Method of Variation of Parameters

To find particular integral of :

$$a\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + cy = X \; or \; f\,(x)$$

$Let \quad C.F. = A\,y_1 + B\,y_2$

$$P.I. = u\,y_1 + v\,y_2$$

$where:$

$$u = \int \frac{-y_2 \,.\, x}{y_1 \,.\, \overline{y_2} - \overline{y_1} \,.\, y_2}\, dx$$

$$v = \int \frac{y_1 \,.\, x}{y_1 \,.\, \overline{y_2} - \overline{y_1} \,.\, y_2}\, dx$$

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Example 20: Solve $\frac{d^2y}{dx^2} + y = csc\ x$

$$(D^2 + 1)_y = csc\ x$$

$$D^2 + 1 = 0 \quad \rightarrow \quad m^2 + 1 = 0 \quad \rightarrow \quad m = \pm i$$

$$C.F. = A\cos x + B\sin x$$

$$y_1 = \cos x \quad \rightarrow \quad \overline{y_1} = -\sin x$$

$$y_2 = \sin x \quad \rightarrow \quad \overline{y_2} = \cos x$$

$$P.I. = u\, y_1 + v\, y_2$$

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$$u = \int \frac{-y_2 \,.\, x}{y_1 \,.\, \overline{y_2} - \overline{y_1} \,.\, y_2}\, dx$$

$$u = -\int \frac{\sin x \,.\, \csc x}{\cos x \,.\, \cos x + \sin x \,.\, \sin x}\, dx$$

$$\cos^2 x + \sin^2 = 1$$

$$= -\int \frac{\sin x \,.\, \frac{1}{\sin x}}{\cos^2 x + \sin^2 x}\, dx = -\int dx = -x$$

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$$v = \int \frac{y_1 \,.\, x}{y_1 \,.\, \overline{y_2} - \overline{y_1} \,.\, y_2}\, dx$$

$$v = \int \frac{\cos x \,.\, \csc x}{\cos x \,.\, \cos x + \sin x \,.\, \sin x}\, dx$$

$$= \int \frac{\cos x \,.\, \frac{1}{\sin x}}{\cos^2 x + \sin^2 x}\, dx = \int \frac{\cot x}{1} dx = \ln \sin x$$

$$P.I. = -x \,.\, \cos x + \ln \sin x \,.\, \sin x$$

$$y = A \cos x + B \sin x - x \,.\, \cos x \,.\, \sin x . \ln \sin x$$

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Example 21: Solve $\frac{d^2y}{dx^2} - 4y = e^{2x}$

$$(D^2 - 4)_y = e^{2x}$$

$$D^2 - 4 = 0 \quad \rightarrow \quad m^2 - 4 = 0 \quad \rightarrow \quad m = \pm 2$$

$$C.F. = c_1\, e^{2x} + c_2\, e^{-2x}$$

$$y_1 = e^{2x} \quad \rightarrow \quad \overline{y_1} = 2e^{2x}$$

$$y_2 = e^{-2x} \quad \rightarrow \quad \overline{y_2} = -2e^{-2x}$$

$$P.I. = u\, y_1 + v\, y_2$$

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$$u = \int \frac{-y_2 \cdot f(x)}{y_1 \cdot \overline{y_2} - \overline{y_1} \cdot y_2} \, dx$$

$$u = \int \frac{-e^{-2x} \cdot e^{2x}}{e^{2x} \cdot (-2e^{-2x}) - 2e^{2x} \cdot e^{-2x}} \, dx$$

$$= \int \frac{-1}{-2-2} \, dx = \frac{1}{4} \int dx = \frac{x}{4}$$

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$$v = \int \frac{y_1 \, . \, f(x)}{y_1 \, . \, \overline{y_2} - \overline{y_1} \, . \, y_2} \, dx$$

$$v = \int \frac{e^{2x} \, . \, e^{2x}}{e^{2x} \, . \, (-2e^{-2x}) - 2e^{2x} \, . \, e^{-2x}} \, dx = \int \frac{e^{4x}}{-2-2} \, dx = -\frac{1}{4} \int e^{4x} dx \qquad = - \frac{e^{4x}}{16}$$

$$P.I. = \frac{x}{4} \, . \, e^{2x} - \frac{e^{4x}}{16} \, . \, e^{-2x} = \frac{x}{4} \, . \, e^{2x} - \frac{e^{2x}}{16}$$

$$y = c_1 \, e^{2x} + c_2 \, e^{-2x} + \frac{x}{4} \, . \, e^{2x} - \frac{e^{2x}}{16}$$

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Example 22 : Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$$(D^2 - 1)\, y = \frac{2}{1 + e^x}$$

$$D^2 - 1 = 0 \quad \rightarrow \quad m^2 - 1 = 0 \quad \rightarrow \quad m = \pm 1$$

$$C.F. = A\, e^x + B\, e^{-x}$$

$$y_1 = e^x \quad \rightarrow \quad \overline{y_1} = e^x$$

$$y_2 = e^{-x} \quad \rightarrow \quad \overline{y_2} = -e^{-x}$$

$$P.I. = u\, y_1 + v\, y_2$$

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$$u = \int \frac{-y_2 \,.\, x}{y_1 \,.\, \overline{y_2} - \overline{y_1} \,.\, y_2} \, dx$$

$$u = \int \frac{-e^{-x} \,.\, \frac{2}{1+e^{x}}}{-e^{x} \,.\, e^{-x} - e^{x} \,.\, e^{-x}} \, dx$$

$$= \int \frac{-e^{-x}}{-2} \,.\, \frac{2}{1+e^{x}} \, dx = \int \frac{e^{-x}}{1+e^{x}} dx = \int \frac{e^{x} \,.\, e^{-x}}{e^{x}\,(1+e^{x})} dx = \int \frac{1}{e^{x}(1+e^{x})} dx$$

$$= \int \left(\frac{1}{e^{x}} - \frac{1}{1+e^{x}}\right) dx = \int e^{-x}\, dx - \int \frac{e^{-x}}{e^{-x}+1} dx = -e^{-x} + ln\,(e^{-x}+1)$$

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$$v = \int \frac{y_1 \,.\, x}{y_1 \,.\, \overline{y_2} - \overline{y_1} \,.\, y_2}\, dx$$

$$v = \int \frac{e^x \,.\, \frac{2}{1+e^x}}{-e^x \,.\, e^{-x} - e^x \,.\, e^{-x}}\, dx = \int \frac{e^x}{-2} \,.\, \frac{2}{1+e^x}\, dx = -\int \frac{e^x}{1+e^x} dx$$

$$= -ln(1+e^x)$$

$$P.I. = [-e^{-x} + ln(1+e^{-x})]e^x - e^{-x} \,.\, ln(1+e^x)$$

$$= -1 + e^x \,.\, ln(1+e^{-x}) - e^{-x} \,.\, ln(1+e^x)$$

$$y = A\, e^x + B\, e^{-x} - 1 + e^x \,.\, ln(1+e^{-x}) - e^{-x} \,.\, ln(1+e^x)$$

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