2) $$\frac{1}{f\,(D)}\,x^n = [f\,(D)]^{-1}\,x^n$$

Expand $[f\,(D)]^{-1}$

$$(1+u)^{-1} = 1 - u + u^2 - u^3 + u^4$$

$$(1-u)^{-1} = 1 + u + u^2 + u^3 + u^4$$

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Example 9 : **Solve** $(D^2 + 2D)_y = x^3 - 3x$

$$m^2 + 2m = 0 \quad \rightarrow \quad m(m+2) = 0 \quad \rightarrow \quad m_1 = 0 \;, \qquad m_2 = -2$$

$$C.F. = c_1 + c_2\, e^{-2x}$$

$$P.I. = \frac{1}{D^2 + 2D} . (x^3 - 3x) = \frac{1}{2D\left(\frac{D}{2} + 1\right)} . (x^3 - 3x)$$

$$= \frac{1}{2D}\left(1 + \frac{D}{2}\right)^{-1}(x^3 - 3x)$$

Note

$$\left(\frac{D}{2} + 1\right) = (u + 1)$$

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$$= \frac{1}{2D}\left[1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \frac{D^4}{16}\right](x^3 - 3x)$$

$$= \frac{1}{2D}\left[(x^3 - 3x) - \frac{1}{2}(3\,x^2 - 3) + \frac{1}{4}(6x) - \frac{1}{8}(6)\right]$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - \frac{3}{2}\,x^2 - \frac{1}{2}\,x^3 + \frac{3}{2}\,x + \frac{3}{4}\,x^2 - \frac{3}{4}\,x\right]$$

$$= \frac{x^4}{8} - \frac{1}{4}\,x^3 - \frac{3}{8}\,x^2 + \frac{3}{8}\,x$$

$$y = c_1 + c_2\,e^{-2x} + \frac{x^4}{8} - \frac{1}{4}\,x^3 - \frac{3}{8}\,x^2 + \frac{3}{8}\,x$$

Example 10: Solve $(2D^2 + 3D + 4)_y = x^2 - 2x$

$$2m^2 + 3m + 4 = 0$$

$$m = \frac{-3 \pm \sqrt{9 - 4 \times 2 \times 4}}{2 \times 2} = \frac{-3 \pm \sqrt{-23}}{4} = \frac{-3}{4} \pm \frac{\sqrt{23}}{4}\, i$$

$$C.F. = e^{\frac{-3}{4}x}\left[A \cos \frac{\sqrt{23}}{4}\, x + B \sin \frac{\sqrt{23}}{4}\, x\right]$$

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$$P.I. = \frac{1}{2D^2 + 3D + 4} . (x^2 - 2x) = \frac{1}{4\left[\frac{D^2}{2} + \frac{3}{4} D + 1\right]} . (x^2 - 2x)$$

$$= \frac{1}{4} \left[1 + \left(\frac{D^2}{2} + \frac{3}{4} D\right)\right]^{-1} . (x^2 - 2x)$$

Note

$$\left[1 + \left(\frac{D^2}{2} + \frac{3}{4} D\right)\right]^{-1} = (u + 1)^{-1}$$

$$= \frac{1}{4} \left[1 - \left(\frac{D^2}{2} + \frac{3}{4} D\right) + \left(\frac{D^2}{2} + \frac{3}{4} D\right)^2\right] . (x^2 - 2x)$$

$$= \frac{1}{4} \left[1 - \frac{D^2}{2} - \frac{3}{4} D + \frac{D^4}{4} + \frac{3}{2} D^3 + \frac{9}{16} D^2\right] . (x^2 - 2x)$$

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$$= \frac{1}{4}\left[x^2 - 2x - \frac{1}{2}(2) - \frac{3}{4}(2x-2) + \frac{9}{16}(2)\right]$$

$$= \frac{1}{4}\left[x^2 - 2x - 1 - \frac{3}{2}x + \frac{3}{2} + \frac{9}{8}\right]$$

$$= \frac{1}{4}\left[x^2 - \frac{7}{2}x + \frac{13}{8}\right]$$

$$y = e^{\frac{-3}{4}x}\left[A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right] + \frac{1}{4}\left[x^2 - \frac{7}{2}x + \frac{13}{8}\right]$$

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Example 11: **Solve** $\frac{d^2y}{dx^2} + 2p\,\frac{dy}{dx} + (p^2 + q^2)y = e^{cx} + p.q.x^2$

$p\,, q$ *and* c *are constants*

$$\left[D^2 + 2p\,D + (p^2 + q^2)\right]_y = e^{cx} + p.q.x^2$$

$$m^2 + 2p\,m + (p^2 + q^2) = 0$$

$$m = \frac{-2p \pm \sqrt{4\,p^2 - 4(p^2 + q^2)}}{2} = \frac{-2p \pm \sqrt{4p^2 - 4p^2 - 4q^2}}{2}$$

$$= \frac{-2p \pm \sqrt{-4q^2}}{2} = -p \pm qi$$

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$$C.F. = e^{-px}\,[A\cos\ qx + B\sin qx]$$

$$P.I. = \frac{1}{[D^2 + 2pD + (q^2 + p^2)]}\,e^{cx} + \frac{1}{[D^2 + 2pD + (q^2 + p^2)]}\,p\,.\,q\,.\,x^2$$

$$= \frac{1}{c^2+2pc+(q^2+p^2)}\,e^{cx} + \frac{1}{(q^2+p^2)\left[\frac{D^2}{q^2+p^2}+\frac{2pD}{q^2+p^2}+1\right]}\,p\,.\,q\,.\,x^2$$

$$= \frac{1}{c^2 + 2pc + q^2 + p^2}\,e^{cx} + \frac{1}{q^2 + p^2}\left[\frac{D^2}{q^2 + p^2} + \frac{2pD}{q^2 + p^2} + 1\right]^{-1} p\,.\,q\,.\,x^2$$

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$$= \frac{e^{cx}}{(c+p)^2 + q^2} + \frac{1}{p^2 + q^2}\left[1 - \left(\frac{D^2}{p^2 + q^2} + \frac{2pD}{p^2 + q^2}\right) + \left(\frac{D^2}{p^2 + q^2} + \frac{2pD}{p^2 + q^2}\right)^2\right] p\,.\,q\,.\,x^2$$

$$= \frac{e^{cx}}{(c+p)^2 + q^2} + \frac{1}{p^2 + q^2}\left[1 - \frac{D^2}{p^2 + q^2} - \frac{2pD}{p^2 + q^2} + \frac{D^4}{(p^2 + q^2)^2} + \frac{4\,p\,D^3}{(p^2 + q^2)^2} + \frac{4\,p^2\,D^2}{(p^2 + q^2)^2}\right] p\,.\,q\,.\,x^2$$

0 0

$$= \frac{e^{cx}}{(c+p)^2 + q^2} + \frac{1}{p^2 + q^2}\left[p\,.\,q\,.\,x^2 - \frac{2pq}{p^2 + q^2} - \frac{4\,p^2\,q\,x}{p^2 + q^2} + \frac{8\,p^3\,q}{(p^2 + q^2)^2}\right]$$

$$= \frac{e^{cx}}{(c+p)^2 + q^2} + \frac{p\,q}{p^2 + q^2}\left[x^2 - \frac{2}{p^2 + q^2} - \frac{4\,px}{p^2 + q^2} + \frac{8\,p^2}{(p^2 + q^2)^2}\right]$$

$$y = e^{-px}\,[A\cos\,qx + B\sin qx] + \frac{e^{cx}}{(c+p)^2 + q^2} + \frac{p\,q}{p^2 + q^2}\left[x^2 - \frac{2}{p^2 + q^2} - \frac{4\,px}{p^2 + q^2} + \frac{8\,p^2}{(p^2 + q^2)^2}\right]$$

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3) $$\frac{1}{f(D^2)} \sin ax = \frac{\sin ax}{f(-a^2)}$$

$$\frac{1}{f(D^2)} \cos ax = \frac{\cos ax}{f(-a^2)}$$

$If \quad f(-a^2) = 0 \quad \rightarrow \quad \frac{1}{f(D^2)} \sin ax = x \frac{\sin ax}{\bar{f}(-a^2)}$

$If \quad \bar{f}(-a^2) = 0 \quad \rightarrow \quad \frac{1}{f(D^2)} \sin ax = x^2 \frac{\sin ax}{\bar{\bar{f}}(-a^2)}$

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Example 12: Solve $\left(D^2+4\right) y = \sin 3x$

$m^2+4=0 \quad \rightarrow \quad m^2=-4 \quad \rightarrow \quad m_1=2i \quad , \quad m_2=-2i$

$C.F. = A\cos 2x + B\sin 2x$

$$P.I. = \frac{1}{D^2+4} . \sin 3x = \frac{\sin 3x}{-(3)^2+4} = -\frac{1}{5} \sin 3x$$

$$y = A\cos 2x + B\sin 2x - \frac{1}{5}\sin 3x$$

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Example 13 : Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \cos 2x$

$$(D^2 + D + 1)\, y = \cos 2x \qquad \rightarrow \qquad m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

$$C.F. = e^{\frac{-1}{2}x} \left[A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right]$$

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$$P.I. = \frac{1}{D^2 + D + 1} . \cos 2x = \frac{\cos 2x}{-(2)^2 + D + 1} = \frac{1}{D - 3} \cos 2x$$

$$= \frac{D + 3}{D^2 - 9} \cos 2x = \frac{D + 3}{-(2)^2 - 9} \cos 2x = \frac{-1}{13} (D + 3) \cos 2x$$

$$= \frac{-1}{13} (-2 \sin 2x + 3 \cos 2x)$$

$$y = e^{\frac{-1}{2}x} \left[A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right] - \frac{1}{13} (-2 \sin 2x + 3 \cos 2x)$$

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Example 14: Solve $(D^2 + a^2)\, y = \sin ax$

$m^2 + a^2 = 0 \quad \rightarrow \quad m^2 = -a^2 \quad \rightarrow \quad m_1 = ai \quad , \quad m_2 = -ai$

$C.F. = A \cos ax + B \sin ax$

$$P.I. = \frac{1}{D^2 + a^2} \,.\, \sin ax = \frac{\sin ax}{-a^2 + a^2} = x \,.\, \frac{1}{2D} \,.\, \sin ax = \frac{-1}{2a} \,.\, x \,.\, \cos ax$$

$$y = A \cos ax + B \sin ax - \frac{x}{2a} \,.\, \cos ax$$

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Example 15 : **Solve** $\left(D^2 - 2D + 3\right) y = 2cos\, 2x$

$$m^2 - 2m + 3 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 3}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm \sqrt{2}\, i$$

$$C.F. = e^x \left[A \cos \sqrt{2}\, x + B \sin \sqrt{2}\, x\right]$$

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$$P.I. = \frac{1}{D^2 - 2D + 3}\,.\,2\cos 2x = \frac{1}{-(2)^2 - 2D + 3}\,.\,2\cos 2x = \frac{1}{-2D - 1}\,.\,2\cos 2x$$

$$= \frac{-2}{2D + 1}\,.\,\cos 2x = \frac{-2\,(2D - 1)}{(2D + 1)(2D - 1)}\,.\,\cos 2x = \frac{-2\,(2D - 1)}{4\,D^2 - 1}\,.\,\cos 2x$$

$$= \frac{-2\,(2D-1)}{4\left[-(2)^2\right]-1}\,.\,\cos 2x = \frac{-2}{-17}\,(2D - 1)\,.\,\cos 2x = \frac{2}{17}\,(-4\sin 2x - \cos 2x)$$

$$y = e^x\left[A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x\right] + \frac{2}{17}\,(-4\sin 2x - \cos 2x)$$

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