2- Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{\emptyset(x,y)}$$

Is called a homogeneous equation if each term of $f(x,y)$ and $\emptyset(x,y)$ is of the same degree.

وتسمى المعادلة التفاضلية متجانسة اذا كان كل حد من حدود البسط والمقام لها نفس الدرجة كما في المثال التالي:

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$

اي ان مجموع الاس في البسط يساوي مجموع الاس في المقام

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In such cases we put

$$\boldsymbol{y = vx}$$

$$\boldsymbol{\frac{dy}{dx} = v + x\,\frac{dv}{dx}}$$

The reduced equation involves ***v*** and *x* only. This new differential equation can be solved by ***variables separable*** method**.**

وبالتعويض عن قيمة *y* ومشتقتها في المعادلة التفاضلية المعطاة نحصل على معادلة تفاضلية بدلالة *x* و *v* يمكن حلها بطريقة فصل المتغيرات.

Example 7:

Solve the equation $(x^2 + y^2)dx - 2xy\,dy = 0$

Solution:

$$(x^2 + y^2)dx = 2xy\,dy \quad \rightarrow \quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

$$y = vx \quad , \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2v} = \frac{x^2(1 + v^2)}{2x^2v} = \frac{1 + v^2}{2v}$$

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$$x\frac{dv}{dx}=\frac{1+v^2}{2v}-v \quad \rightarrow \quad x\frac{dv}{dx}=\frac{1-v^2}{2v}$$

$$\int\frac{2v}{1-v^2}\,dv=\int\frac{dx}{x} \quad \rightarrow \quad -ln\left(1-v^2\right)=ln\,x+ln\,c$$

$$ln\frac{1}{1-v^2}=ln\,cx \quad \rightarrow \quad \frac{1}{1-v^2}=cx \quad \rightarrow \quad \frac{1}{1-\frac{y^2}{x^2}}=cx$$

$$\frac{x^2}{x^2-y^2}=cx \quad \rightarrow \quad x^2=cx\left(x^2-y^2\right) \quad \rightarrow \quad x=c\left(x^2-y^2\right)$$

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Example 8:

Solve the equation $(2xy + x^2)\frac{dy}{dx} = 3\,y^2 + 2xy$

Solution:

$$(2xy + x^2)\frac{dy}{dx} = 3\,y^2 + 2xy \quad \rightarrow \quad \frac{dy}{dx} = \frac{3\,y^2 + 2xy}{2xy + x^2}$$

$$y = vx \quad , \quad \frac{dy}{dx} = v + x\,\frac{dv}{dx}$$

$$v + x\,\frac{dv}{dx} = \frac{3\,v^2\,x^2 + 2\,vx^2}{2x^2v + x^2} = \frac{x^2\,(3\,v^2 + 2v)}{x^2\,(2v + 1)}$$

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$$v + x\frac{dv}{dx} = \frac{3\,v^2 + 2\,v}{2v + 1} \qquad \rightarrow \qquad x\frac{dv}{dx} = \frac{3\,v^2 + 2\,v}{2v + 1} - v$$

$$x\frac{dv}{dx} = \frac{3\,v^2 + 2\,v - v\,(2v + 1)}{2v + 1} \qquad \rightarrow \quad x\frac{dv}{dx} = \frac{3\,v^2 + 2\,v - 2\,v^2 - v}{2v + 1}$$

$$x\frac{dv}{dx} = \frac{v^2 + v}{2\,v + 1} \qquad \rightarrow \qquad \frac{x}{dx} = \left(\frac{v^2 + v}{2\,v + 1}\right)\frac{1}{dv}$$

$$\frac{dx}{x} = \frac{2\,v + 1}{v^2 + v}\,dv$$

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$$\int \frac{2\,v+1}{v^2+v}\,dv = \int \frac{dx}{x}$$

$$ln\,(v^2+v) = ln\,x + ln\,c \qquad \rightarrow ln\,(v^2+v) = ln\,cx$$

$$V^2+v = cx$$

$$\frac{y^2}{x^2}+\frac{y}{x} = cx$$

$$y^2+xy = c\,x^3$$

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Example 9:

Solve the equation $x\,(y-x)\,\frac{dy}{dx} = y\,(y+x)$

Solution:

$$\frac{dy}{dx} = \frac{y\,(y+x)}{x\,(y-x)} = \frac{y^2 + xy}{xy - x^2}$$

$$y = vx \quad , \qquad \frac{dy}{dx} = v + x\,\frac{dv}{dx}$$

$$v + x\,\frac{dv}{dx} = \frac{vx\,(vx+x)}{x\,(vx-x)} = \frac{v^2\,x^2 + v\,x^2}{v\,x^2 - x^2} = \frac{v\,x^2\,(v+1)}{x^2\,(v-1)}$$

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$$v + x\frac{dv}{dx} = \frac{v\,(v+1)}{v-1} \quad \rightarrow \quad x\frac{dv}{dx} = \frac{v^2+v}{v-1} - v$$

$$x\frac{dv}{dx} = \frac{\cancel{v^2} + v - \cancel{v^2} + v}{v-1} = \frac{2v}{v-1}$$

$$\frac{x}{dx} = \frac{2v}{v-1} - \frac{1}{dv} \quad \rightarrow \quad \frac{dx}{x} = \frac{(v-1)\,dv}{2v}$$

$$\frac{dx}{x} = \frac{\cancel{v}}{2\cancel{v}}\,dv - \frac{1}{2v}\,dv$$

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$$\int \frac{dx}{x} = \frac{1}{2}\int dv - \frac{1}{2}\int \frac{dv}{v}$$

$$ln\, x = \frac{1}{2}\, v - \frac{1}{2}\, ln\, v + c$$

$$ln\, x = \frac{1}{2}\frac{y}{x} - \frac{1}{2}\, ln\, \frac{y}{x} + c$$

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Equations Reducible to Homogeneous Form

تحويل المعادلات الغير متجانسة الى معادلات متجانسة

The equations of form

$$\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C} \qquad \text{..............................(1)}$$

Can be reduced to the homogenous form by the substitution:

$x = X+h$, $y = Y+k$ (***h,k*** being constant)

اذا كانت المعادلة التفاضلية المعطاة بصيغة مشابه الى صيغة المعادلة (1) فيمكن تحويلها الى المعادلة تفاضلية متجانسة بالتعويض عن قيم (x,y) كما في اعلاه.

بحيث:

$$\frac{dy}{dx} = \frac{dY}{dX}$$

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The given differential equation reduces to: فتصبح المعادلة (1) بالشكل التالي

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C};\qquad \frac{dY}{dX} = \frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C} \quad \text{.................................(2)}$$

Choose h, k so that:

$ah + bk + c = 0$, $Ah + Bk + C = 0$

Then the equation (2) becomes homogeneous,

$$\frac{dY}{dX} = \frac{aX+bY}{AX+BY} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

تستخدم هذه الحالة عندما يكون :

$\frac{a}{A} \neq \frac{b}{B}$ then use Y= vX, $\frac{dY}{dx} = v + X\frac{dv}{dx}$

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اما اذا كانت :

$$\frac{a}{A} = \frac{b}{B}$$

Then put $ax + by = Z \rightarrow a + b\frac{dY}{dx} = \frac{dz}{dx}$

وبعد التعويض في المعادلة التفاضلية المعطاه تستخدم طريقة فصل المتغيرات.

Example 10: Solve the following equation $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution:

$a = 1,\ A = 2,\ b = 2,\ B = 1$

$\frac{a}{A} = \frac{1}{2}, \quad \frac{b}{B} = \frac{2}{1}, \qquad \therefore \frac{a}{A} \neq \frac{b}{B}$

$x = X + h \qquad , \quad y = Y + k$

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$$\frac{dY}{dX}=\frac{a(X+h)+b(Y+k)+c}{A(X+h)+B(Y+k)+C}, \quad \frac{dY}{dX}=\frac{(X+h)+2(Y+k)-3}{2(X+h)+B(Y+k)-3}, \quad \frac{dY}{dX}=\frac{X+2Y+h+2k-3}{2X+Y+2h+k-3}$$

$ah + bk + c = 0$, $Ah + Bk + C = 0$

Let, $h + 2k - 3 = 0$ ……….(a) multiply by 2 & Subtract (a) form (b)

$2h + k - 3 = 0$ ……….(b)

$3k - 3 = 0 \rightarrow k = 1$, from Eq. (a) or (b) $h+2-3=0 \rightarrow h=1$

$x=X+1$, $y=Y+1$

$$\frac{dY}{dX}=\frac{aX+bY}{AX+BY}, \quad \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

تستخدم هذه الحالة عندما يكون :

$\frac{a}{A} \neq \frac{b}{B}$ then use Y= vX, $\frac{dY}{dx} = v + X\frac{dv}{dx}$

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$$v + X\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{1+2v}{2+v} \rightarrow X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1+2v-2v-v^2}{2+v} = \frac{1-v^2}{2+v}$$

$$\int \frac{2+v}{1-v^2} dv = \int \frac{dx}{x}, \qquad \int \frac{2+v}{(1-v)(1+v)} dv = \ln x + \ln c$$

$$\int \frac{2+v}{(1-v)(1+v)} dv = \int \left(\frac{A}{1-v} + \frac{B}{1+v}\right) dv = \int \left(\frac{A(1+v)+B(1-v)}{(1-v)(1+v)}\right) dv$$

$$2 + v = A\,(1+v) + B(1-v)$$
$$2 + v = A + Av + B - Bv$$
$$2 = A + B \rightarrow A = 2 - B$$
$$1 = A - B \rightarrow 1 = 2 - B - B \rightarrow B = {}^{1}/_{2}, \; and \quad A = 2 - {}^{1}/_{2} \rightarrow A = {}^{3}/_{2}$$

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$$\frac{3}{2}\int\frac{1}{1-v}\,dv+\frac{1}{2}\int\frac{1}{1+v}\,dv=lnX+\ln c,\qquad -\frac{3}{2}\ln(1-v)+\frac{1}{2}\ln(1+v)=\ln Xc$$

$$1/2\,[\ln(1+v)-3\ln(1-V)]=\ln Xc\,,\qquad 1/2\,[\ln(1+v)-\ln(1-v)^3]=\ln Xc$$

$$1/2\ln\frac{1+v}{(1-v)^3}=\ln Xc\,,\qquad \ln\sqrt{\frac{1+v}{(1-v)^3}}=\ln Xc$$

$$\sqrt{\frac{1+v}{(1-v)^3}}=Xc\rightarrow\frac{1+v}{(1-v)^3}=X^2C^2$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3}=X^2C^2\rightarrow\frac{\frac{X+Y}{X}}{\left(\frac{X-Y}{X}\right)^3}=C^2X^2\rightarrow\frac{\frac{X+Y}{X}}{\frac{(X-Y)^3}{X^3}}=C^2X^2\rightarrow\frac{X+Y}{(X-Y)^3}=C^2$$

$$X+Y=C^2\,(X-Y)^3$$

$x=X+h\rightarrow x=X+1\rightarrow X=x-1$ ***and*** $y=Y+k\rightarrow y=Y+1\rightarrow Y=y-1$

$x-1+y-1=C^2(x-1-y+1)^{\,3}$

$\boldsymbol{x+y-2=C^2(x-y)^{\,3}}$

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Example 11: Solve the following equation $(x+2y)(dx-dy)=dx+dy$

Solution:

$$x\,dx - x\,dy + 2y\,dx - 2y\,dy - dx - dy = 0$$
$$(x+2y-1)dx - (x+2y+1)dy = 0$$

$$\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1}$$

***a* = 1 , *A* = 1, *b* = 2, *B* = 2**

$$\frac{a}{A} = \frac{1}{1} = 1\,, \qquad \frac{b}{B} = \frac{2}{2} = 1, \qquad \therefore \frac{a}{A} = \frac{b}{B}$$

Then use: $ax + by = Z \quad \rightarrow \quad a + b\frac{dy}{dx} = \frac{dz}{dx}$

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$$1+2\frac{dy}{dx}=\frac{dZ}{dx} \rightarrow 2\frac{dy}{dx}=\frac{dZ}{dx}-1 \rightarrow \frac{dy}{dx}=\frac{1}{2}\frac{dZ}{dx}-\frac{1}{2}$$

[a = 1 , A = 1, b = 2, B = 2]

$ax + by = Z \rightarrow a+b\frac{dy}{dx}=\frac{dz}{dx}$

$$z = x + 2y \rightarrow \frac{dy}{dx}=\frac{z-1}{z+1}$$

$$\frac{Z-1}{Z+1}=\frac{1}{2}\frac{dZ}{dx}-\frac{1}{2}, \quad \frac{1}{2}\frac{dZ}{dx}=\frac{Z-1}{Z+1}+\frac{1}{2}, \quad \frac{dZ}{dx}=\frac{2(Z-1)}{Z+1}+1$$

$$\frac{dZ}{dx}=\frac{2Z-2+Z+1}{Z+1}=\frac{3Z-1}{Z+1} \rightarrow \frac{Z+1}{3Z-1}dz = dx$$ multiply by 3/3

$$\frac{3Z+3}{3(3Z-1)}dZ = dx, \quad \frac{3Z+3+1-1}{3(3Z-1)}dZ = dx, \quad \frac{(3Z-1)+4}{3(3Z-1)}dZ = dx$$

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$$\frac{3Z-1}{3(3Z-1)}dZ + \frac{4}{3(3Z-1)}dZ = dx$$

$$\frac{1}{3}\int dZ + \frac{4}{3}\int \frac{dz}{3Z-1} = \int dx$$

$$\frac{Z}{3} + \frac{4}{9}\ln(3Z-1) = x + c$$

$$3\,Z + 4\ln(3Z-1) = 9\,x + 9c$$

$$3(x+2y) + 4\ln(3x+6y-1) = 9X + 9c$$

$$4\ln(3x+6y-1) = 6\,x\ - 6y + 9c$$

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