



Simultaneous Differential Equations

المعادلات التفاضلية الانية

If two or more dependent variables are a functions of a single independent variable, the equations involving their derivative are called simultaneous equations,

وهي المعادلات التي تتضمن اكثر من متغير اساسي مثل x, y وتحتوي على متغير غير اساسي واحد. كما موضح في المعادلات التالية:

$$\frac{dx}{dt} + 4y = t ;$$

$$\frac{dy}{dt} + 2x = e^t$$

حيث ان كل من x, y متغير معتمد بينما تمثل t متغير مستقل.

The method of solving these equations is based on process of elimination.

ولحل هذا النوع من المعادلات نعتمد على طريقة الحذف و الاختزال.



Example 23: Solve $\frac{dx}{dt} = -wy$, $\frac{dy}{dt} = wx$

On putting $\frac{d}{dt} = D$ in the equations, we have

$Dx + wy = 0$ (1) $\times w$

$Dy - wx = 0$ (2) $\times D$

$w Dx + w^2 y = 0$ (3)

$-w Dx + D^2 y = 0$ (4)

----- بالجمع

$w^2 y + D^2 y = 0 \rightarrow (w^2 + D^2) y = 0$ (5)



$$D^2 + w^2 = 0 \quad \rightarrow \quad D^2 = -w^2 \quad \rightarrow \quad D = \pm wi$$

$$y = A \cos wt + B \sin wt \dots \dots \dots (6)$$

$$\frac{dy}{dt} = Dy = -Aw \sin wt + Bw \cos wt$$

On putting the value of Dy in (2), we get $-wx - Aw \sin wt + Bw \cos wt = 0$

$$-Aw \sin wt + Bw \cos wt = wx$$

$$x = -A \sin wt + B \cos wt \dots \dots \dots (7)$$

On squaring (6) and (7) and adding, we get
 $x^2 + y^2 = A^2 (\cos^2 wt + \sin^2 wt) + B^2 (\cos^2 wt + \sin^2 wt)$
 $x^2 + y^2 = A^2 + B^2 \dots \dots \dots$ This is equation of circle



Example 24 : Solve the following simultaneous equations

$$\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 0 ; \quad \frac{dy}{dt} + 5x + 3y = 0$$

On putting $\frac{d}{dt} = D$ in the equations, we have

$$D_x + D_y + 2x + y = 0$$

$$D_y + 5x + 3y = 0$$

$$(D+2) x + (D+1) y = 0 \dots\dots\dots(1)$$

$$5 x + (D+3) y = 0 \dots\dots\dots(2)$$

Multiplying (1) by 5 and (2) by (D+2), we get

$$(D + 2) x + (D + 1)y = 0 \quad \times 5$$

$$5x + (D + 3) y = 0 \quad \times (D + 2)$$



$$5(D + 2)x + 5(D + 1)y = 0 \dots\dots\dots(3)$$

$$5(D + 2)x + (D + 2)(D + 3)y = 0 \dots\dots\dots(4)$$

----- subtracting (3) from (4) **بالطرح**
 $(D + 2)(D + 3)y - 5(D + 1)y = 0$

$$(D^2 + 5D + 6 - 5D - 5)y = 0$$

$$(D^2 + 1)y = 0 \rightarrow D^2 + 1 = 0 \rightarrow D^2 = -1 \rightarrow D = \pm i$$

$$y = A \cos t + B \sin t$$

$$\frac{dy}{dt} = Dy = -A \sin t + B \cos t$$



Putting the value of Dy and y in (2), we get

نعوض عن قيمة y و Dy في المعادلة رقم (2)

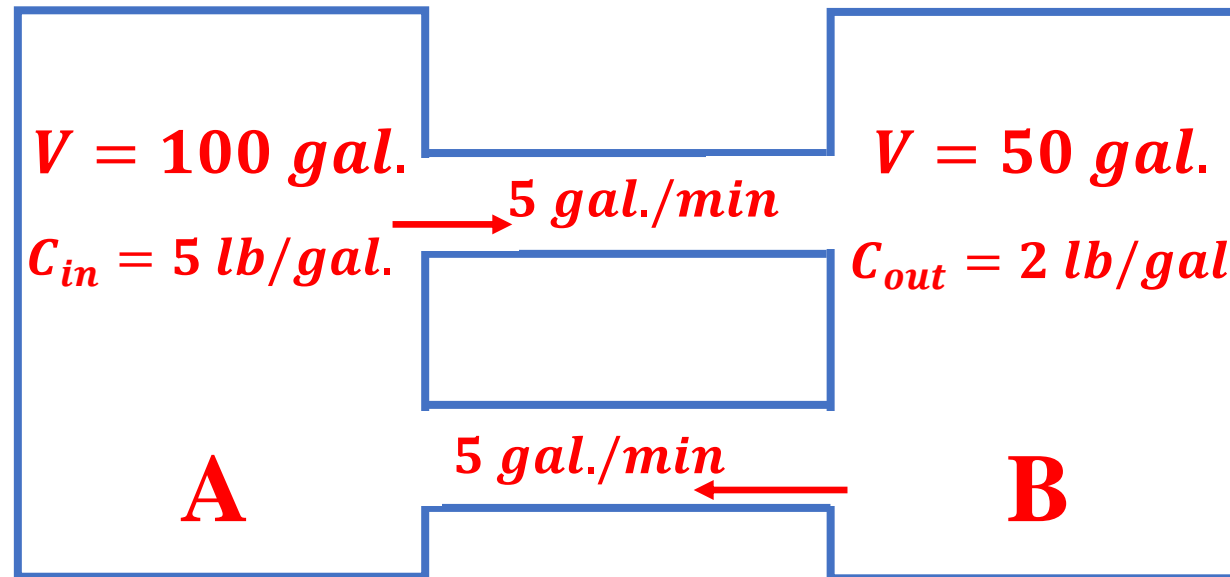
$$5x - A \sin t + B \cos t + 3A \cos t + 3B \sin t = 0$$

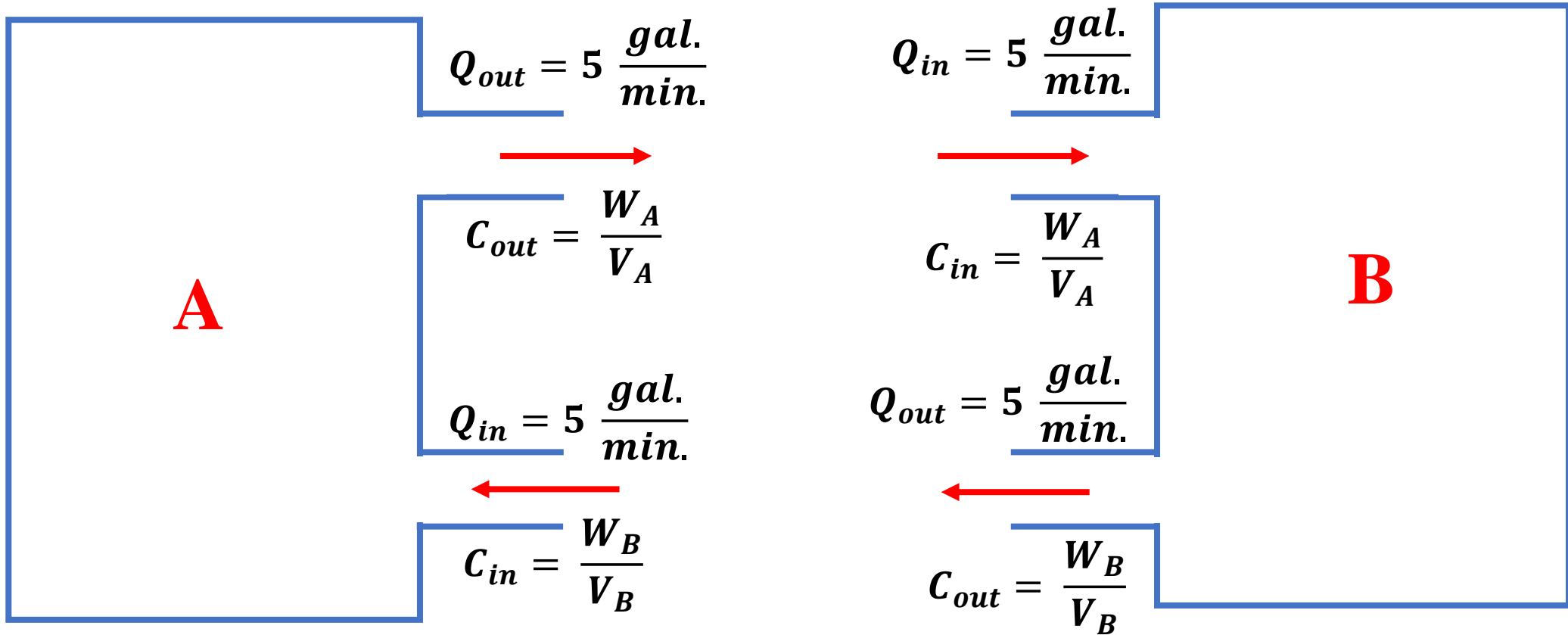
$$5x = A \sin t - B \cos t - 3A \cos t - 3B \sin t$$

$$x = \frac{1}{5} [(A - 3B) \sin t - (3A + B) \cos t]$$

Example 25:

Find amount of a salt in the tank A and tank B with respect to time if after 10 min. the amount of salt in the tank A and tank B equal 100 lb and 60 lb respectively.







$$\frac{dW_A}{dt} = Q_{in} \cdot C_{in} - Q_{out} \cdot C_{out}$$

$$\frac{dW_A}{dt} = 5 \cdot \frac{W_B}{50} - 5 \cdot \frac{W_A}{100}$$

$x \rightarrow (W_A)$ كمية الملح في الخزان A

$y \rightarrow (W_B)$ كمية الملح في الخزان B

$$\frac{dx}{dt} = 5 \cdot \frac{y}{50} - 5 \cdot \frac{x}{100}$$

$$\frac{dx}{dt} = \frac{y}{10} - \frac{x}{20} \dots \dots \dots (1)$$





$$\frac{dW_B}{dt} = Q_{in} \cdot C_{in} - Q_{out} \cdot C_{out}$$

$$\frac{dW_B}{dt} = 5 \cdot \frac{W_A}{100} - 5 \cdot \frac{W_B}{50}$$

$$\frac{dy}{dt} = \frac{x}{20} - \frac{y}{10} \dots\dots\dots (2)$$

$$Dx = \frac{y}{10} - \frac{x}{20} \rightarrow 20 Dx = 2y - x$$

$$(20D + 1)x - 2y = 0 \dots\dots\dots (3)$$



$$Dy = \frac{x}{20} - \frac{y}{10} \quad \rightarrow \quad 20 D_y = x - 2y$$

$$(20D + 2)y - x = 0 \quad \dots \dots \dots (4)$$

$$(20D + 1)x - 2y = 0$$

$$(20D + 2)y - x = 0 \quad \times (20D + 1)$$

$$~~(20D + 1)x - 2y = 0~~$$

$$-\cancel{(20D + 1)x} + (20D + 2)(20D + 1)y = 0$$

بالجمع

$$(20D + 2)(20D + 1)y - 2y = 0$$



$$(400 D^2 + 60D + 2 - 2)y = 0 \quad \rightarrow \quad (400D^2 + 60D)y = 0$$

$$400m^2 + 60m = 0 \quad \rightarrow \quad m(400m + 60) = 0$$

$$m_1 = 0 \quad , \quad m_2 = \frac{-60}{400} = -0.15$$

$$y = c_1 + c_2 e^{-0.15 t}$$

$$\text{at } t = 0 \quad , \quad y = W_B = C_{out} \times V_B = 2 \times 50 = 100$$

$$100 = c_1 + c_2 e^0 \quad \rightarrow \quad c_1 = 100 - c_2$$

$$y = 100 - c_2 + c_2 e^{-0.15 t}$$



$$\text{at } t = 10, \quad y = 60$$

$$60 = 100 - c_2 + c_2 e^{-0.15(10)} \quad \rightarrow \quad c_2 = 51.5$$

$$c_1 = 100 - 51.5 = 48.5$$

$$y = 48.5 + 51.5 e^{-0.15 t},$$

$$W_B = 48.5 + 51.5 e^{-0.15 t}$$

$$\frac{dy}{dt} = -7.725 e^{-0.15 t}$$



بالتعويض في المعادلة رقم (2)

$$-7.725 e^{-0.15 t} = \frac{x}{20} - \left(\frac{48.5 + 51.5 e^{-0.15 t}}{10} \right)$$

$$-154.5 e^{-0.15 t} = x - 97 - 103 e^{-0.15 t}$$

$$x = 97 - 51.5 e^{-0.15 t}$$

$$W_A = 97 - 51.5 e^{-0.15 t}$$