4) $$\frac{1}{f\,(D)}\,.\,e^{ax}.\,\varphi(x) = e^{ax}\,.\,\frac{1}{f(D+a)}\,.\,\varphi(x)$$

Example 15 : Solve $(D^2 - 5D + 6)y = e^x \cos 2x$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0 \quad \rightarrow \quad m_1 = 2 \quad , \quad m_2 = 3$$

$$C.F. = c_1\, e^{2x} + c_2\, e^{3x}$$

Email: iibrahim@uomosul.edu.iq

$$P.I. = \frac{1}{D^2 - 5D + 6} \cdot e^x \cos 2x = e^x \cdot \frac{1}{(D+1)^2 - 5(D+1) + 6} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{D^2+2D+1-5D-5+6} \cdot \cos 2x = e^x \cdot \frac{1}{D^2-3D+2} \cdot \cos 2x$$

$$= e^x \cdot \frac{1}{-(2)^2 - 3D + 2} \cdot \cos 2x = e^x \cdot \frac{1}{-3D-2} \cdot \cos 2x$$

$$= -e^x \cdot \frac{1}{3D+2} \cdot \cos 2x = -e^x \cdot \frac{(3D-2)}{(3D+2)(3D-2)} \cdot \cos 2x$$

Email: iibrahim@uomosul.edu.iq

$$= -e^{x} \cdot \frac{3D-2}{9D^{2}-4} \cdot \cos 2x = -e^{x} \cdot \frac{3D-2}{9\,(-4)-4} \cdot \cos 2x$$

$$= \frac{e^{x}}{40}\,(3D-2)\cos 2x = \frac{e^{x}}{40}\,(-6\sin 2x - 2\cos 2x)$$

$$= \frac{-e^{x}}{20}\,(3\sin 2x + \cos 2x)$$

$$y = c_1\, e^{2x} + c_2\, e^{3x} - \frac{e^{x}}{20}\,(3\sin 2x + \cos 2x)$$

Email: iibrahim@uomosul.edu.iq

Example 16 : Solve $\left(D^2 - 4D + 3\right) y = 2xe^{3x} + 3e^{x} \cos 2x$

$$m^2 - 4m + 3 = 0$$

$$(m-1)(m-3) = 0 \quad \rightarrow \quad m_1 = 1 \quad , \quad m_2 = 3$$

$$C.F. = c_1 e^{x} + c_2 e^{3x}$$

Email: iibrahim@uomosul.edu.iq

$$P.I. = \frac{1}{D^2 - 4D + 3} \cdot 2xe^{3x} + \frac{1}{D^2 - 4D + 3} \cdot 3e^x \cos 2x$$

$$= 2\, e^{3x} \cdot \frac{1}{(D+3)^2 - 4\,(D+3) + 3} \cdot x + 3e^x \cdot \frac{1}{(D+1)^2 - 4\,(D+1) + 3} \cdot \cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{D^2 + 6D + 9 - 4D - 12 + 3} \cdot x + 3e^x \cdot \frac{1}{D^2 + 2D + 1 - 4D - 4 + 3} \cdot \cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{D^2 + 2D} \cdot x + 3e^x \cdot \frac{1}{D^2 - 2D} \cdot \cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{2D\left(\frac{D}{2} + 1\right)} \cdot x + 3e^x \cdot \frac{1}{-(2)^2 - 2D} \cdot \cos 2x$$

Email: iibrahim@uomosul.edu.iq

$$= e^{3x} \cdot \frac{1}{D}\left[1+\frac{D}{2}\right]^{-1} \cdot x - \frac{3}{2}\, e^{x} \cdot \frac{1}{2+D} \cdot cos2x$$

$$= e^{3x} \cdot \frac{1}{D}\left[1-\frac{D}{2}+\frac{D^2}{4} - \;\dots\dots\right] x - \frac{3}{2} \cdot e^{x} \cdot \frac{2-D}{4-D^2} \cdot cos\,2x$$

$$= e^{3x} \cdot \frac{1}{D}\left[x-\frac{1}{2}\right] - \frac{3}{2}\, e^{x} \cdot \frac{2-D}{4+(2)^2} \cdot cos\,2x$$

$$= e^{3x}\left[\frac{x^2}{2}-\frac{x}{2}\right] - \frac{3}{16} e^{x}[2\,cos\,2x + 2\,sin\,2x] = \frac{e^{3x}}{2}(x^2 - x) - \frac{3e^{x}}{8}(cos\,2x + sin\,2x)$$

Email: iibrahim@uomosul.edu.iq

The Homogeneous Linear Equations

$$a_n\, x^n \frac{d^n y}{dx^n} + a_{n-1}\, x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ \dots\dots\dots\dots + a_0\, y = \emptyset\,(x)$$

$where\ \ a_0\,, a_1\,, a_2\,, \dots\dots\dots\dots\dots\ \ are\ constants, is\ called\ a\ Homogeneous\ Equation$

يجب ان يكون الاس ل x ودرجة المشتقة المضروبة بها من نفس الدرجة

$put\ \ x = e^z,\qquad z = lnx \quad , \quad \frac{d}{dz} = D \quad , \quad \frac{dz}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz} \quad \rightarrow \quad x\frac{dy}{dx} = \frac{dy}{dz} \quad \rightarrow \quad \boxed{x.\frac{dy}{dx} = \frac{dy}{dz} = Dy}$$

Engineering Analysis

Dr. Ibrahim Al-Hafidh

Email: iibrahim@uomosul.edu.iq

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dx}\right) = \frac{-1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= \frac{-1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{1}{x} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}\left(D^2 - D\right)\ y$$

$$x^2\frac{d^2y}{dx^2} = D\,(D-1)\,y$$

$$x^3\frac{d^3y}{dx^3} = D\,(D-1)(D-2)\,y$$

Email: iibrahim@uomosul.edu.iq

Example 17 : Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x$

$$x\frac{dy}{dx}=D_y \quad , \qquad x^2\frac{d^2y}{dx^2}=D(D-1)\ y$$

$$put \quad x=e^z \quad , \qquad \ln x=\ln e^z \quad \rightarrow \quad z=\ln x$$

$$D(D-1)_y+D_y+\mathrm{y}=e^z \quad \rightarrow \quad (D^2-\cancel{D}+\cancel{D}+1)\ y=e^z$$

$$(D^2+1)\ y=e^z$$

$$D^2+1=0 \quad \rightarrow \quad m^2+1=0 \quad \rightarrow \quad m=\pm i$$

Email: iibrahim@uomosul.edu.iq

$$C.F. = A\cos z + B\sin z$$

$$P.I. = \frac{1}{D^2+1}e^z = \frac{1}{(1)^2+1}e^z = \frac{1}{2}e^z$$

$$y = A\cos z + B\sin z + \frac{1}{2}e^z$$

$$y = A\cos(\ln x) + B\sin(\ln x) + \frac{1}{2}x$$

Email: iibrahim@uomosul.edu.iq

Example 18 : Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = sin\,2[ln\,(1+x)]$

$$(1+x)\frac{dy}{dx} = D_y \quad , \quad (1+x)^2\frac{d^2y}{dx^2} = D(D-1)_y$$

$$put \quad 1+x = e^z \quad , \quad ln\,(1+x) = ln\,e^z \quad \rightarrow \quad z = ln\,(1+x)$$

$$D(D-1)_y + D_y + y = sin\,2z \quad \rightarrow \quad (D^2 - \cancel{D} + \cancel{D} + 1)_y = sin\,2z$$

$$(D^2+1)_y = sin\,2z$$

$$D^2+1 = 0 \quad \rightarrow \quad m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$$

Email: iibrahim@uomosul.edu.iq

$$C.F. = A\cos z + B\sin z$$

$$P.I. = \frac{1}{D^2+1}\sin 2z = \frac{1}{-(2)^2+1}\sin 2z = \frac{-1}{3}\sin 2z$$

$$y = A\cos z + B\sin z - \frac{1}{3}\sin 2z$$

$$y = A\cos[\ln(1+x)] + B\sin[\ln(1+x)] - \frac{1}{3}\sin 2[\ln(1+x)]$$

Email: iibrahim@uomosul.edu.iq

Example 19 : Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\ln x^2)$

$x\frac{dy}{dx} = D\,y$, $x^2\frac{d^2y}{dx^2} = D\,(D-1)\,y$

put $x = e^z$, $\ln x = \ln e^z$ $\rightarrow$ $z = \ln x$ $\rightarrow$ $2z = 2\ln x = \ln x^2$

$D\,(D-1)\,y + D\,y + y = \sin 2z$ $\rightarrow$ $(D^2 - D + D + 1)\,y = \sin 2z$

$(D^2 + 1)\,y = \sin 2z$

$D^2 + 1 = 0$ $\rightarrow$ $m^2 + 1 = 0$ $\rightarrow$ $m = \pm i$

Email: iibrahim@uomosul.edu.iq

$$C.F. = A\cos z + B\sin z$$

$$P.I. = \frac{1}{D^2+1}\sin 2z = \frac{1}{-(2)^2+1}\sin 2z = \frac{-1}{3}\sin 2z$$

$$y = A\cos z + B\sin z - \frac{1}{3}\sin 2z$$

$$y = A\cos(\ln x) + B\sin(\ln x) - \frac{1}{3}\sin(\ln x^2)$$

Email: iibrahim@uomosul.edu.iq