$	undef	$+$	0	$-$

Thus the solution set of the given inequality is the interval $(1, 7/5]$.

See Figure P.4 for graphs of the solutions.



Examples Solving a quadratic inequality $x^2 - 3x + 2 > 0$.

Solution:

Method[1]: we can write $x^2 - 3x + 2$ as form $(x-2)(x-1)$.

Since $(x-2)(x-1) > 0$ means two terms $(x-2)$ and $(x-1)$ either positive or both negative.

$\Longrightarrow (x-2) > 0$ and $(x-1) > 0$ or $(x-2) < 0$ and $(x-1) < 0$

$\Longrightarrow x > 2$ and $x > 1$ or $x < 2$ and $x < 1$.

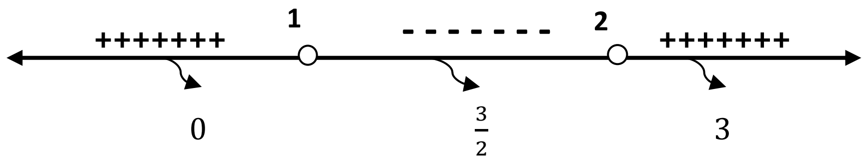
$\therefore$ The solution set in this case is or the solution set in this case is

$S_1 = (2, \infty)$ $S_2 = (-\infty, 1)$

$\therefore$ Then solution set is $S_1 \cup S_2 = (-\infty, 1) \cup (2, \infty) = \mathbb{R} \backslash [1, 2]$.

Method²:

$x = 1, 2$



Then the solution set is $(-\infty, 1) \cup (2, \infty)$

Examples Solving a quadratic inequality $x^2 - 3x - 4 < 0$.

Solution:

Method¹: we can write $x^2 - 3x - 4$ as form $(x-4)(x+1)$.

Since $(x-4)(x+1) < 0$ means two terms $(x-4)$ and $(x+1)$ as follows

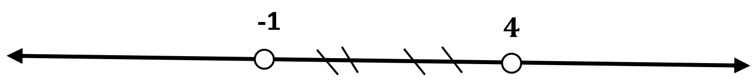
$\Longrightarrow (x-4) < 0$ and $(x+1) > 0$ or $(x-4) > 0$ and $(x+1) < 0$

$\Longrightarrow x < 4$ and $x > -1$ or $x > 4$ and $x < -1$.

$\therefore$ The solution set in this case is or the solution set in this case is

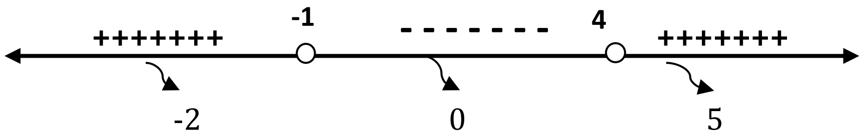
$S_1 = (-1, 4)$ $S_2 = \{\phi\}$

$\therefore$ Then the solution set is $S_1 \cup S_2 = (-1, 4) \cup \{\phi\} = (-1, 4)$.



Method^{2}:

$x = -1, 4$



Then the solution set is $(-1,4)$

Absolute value

1.11 Definition The ***absolute value*** of a number x, denoted by is defined by the formula

$$|x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}.$$

1.12 Example $|2| = 2,\ |-5| = -(-5) = 5.$

Some properties of the absolute value

Let a, b and x be any real numbers then:

1) $|x| = \sqrt{x^2}$

2) $|ab| = |a||b|$

3) $|a+b| \leq |a| + |b|$

4) $|a-b| \geq ||a| - |b||$

5) $|x| \leq a$ if and only if $x \leq a$ and $x \geq -a$ $(\, or\ -a \leq x \leq a)$

6) $|x| > a$ if and only if $x \geq a$ or $x \leq -a$

1.13 Example Solve: (a) $|2x+5| = 3$ (b) $|3x-2| \leq 1$.

Solution

(a) $|2x+5| = 3 \iff 2x+5 = \pm 3$. Thus, either $2x = -3-5 = -8$ or $2x = 3-5 = -2$. The solutions are $x = -4$ and $x = -1$.

(b) $|3x-2| \leq 1 \iff -1 \leq 3x-2 \leq 1$. We solve this pair of inequalities:

$$\left\{ \begin{array}{r} -1 \leq 3x-2 \\ -1+2 \leq 3x \\ 1/3 \leq x \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{r} 3x-2 \leq 1 \\ 3x \leq 1+2 \\ x \leq 1 \end{array} \right\}.$$

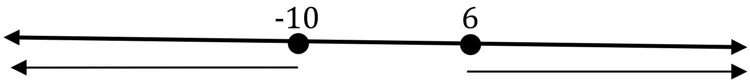
Thus the solutions lie in the interval $[1/3, 1]$.

Example Solve the inequality $|x+2| \geq 8$.

Solution: $|x+2| \geq 8$ if and only if $x+2 \geq 8$ or $x+2 \leq -8$

$$\Rightarrow \quad x \geq 6 \qquad \text{or} \quad x \leq -10$$

Then the solution set is $(-\infty, -10] \cup [6, \infty)\}$.



Chapter two

Relations & Functions

Cartesian Product of Sets

A cartesian product of two non-empty sets A and B is the set of all possible ordered pairs where the first component of the pair is from A, and the second component of the pair is from B. The set of ordered pairs thus obtained is denoted by **A×B**.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

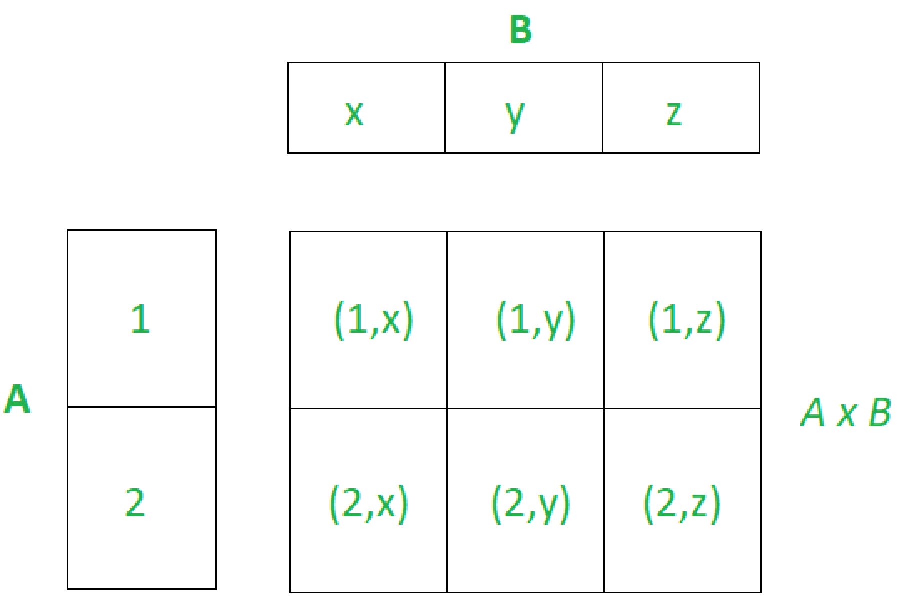
Example:

Let A = {1, 2} and B = {4, 5, 6}

A × B = {(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6)}

Here the first component of every ordered pair is from set A the second component is from set B.

The Cartesian Product of two sets can be easily represented in the form of a matrix where both sets are on either axis, as shown in the image below. Cartesian Product of A = {1, 2} and B = {x, y, z}



Properties of Cartesian Product

1. The Cartesian Product is non-commutative: $A \times B \neq B \times A$

Example:

$A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

$B \times A = \{(a, 1), (b, 1), (b, 1), (b, 2)\}$

Therefore as $A \neq B$ we have $A \times B \neq B \times A$

2. $A \times B = B \times A$, only if $A = B$

Proof:

Let $A \times B = B \times A$ then we have

$A \subseteq B$ and $B \subseteq A$, it follows that $A = B$

3. The cardinality of the Cartesian Product is defined as the number of elements in A × B and is equal to the product of cardinality of both sets: $|A \times B| = |A| * |B|$

Proof:

Let $a \in A$ then the number of ordered pair (a, b) such that $b \in B$ is $|B|$.

Therefore we have $|B|$ choices for b for each a where $a \in A$ therefore the number of element in $A \times B$ is $|A| * |B|$.

4. $A \times B = \{\emptyset\}$, if either $A = \{\emptyset\}$ or $B = \{\emptyset\}$

Proof:

We know $|\{\emptyset\}| = 0$.

Now we have $|A \times B| = |\{\emptyset\}| = 0$

As $|A \times B| = |A| * |B|$, we get $|A| * |B| = 0$

Thus atleast one of $|A|$ or $|B|$ should be equal to 0

Hence either $A = \{\emptyset\}$ or $B = \{\emptyset\}$

Sample Problems on Ordered Pairs and Cartesian Product of Sets

Problem 1: Find the value of x and y given $(2x - y,\ 25) = (15,\ 2x + y)$?

Solution:

As we know from the property of ordered pairs, $2x - y = 15$ and $25 = 2x + y$.

Solving the linear equations we have $x = 10$ and $y = 5$.

Problem 2. Given A = {2, 3, 4 , 5} and B = {4 , 16 , 23}, a ∈ A, b ∈ B, find the set of ordered pairs such that $a^2 < b$?
Solution:
As $2^2 < 16$ and 23, $3^2 < 16$ and 23, $4^2 < 23$
We have the set of ordered pairs such that $a^2 < b$ is {(2, 16), (2, 23), (3, 16), (2, 23), (4, 23)}
Problem 3. If A = {9, 10} and B = {3, 4, 6}, find A × B and |A × B|?
Solution:
A × B = {(9, 3), (9, 4), (9, 6), (10, 3), (10, 4), (10, 6)}

*|A × B| = |A| * |B| = 2 * 3 = 6*

Problem 4. If A × B = {(a, x), (a, y), (b, x), (b, y)}, find A and B?
Solution:
We know A is the set of all first components in ordered pairs of A × B and

B is the set of the second component in the ordered pair of A × B.

Therefore A = {a, b} and B = {x, y}

Problem 5. Given A × B has 15 ordered pairs and A has 5 elements, find the number of elements in B?
Solution:
*We know |A × B| = |A| * |B|, 15 = 5 * |B|*

Therefore B has 15 / 5 = 3 elements.

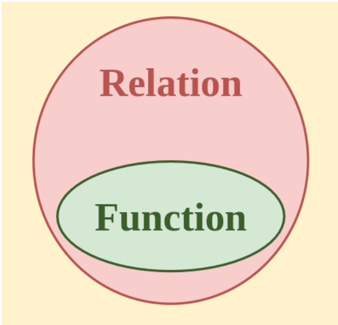
Relation Definition

Relation is defined on a non-empty set A to no-empty set B such that Relation from A to B is a subset of Cartesian Product of A and B i.e. R ⊆ A × B.
For Example if **A = {a, b, c}** and **B = {p, q, r},**
Then **A × B = {(a, p), (a, q), (a, r), (b, p), (b, q), (b, r), (c, p), (c, q), (c, r)}**.
If there is another set are that is defined as **R = {(a, p), (b, q), (c, r)}** then we see that R is a subset of **A × B**.
Hence, R is a relation from set A to set B.

Function Defintion

Function is a special type of relation in which the two entities are exclusively related to each other only. A relation from set A to set B is defined as function if all the elements of set A are related to at least one element of B and no two elements of B are related to a single element of A.
Here, the element of set A are called pre-image and the element of set B are called image. Hence a function from A to B is defined only when each pre-image in set A has an image in B and no two different images in B has a single pre-image in A.

It should be noted that every function is a relation but not every relation is a function.



Representation of Relation and Function

Relation and Function are in general the same with some basic difference. They both take input, process it and relates to output. They can be represented in the following forms:

Roster Form

In roster form the elements of two sets among which relation is defined are written in the form of ordered pair.

For Example, if A = {-1, 0, 1, 2} and B = {0, 1, 2, 3, 4} and set A is related to set B as a^2 = b where a is an element of set A and b is an element of set B then R = {(-1, 1), (0, 0), (1, 1), (2, 4)}.

Set Builder Form

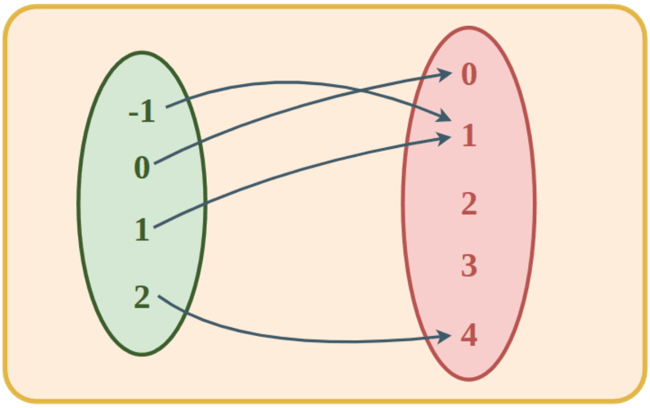
In set builder form the relation is not written in expanded pair form rather it is written in compressed form using an algebraic expression to define the relation between two sets.

For Example, if A = {-1, 0, 1, 2} and B = {0, 1, 2, 3, 4} and set A is related to set B as a^2 = b where a is an element of set A and b is an element of set B then Relation in Set Builder form is given as R = {(a, b): a ϵ A, b ϵ B and b = a^2}

Arrow Diagram

In arrow diagram the relation is shown using connecting the elements of the sets which are contained in the box using arrows.

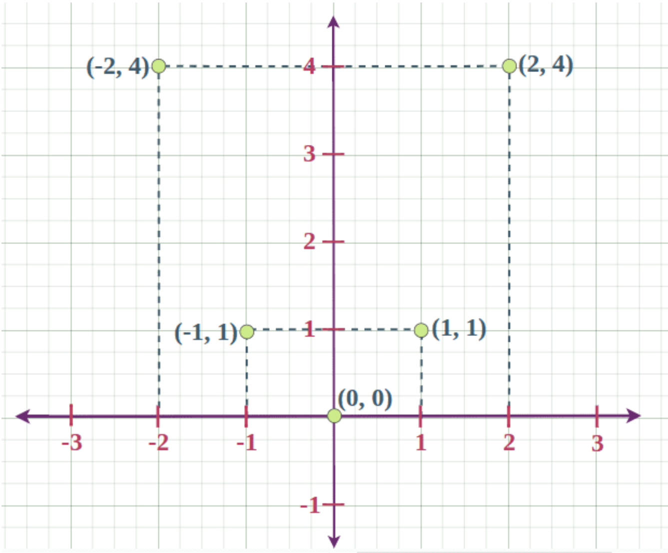
For Example, if A = {-1, 0, 1, 2} and B = {0, 1, 2, 3, 4} and set A is related to set B as a^2 = b where a is an element of set A and b is an element of set B then Relation in Arrow Diagram is given as follows:



Lattice Diagram

In lattice diagrams the elements which are linked to each other by a relation are plotted on cartesian plane.

For Example, if A = {-1, 0, 1, -2, 3} and B = {0, 1, 2, 3, 4, 9} and set A is related to set B as a2 = b where a is an element of set A and b is an element of set B then Relation in Lattice Diagram is given as follows.



Terms Related to Relation and Function

Some of the commonly used terms associated with Relation and Function are discussed below:

Domain

Domain of Relation or a function is the set of inputs for which the outputs are obtained. For Example, in A = {-1, 0, 1, 2} and B = {0, 1, 2, 3, 4} and set A is related to set B as a2 = b, the set A is the domain of the relation.

Codomain

Codomain is the set of outputs or the image of the relation and function. Codomain may contain exact or more number of elements than the output. For Example, in A = {-1, 0, 1, 2} and B = {0, 1, 2, 3, 4} and set A is related to set B as $a^2 = b$, set B is the codomain. In set B there is element 3 which is not a perfect square hence it will not have a pre-image.

Range

Range is the set of all outputs which has a pre image. In range all elements are related. Hence, it has has exact number of elements for which relation is defined. Thus, Range is subset of codomain. For Example, in A = {-1, 0, 1, 2} and B = {0, 1, 2, 3, 4} and set A is related to set B as $a^2 = b$, Range is {0, 1, 2, 4}.

Cartesian Product

Let's assume A and B to be two non-empty sets, the sets of all ordered pairs (x, y) where $x \in A$ and $y \in B$ is called a Cartesian product of the sets. $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

Types of Relation and Function

Relation and Function are classified on the basis of the input it take and output it gives for a given relation. The different types of Relation and Function are discussed separately below:

Types of Relation

There are eight different types of relations which are listed below:

- **Empty Relation-** There is no relation between any elements of a set.
- **Universal Relation-** Every element of the set is related to each other.
- **Identity Relation-** In an identity relation, every element of a set is related to itself only.
- **Inverse Relation-** Inverse relation is seen when a set has elements that are inverse pairs of another set.
- **Reflexive Relation-** In a reflexive relation, every element maps to itself.
- **Symmetric Relation-** In asymmetric relation, if a=b is true then b=a is also true.
- **Transitive Relation-** For transitive relation, if $(x, y) \in R$, $(y, z) \in R$, then $(x, z) \in R$.
- **Equivalence Relation-** A relation that is symmetric, transitive, and reflexive at the same time.

Types of Function

We know that in a function no two images can have one common pre-image and all the pre-images must have an image. A function 'f' defined from A × B is classified as follows

- **One-One Function(Injection):** A function 'f' from A to B is said to be One-One or Injection if each element of A is mapped with a different element in B. One-One Function is also called an Injective Function.

- **Many-One Function:** A function 'f' from A to B is said to be Many-One or Injection if two or more elements of A is mapped with a common element in B. It means two elemts in A can have common image in B.
- **Onto Function(Surjection):** A function 'f' from A to B is said to be Onto Function if all the elements of set B has a pre-image in set A i.e. no element in set B remains unmapped.
- **Into Function:** A function 'f' from A to B is said to be Into Function if at least one image in set B does not have a pre-image in set A i.e. one element of set B remains unmapped.
- **One-One Onto Function:** A function 'f' from A to B is said to be One-One Onto function if all the elements of set A has a unique image in set B and all the elements of set B has a pre-image in set A. This type of function exhibits characteristics of One One Function and Onto Function. One-One Onto Function is also called Bijection Function.

Apart from the above types of function there are other standard functions such as constant function, algebraic function, greatest integer function, logarithmic function and trigonometric function etc which can be studied from the link attached below:

Difference Between Relation and Function

Relation and Function are basically the same but they differ from each other in some manner. Let's understand the difference between them from the table given below:

Relation	Function
Relation is defined a non-empty set A to non-empty set B such that Relation from A to B is a subset of A × B i.e. R ⊆ A × B	Function is a special type of relation from set A to set B where all the elements of set A are related to some or all elements of set B and no two elements of set B is related to a common element in set A.
In case of relation of the pre-image may or may not have an image	In case of function all pre-image must have an image
In case of relation a pre-image can have two or more image	In case of function no two image can have a common pre-image
A relation can or can not be a function	A function is always a relation.

Introduction to Domain and Range

To understand the concept of Domain and range of a Relation first, we have to learn about Relation. A relation is a set of rules which relates the value of one set to the value of other sets.
The domain of a relation is the set of values that we take as input and the range is the set of the values which are obtained in the form of the answers to the relation.

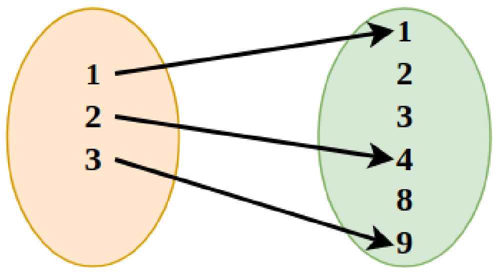
In this article, we will learn about the Domain and Range of Relations, its examples, and others.

What is a Relation?

For any two non-empty sets A and B, we define the relation R as the subset of the Cartesian product of A × B where each member of set A is related to the member of set B through some unique rule.

We defined relation as,

$$\mathbf{R = \{(x,y) : x \in A \text{ and } y \in B\}}$$



Types of Relations

There are the following types of relations between two sets:

- Universal relation
- Empty relation
- Identity relation
- Symmetric relation
- Inverse relation
- Reflexive relation
- Transitive relation

Universal Relations

A relation is called a universal connection if each element of set A is related to another element of set A i.e. R = A × A.

Empty Relations

The relation wherein there is no connection between any components of a set is called an empty relation. An empty relation is also called void relation.

Consider if set A = {2, 3} then, empty relation is R = {m,n} where, |m-n| = 8.

Identity Relations

The relation where each component of a set is identified with itself is called Identity Relations.

Consider a set A = {1,2,3}, the identity relation will be I = {1,1}, {2,2}, {3,3}.

I = {(m,m), m ∈ A}

Symmetric Relations

A relation is symmetric if a = b holds true then b = a also holds true. A connection R is symmetric just if (b, a) ∈ R is then (a,b) ∈ R.

R = {(1, 2), (2, 1)} is symmetric relation set for a set A = {1, 2}.

Inverse Relations

Inverse relation occurs when a set has inverse pairs of another set. i.e. if R ∈ A × B then the inverse relation is R^{-1} = {(b,a) such that (a,b) ∈ R}
Consider if set A = {(1,2),(3,4)}, then inverse relation will be R-1 = {(2,1), (4,3)}.

Reflexive Relations

A relation where each component of a set A is mapped to itself is called reflexive relation, i.e. for every x ∈ A, (a, a) ∈ R

Example: For set A = {-1, 4}. Reflexive relation is R = {(-1, -1), (4, 4), (-1, 4), (4, -1)}

Transitive Relations

If for any relation (m,n) ∈ R and (n,p) ∈ R, then if (m,p) ∈ R is

Consider **aRb and bRc ⇒ aRc ∀ a, b, c ∈ A**

Domain and Range of a Relation

As we know any set of ordered pairs that are related to a unique relation we have domain and range i.e. for R such that R(A×B) such that {(a, b) where a ∈ A and b ∈ B} we have domain and range of R.

Here the set of values of A is called the domain and the set of values of b is called the range.

Domain of a Relation

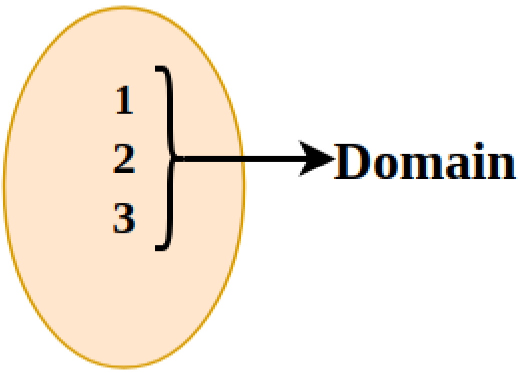
Domain of any Relation is the set of input values of the relation. For example, if we take two sets A and B, and define a relation R: {(a,b): a ∈ A, b ∈ B} then the set of values of A is called the domain of the function.

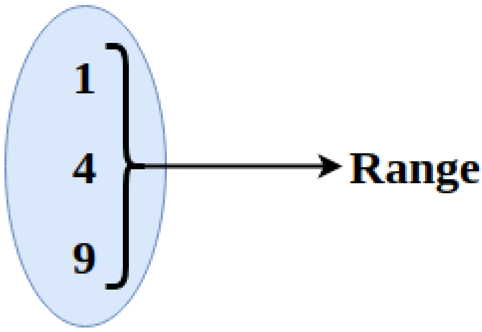
The image given below represents the domain of a relation.

Range of a Relation

Range of any Relation is the set of output values of the relation. For example, if we take two sets A and B, and define a relation R: {(a,b): a ∈ A, b ∈ B} then the set of values of B is called the domain of the function.

The image given below represents the range of a relation.

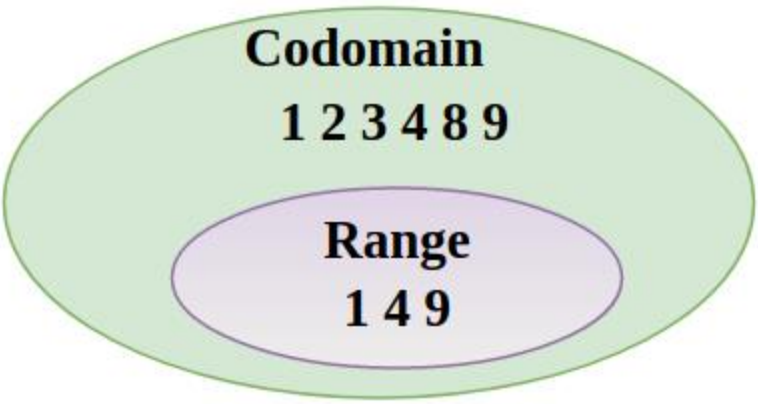




Codomain of Relation

We define the codomain of the relation R as the set B of the cartesian product A×B on which the relation is defined. Now it is clear that the range of the function is the proper subset of Codomain.

Range ⊆ Codomain



Example: Take a set S = {4, 5, 6, 9,10,11,12,13, 17} and define a relation A from S to S such that in the ordered pair (x,y) in A, y is two more than x.
Solution:
We define R as,

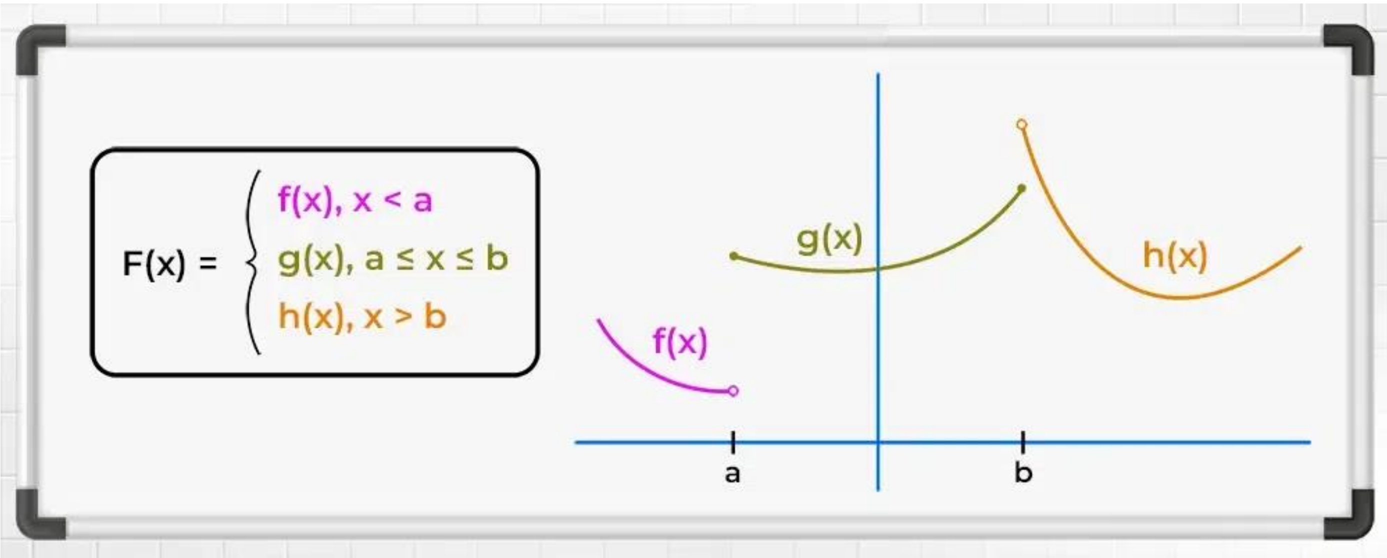
R = {(9,11), (10,12), (11,13)}

Thus,

- *Domain of R is (9, 10, 11)*
- *Range of R is (11, 12, 13)*
- *Codomain is (4, 5, 6, 9,10,11,12,13, 17)*

Piecewise Function

Piecewise Function is a function that behaves differently for two types of input. As we know a function is a mathematical object which associates each input with exactly one output. For example: If a function takes on any input and gives the output as 3. It can be represented in mathematical form as f(x) = 3. But in the case of the Piecewise function, it is defined by individual expressions for each interval.
Piecewise functions are very useful tools as these can be used to model real-life scenarios in terms of mathematical functions and then solve that problem. In this article, we will study about **piecewise function, how to graph a piecewise function, and how to evaluate it.**



What is Piecewise Function?

Piecewise Function is a function that is defined differently on a sequence of intervals. In other words, a piecewise function is a mathematical function that is defined by multiple sub-functions, with each sub-function being valid only in a certain interval or region of the domain. In other words, a piecewise function is a function that is defined differently on different parts of its domain.

The general piecewise function can be written mathematically as:

$$\mathbf{f(x)} = \begin{cases} f_1(x), & \text{if } x < a \\ f_2(x), & \text{if } a \le x < b \\ f_3(x), & \text{if } b \le x \end{cases}$$

Where,

- ***$f_1(x)$, $f_2(x)$,** and **$f_3(x)$** are three different functions, and*
- ***a, b,** and **c** are some real numbers.*

The above expression for piecewise function means that for x less than a, the function takes on the value of $f_1(x)$, for x between a and b, it takes on the value of $f_2(x)$, and for x greater than or equal to b, it takes on the value of $f_3(x)$.\\

Piecewise Function Definition:

A piecewise function, also known as a piecewise-defined function, is a function that is defined by different expressions or formulas on different intervals of its domain. This enables us to accurately describe functions that exhibit varying behaviors or properties within different ranges of input values.

Domain and Range of Piecewise Function

Domain and Range of a piecewise function can be calculated using the domain and range of the individual pieces and taking the union of that range and domains.

Example: Find the Range and Domain of function f(x) which is defined as follows:

$$\mathbf{f(x)} = \begin{cases} x, & x < 1 \\ 2, & 1 \le x \le 5 \\ x^2, & x > 5 \end{cases}$$

Solution:

*As the function is defined for all the real numbers, so its domain is **R** if is defined for some portion of R then that portion becomes its domain.*

Now for Range of function, for $x<2$, $f(x) = x$, thus range for this part is $x<2$

For $2 \le x \le 5$, $f(x) = 2$, thus its range is 2 as it is a constant function for this interval.

For $x > 5$, $f(x) = x^2$, $f(5) = 25$ and x^2 is continuous and increasing function for $x > 0$, thus range is $x>25$.

Now, the union of all the ranges is {x<2} U {2} U {x>25} = ***(-∞, 2] U (25, ∞)***

Piecewise Function Graph

To graph the Piecewise Function, we just need to graph the function individually for all the different intervals it is defined.

Example: Plot the graph of the function defined as follows:

$$\mathbf{f(x)} = \begin{cases} x, & x < -1 \\ 2, & -1 \leq x \leq 2 \\ x^2, & x > 2 \end{cases}$$

Solution:

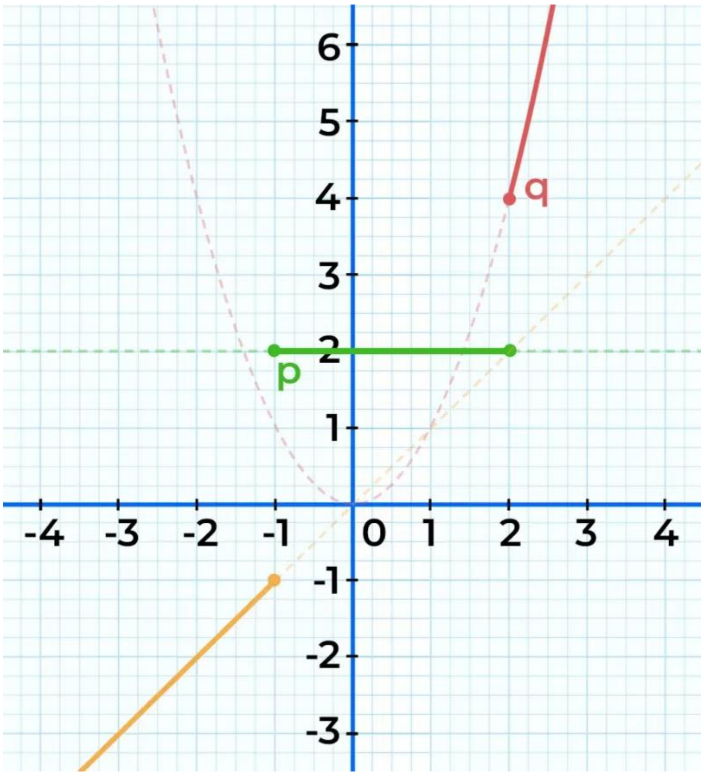
As the domain of the function is the complete set of real numbers, thus there is no such values in R for which the function is not defined.

Now, for the first piece of graph for x < -1, is given as f(x) = x, which can be easily plotted. So the graph of a function for x < -1 is a straight line with a slope of 1 that passes through the origin.

For the second piece of the graph for -1 ≤ x ≤ 2, the given function is a constant function as f(x) = 2. So the graph of a function for -1 ≤ x ≤ 2 is again a straight horizontal line which is at a 2 unit distance from the x-axis.

For the third piece of the graph for x > 2, the given function is a parabolic curve that opens upwards and x^2 *is a increasing and continuous function, so the graph starts at the point (2, 4) goes in the upward direction as parabolic curve.*

Plot these three pieces of the graph to obtain the required graph of the function.



Examples of Piecewise Function

There are many famous examples of piecewise functions, some of which are as follows:

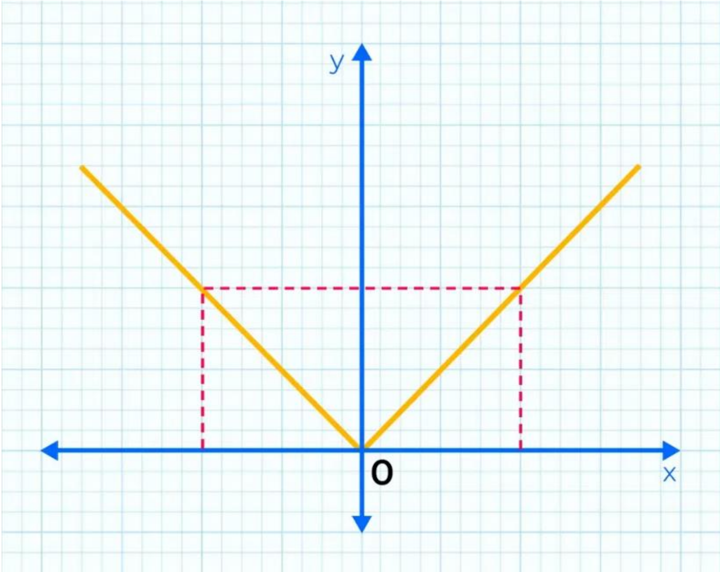
- Modulus Function
- Floor Function
- Ceiling Function
- Unit Step Function
- Signum Function

Modulus Function

It is also called the absolute value function, and it is defined in two pieces as follows:

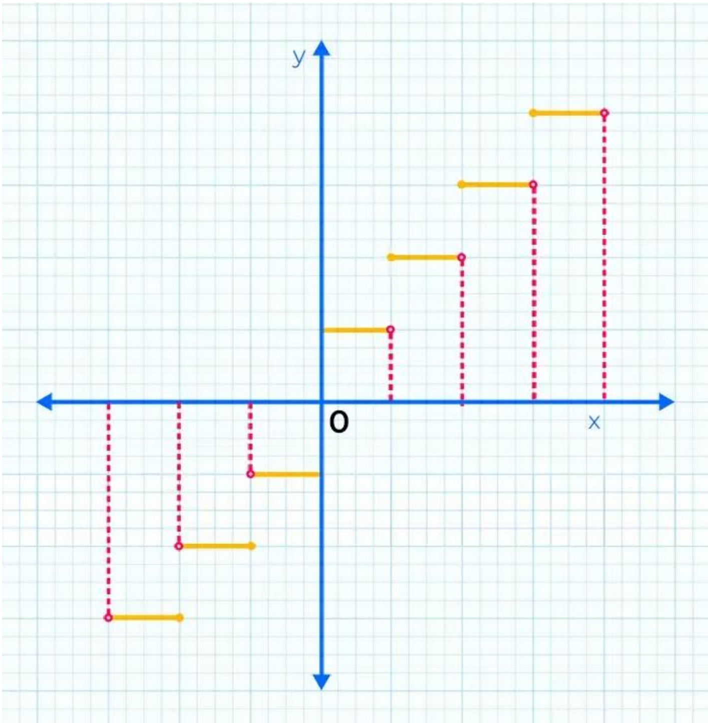
$$\mathbf{f(x)} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

The graph of this function is as follows:

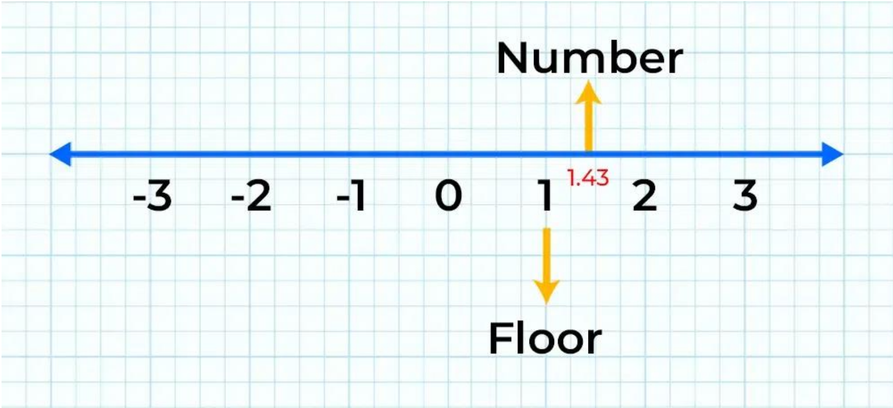


Greatest Integer Function or Floor Function

The floor function also called the greatest integer function or integer value, gives the largest integer less than or equal to x. The domain for this function is all the real numbers **R** while the range of this function is all the integers **Z**.



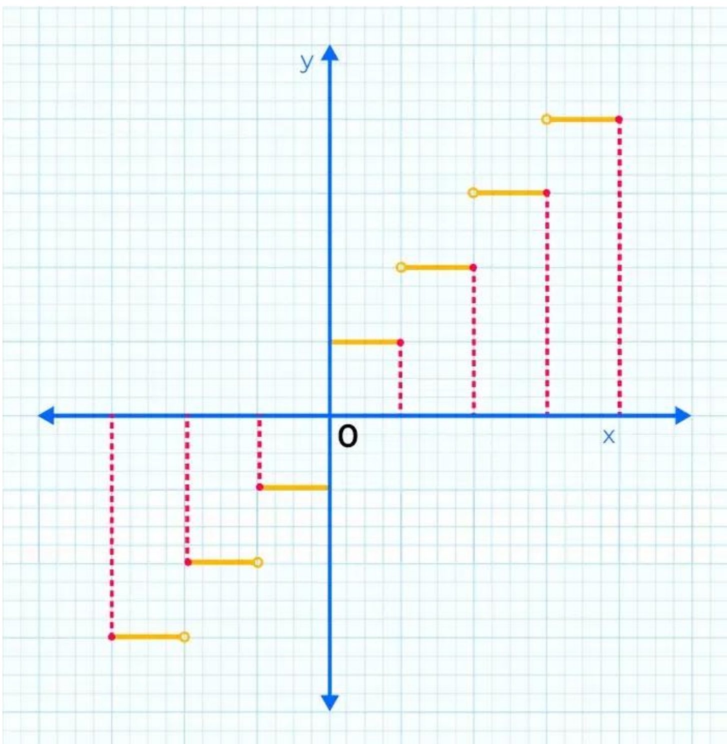
Question 1: What is the Floor of 1.43?
Solution:
Floor of a number is the ***greatest Integer*** *lesser or equal to that number. Therefore, here the Floor of 1.43 is 1.*



Question 2: What is the Floor of -5.66?
Solution:
On Negative axis, the greatest Integer lesser than -5.66 is -6.
Hence, ***-6*** *is the Floor of -5.66.*

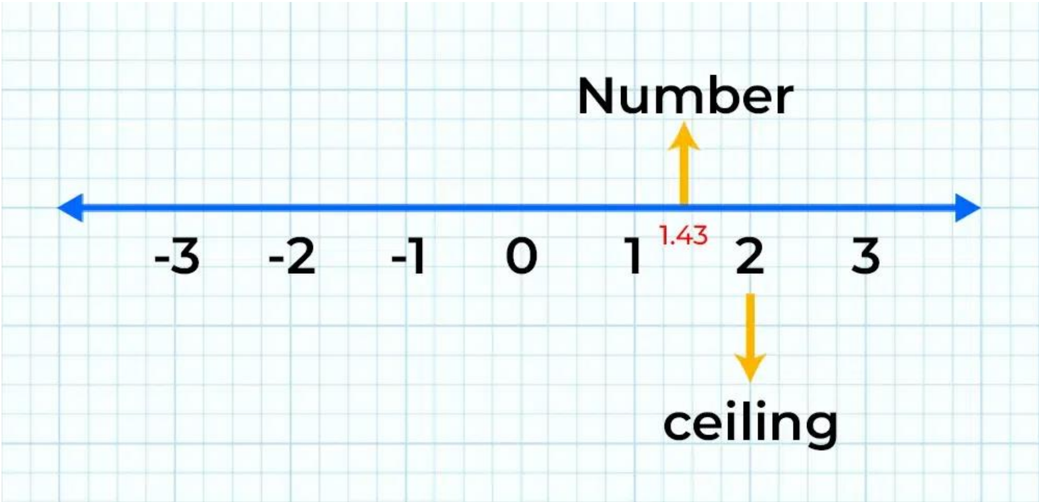
Least Integer Function or Ceiling Function

This function returns the smallest successive integer. The ceiling function of a real number x is the least integer that is greater than or equal to the given number x. The domain for this function is **R** and range **Z**.



Similar to the floor function, the domain of the ceiling function is **R** and the range is all the integers **I.**

Example 1: What is the Ceiling of 1.43?
Solution:
*The ceiling of the 1.43 should be its **smallest successive Integer,** hence the ceiling of 1.43 is 2.*



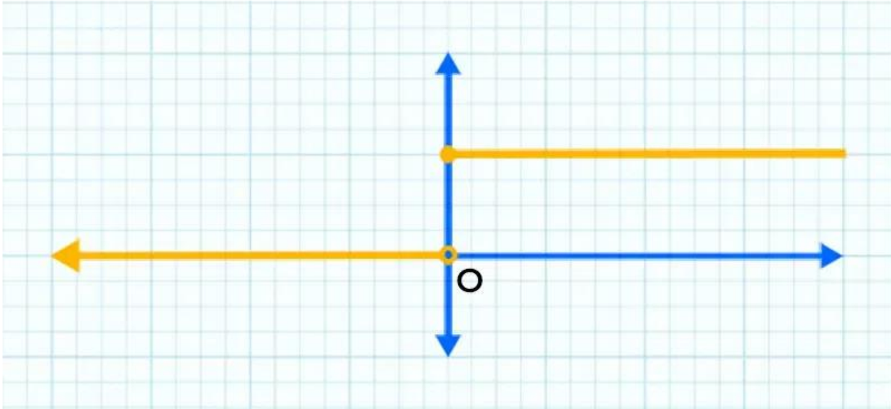
Example 2: What is the ceiling of -7.8?
Solution:
The smallest successive Integer of -7.8 is -7.
*Hence, **-7** is the ceiling of -7.8.*

Unit Step Function

Unit Step Function is yet another type of function used a lot in Signals and systems studies. It is defined as,

$$\mathbf{f(x)} = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

This function has no value at x = 0. It is called a step function because, at t = 0, it takes a step from 0 to 1. The domain for this function is **R – {0}** and range **{0,1}**.



Signum Function

This function shows the polarity of the input number if the number is negative function spits out -1 as output and if the number is positive the signum function spits out +1 as output and for 0 which is neutral in nature signum function spits 0 as output.
Mathematically signum function is defined as

$$\mathbf{sgn(x)} = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Evaluating Piecewise Functions

Let's consider the following examples of piecewise functions to evaluate their value at any given point.

Example: Find the value of the following function at x = -2 and x = 10.

$$\mathbf{f(x)} = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x, & \text{otherwise} \end{cases}$$

Solution:

at x = -2, x < 0 so the f(-2) = -2.
at x = 10, x > 0, so f(10) = 10^2 *= 100.*

Example: An arcade game charges the following prices depending on the length of time:

1. **Up to 6 minutes costs Rs.10**
2. **Over 6 and up to 15 minutes costs Rs.15**
3. **Over 15 minutes costs Rs.15 plus Rs.1 per minute above 15 minutes**

Represent this as a piecewise function and tell the price charged if Anil played the game for 13 minutes and Raju played for 20 minutes.

Solution:

These kind of prices charges can be represented as,

$$f(t) = \begin{cases} 10, & \text{if } t \leq 6 \\ 15 & \text{if } t \leq 15 \text{ and } t6 \\ 15 + 1(t - 15), & \text{otherwise} \end{cases}$$

at x = 13, f(13) = Rs.15 and x = 20, f(20) = 15 + 1(20 – 15) = 20.

Piecewise Continuous Function

A **Piecewise Continuous Function** is a function that is **continuous** across its entire domain i.e., each piece of the function is continuous itself and all the intersection points are the same for each piece so where each piece ends another piece of function starts from there in the graph.

An example of a **Piecewise Continuous Function** is given as follows:

$$\mathbf{f(x)} = \begin{cases} x^2, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

This function is continuous across the entire domain because the limit of f(x) as x approaches 0 from the left (for x < 0) is equal to the limit of f(x) as x approaches 0 from the right (For x ≥ 0), and both limits are equal to 0. Therefore, f(x) is continuous at x = 0.

Piecewise Continuous Function Graph

A Piecewise Continuous Function Graph is given below. The graph of a piecewise continuous function often resembles a series of connected segments, each representing the function's behavior within a specific interval. At the points where intervals meet, the function may experience jumps, breaks, or other types of discontinuities.

$$f(x) = \begin{cases} x + 3, & x < -1 \\ 2, & -1 \leq x \leq 2 \\ x^2 - 2, & x > 2 \end{cases}$$