Applications of Differential Equation تطبيقات على المعادلات التفاضلية

1- Electrical Circuit.

2- Mechanics.

3- Heat Conduction.

4- Springs.

Example 17:

The acceleration and velocity of a body falling in the air approximately satisfy the equation: $Acceleration = g - k\,v^2$, Where: *v* is the velocity of the body at any time *(t)*. And *g*, *k* are constant.

Find the distance traversed as a function of the time , if the body falls from the rest. Show the value of *v* will never exceed $\sqrt{g/k}$.

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Solution: $Acceleration = g - k\,v^2 \quad or \quad \frac{dV}{dt} = g - k\,v^2 \quad or \quad \frac{dv}{g-k\,v^2} = dt$

$$\rightarrow \quad \left[\frac{1}{(\sqrt{g} + \sqrt{k}\,v)(\sqrt{g} - \sqrt{k}\,v)}\right] dv = dt$$

$$\frac{1}{2\sqrt{g}}\left[\frac{1}{\sqrt{g} + \sqrt{k}\,v} + \frac{1}{\sqrt{g} - \sqrt{k}\,v}\right] dv = dt$$

On integrating,

$$\frac{1}{2\sqrt{g}}\int \frac{dv}{\sqrt{g} + \sqrt{k}\,v} + \frac{1}{2\sqrt{g}}\int \frac{dv}{\sqrt{g} - \sqrt{k}\,v} = \int dt$$

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$$\frac{1}{2\sqrt{g}}\cdot\frac{1}{\sqrt{k}}\cdot ln\left(\sqrt{g}+\sqrt{k}\,v\right)-\frac{1}{2\sqrt{g}}\cdot\frac{1}{\sqrt{k}}\cdot ln\left(\sqrt{g}-\sqrt{k}\,v\right)=t+c$$

when $t=0$ *and* $v=0$

$$\frac{1}{2\sqrt{g}}\cdot\frac{1}{\sqrt{k}}\cdot ln(\sqrt{g}+0)-\frac{1}{2\sqrt{g}}\cdot\frac{1}{\sqrt{k}}\cdot ln(\sqrt{g}-0)=0+c$$

$$\frac{1}{2\sqrt{gk}}\left[ln\sqrt{g}-ln\sqrt{g}\right]=c$$

$$\frac{1}{2\sqrt{gk}}\cdot ln\frac{\sqrt{g}}{\sqrt{g}}=c \quad\rightarrow\quad \frac{1}{2\sqrt{gk}}\cdot \underbrace{ln\,1}_{0}=c \quad\rightarrow\quad c=0$$

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$$\frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} + \sqrt{k}\,v\right) - \frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} - \sqrt{k}\,v\right) = t$$

$$\frac{1}{2\sqrt{gk}}\left[ln\,\frac{\sqrt{g} + \sqrt{k}\,v}{\sqrt{g} - \sqrt{k}\,v}\right] = t \qquad \rightarrow \qquad ln\,\frac{\sqrt{g} + \sqrt{k}\,v}{\sqrt{g} - \sqrt{k}\,v} = 2\sqrt{gk}\,t$$

$$e^{ln\frac{\sqrt{g} + \sqrt{k}\,v}{\sqrt{g} - \sqrt{k}\,v}} = e^{2\sqrt{gk}\,t}$$

$$\frac{\sqrt{g} + \sqrt{k}\,v}{\sqrt{g} - \sqrt{k}\,v} = e^{2\sqrt{gk}\,t} \qquad \rightarrow \sqrt{g} + \sqrt{k}\,v = e^{2\sqrt{gk}\,t}\left(\sqrt{g} - \sqrt{k}\,v\right)$$

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$$\sqrt{g} + \sqrt{k}\, v = \sqrt{g}\,.\,e^{2\sqrt{gk}\,t} - \sqrt{k}\,.\,v\,.\,e^{2\sqrt{gk}\,t}$$

$$\sqrt{k}\,.\,v + \sqrt{k}\,.\,v.\,e^{2\sqrt{gk}\,t} = \sqrt{g}\,.\,e^{2\sqrt{gk}\,t} - \sqrt{g}$$

$$\sqrt{k}\,.\,v\left[1 + e^{2\sqrt{gk}\,t}\right] = \sqrt{g}\left[e^{2\sqrt{gk}\,t} - 1\right]$$

$$\frac{\sqrt{k}\,v}{\sqrt{g}} = \frac{e^{2\sqrt{gk}\,t} - 1}{e^{2\sqrt{gk}\,t} + 1} \times \frac{e^{-2\sqrt{gk}\,t}}{e^{-2\sqrt{gk}\,t}}$$

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$$\frac{\sqrt{k}\,v}{\sqrt{g}} = \frac{e^{\sqrt{gk}\,t} - e^{-\sqrt{gk}\,t}}{e^{\sqrt{gk}\,t} + e^{-\sqrt{gk}\,t}} = \tanh\sqrt{gk}\,t$$

$$\sqrt{k}\,v = \sqrt{g}\,.\tanh\sqrt{gk}\,t$$

$$v = \sqrt{\frac{g}{k}}\,.\tanh\sqrt{gk}\,t$$

$$\frac{dx}{dt} = \sqrt{\frac{g}{k}}\,.\tanh\sqrt{gk}\,t$$

$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

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$$\int dx = \sqrt{\frac{g}{k}} \int \tanh \sqrt{gk}\, t\, dt$$

$$x = \sqrt{\frac{g}{k}} \cdot \frac{1}{\sqrt{gk}} \cdot \ln \cosh \sqrt{gk}\, t + c$$

$$x = \frac{1}{k} \cdot \ln \cosh \sqrt{gk}\, t + c$$

$$at \quad t = 0 \quad and \quad x = 0 \quad \rightarrow \quad c = 0$$

$$x = \frac{1}{k} \cdot \ln \cosh \sqrt{gk}\, t$$

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Example 18:

The initial temperature of body is 130 ℃ , it put in media of 10 ℃ temperature. Find the temperature of the body after 30 min. , if the temperature of the body became 80 ℃ , after put it in this media for 15 min.

$$\text{Ues Newtens law of cooling: } \frac{dT}{dt} = -k\,(T - T_a)$$

Where

$k \rightarrow$ *constant*

$T \rightarrow$ *temperature of body*

$T_a \rightarrow$ *media temperature*

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Solution: $\frac{dT}{dt} = -k\,(T - T_a) \rightarrow \frac{dT}{dt} = -k\,(T - 10)$

$$\int \frac{dT}{T-10} = -k \int dt$$

$$ln\,(T-10) = -kt + c \quad \rightarrow \quad e^{ln\,(T-10)} = e^{-kt+c} \quad \rightarrow \quad T - 10 = e^{-kt} + e^{c}$$

$$T - 10 = e^{-kt} . e^{c}$$

$$let\ \ e^{c} = A$$

$$T - 10 = e^{-kt} . A \quad \rightarrow \quad T = A\, e^{-kt} + 10$$

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$$at \quad t = 0 \quad and \quad T = 130\,℃$$

$$130 = A\, e^{0} + 10 \qquad \rightarrow \qquad A = 120$$

$$T = 120\, e^{-kt} + 10$$

$$at \quad t = 15\, min. \quad and \quad T = 80\,℃$$

$$80 = 120\, e^{-k\,(15)} + 10 \qquad \rightarrow \qquad e^{-k(15)} = \frac{80 - 10}{120} = 0.5833$$

$$ln\left(e^{-k(15)}\right) = ln\,(0.5833) \quad \rightarrow \quad -k\,(15) = ln\,(0.5833) \quad \rightarrow \quad k = 0.036$$

$$T = 120\, e^{-0.036t} + 10$$

$$at \quad t = 30\, min.$$

$$T = 120\, e^{-0.036\,(30)} + 10 = 50.75$$

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