The Derivative : is the slope and we can define as :

$$f(x)=\frac{dy}{dx}=m=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Where " m " is the slope , if this derivative exists then we say f is " differentiable " .

EXAM : Find to the function $\frac{dy}{dx}$ $f(x)=\sqrt{x}$

Solution :

$$\frac{dy}{dx}=f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\bullet\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$=\lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

EXAM : Find $\frac{dy}{dx}$ to the function

1) $f(x)=x^2$

Solution :

$$f(x+\Delta x)=(x+\Delta x)^2=x^2+2x\Delta x+\Delta^2 x$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+\Delta^2 x-x^2}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta^2 x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x[2x+\Delta x]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}2x+\Delta x=2x+0=2x$$

2) $f(x)=\frac{1}{x}$

Solution :

$$f(x+\Delta x)=\frac{1}{x+\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{x-x-\Delta x}{x(x+\Delta x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)}=\frac{-1}{x^2}$$

RULES OF DERIVATIVE :

1) $y=f(x)=C \Rightarrow y'=0$

2) $y=f(x)=x \Rightarrow y'=1$

3) $y=f(x)=x^n \Rightarrow y'=nx^{n-1}$

4) $y=f(x)=Cg(x) \Rightarrow y'=Cg'(x)$

5) $y=f(x)\pm g(x) \Rightarrow y'=f'(x)\pm g'(x)$

6) $y=f(x)\bullet g(x) \Rightarrow y'=f(x)g'(x)+g(x)f'(x)$

7) $y=\frac{f(x)}{g(x)} \Rightarrow y'=\frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$

8) $y=f(x)=C(g(x))^n \Rightarrow y'=C(g(x))^{n-1}\bullet g'(x)$

Proof :

1) $f(x)=C \Rightarrow f(x+\Delta x)=C$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{C-C}{\Delta x}=0$$

2) $f(x)=x \Rightarrow f(x+\Delta x)=x+\Delta x$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

NOTE

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}y^3+\cdots+y^n$$

3) $f(x) = x^n$

$$f(x+\Delta x) = (x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta^2 x + \ldots + \Delta^n x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{x^n} + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta^2 x + \ldots + \Delta^n x - \cancel{x^n}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\,[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}\Delta^2 x + \ldots + \Delta^{n-1} x]}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}\Delta^2 x + \ldots + \Delta^{n-1} x$$

$$= nx^{n-1} + 0 + 0 + \ldots + 0 = nx^{n-1}$$

EXAM : Find $\frac{dy}{dx}$ to the function

1) $y = x^6 \Rightarrow y' = 6x^5$

2) $y = 3x^4 \Rightarrow y' = 3.4x^3 = 12x^3$

3) $y = 3x^4 + 2x^2 - 7x + 10 \Rightarrow y' = 12x^3 + 4x - 7$

4) $y = (x+3)(x^2+2) \Rightarrow y' = (x+3)(2x) + (x^2+2).1$

5) $y = \frac{(x+3)}{(x^2+2)} \Rightarrow y' = \frac{(x^2+2).1 - (x+3)(2x)}{(x^2+2)^2}$

6) $y = (3x^2+4x)^5 \Rightarrow y' = 5(3x^2+4x)^4(6x+4)$

7) $y = (3x^2+4x)^5(x^3+7)^3 \Rightarrow y' = (3x^2+4x)^5.3(x^3+7)^2.(3x^2) +$

$+ (x^3+7)^3 5(3x^2+4x)^4(6x+4)$

IMPLICIT DIFFERENTIATION الاشتقاق الضمني

نستخدم الاشتقاق الضمني في حالة إعطاء الدالة بمتغيرين او أكثر ، حيث نشتق المتغير المستقل(Independent) اشتقاقاً صريحاً ، اما المتغير المعتمد(Dependent) نشتقه اشتقاقاً ضمنياً ، حيث أن المتغير المعتمد هو المتغير الذي يعتمد على متغيرات اخرى اما المتغير المستقل فهو المتغير الذي يُشتق بالنسبة له . مثل :

$\frac{dy}{dx}$ y (dependent) & x (independent)

$\frac{dz}{dy}$ z (dependent) & y (independent)

EXAM :

If $x^2 + y^2 = 9$, *Find* $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

Solution :

$$2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow \boxed{y' = -\frac{x}{y}}$$

$$y'' = -\frac{(y.1 - xy')}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{\frac{x^2+y^2}{y}}{y^2}$$

$$= -\frac{x^2+y^2}{y^3} = -\frac{9}{y^3}$$

EXAM :

If $2y^2 = x^3$ *, then prove that* $\frac{d^2y}{dx^2} = \frac{3}{8}\frac{x}{y}$

solution :

$$4yy' = 3x^2 \Rightarrow \boxed{y' = \frac{3}{4}\frac{x^2}{y}}$$

$$y'' = \frac{3}{4}\frac{y.2x - x^2.y'}{y^2} = \frac{3}{4}\frac{2yx - \frac{3}{4}\frac{x^4}{y}}{y^2} = \frac{3}{4}\frac{\frac{8xy^2 - 3x^4}{4y}}{y^2}$$

$$= \frac{3}{4}\frac{x(8y^2 - 3x^3)}{4y^3} = \frac{3}{16}\frac{x(8y^2 - 6y^2)}{y^3} \qquad \boxed{\because x^3 = 2y^2}$$

$$= \frac{3}{16}\frac{x(2y^2)}{y^3} = \frac{3}{8}\frac{x}{y}$$

EXAM :

If $y^3 - x^3 = a^3$, *prove that* :

$$\frac{d^2x}{dy^2} = -\frac{2ya^3}{x^5}$$

a is constant.

$$3y^2 - 3x^2x' = 0 \Rightarrow 3y^2 = 3x^2x' \Rightarrow \boxed{x' = \frac{y^2}{x^2}}$$

$$x'' = \frac{x^2.2y - y^2.2xx'}{x^4} = \frac{2yx^2 - .2xy^2\frac{y^2}{x^2}}{x^4} = \frac{2yx^2 - \frac{2y^4}{x}}{x^4}$$

$$= \frac{2yx^3 - 2y^4}{x^5} = \frac{2y(x^3 - y^3)}{x^5} = \frac{2y(-a^3)}{x^5} = \frac{-2ya^3}{x^5}$$

CHAIN RULE قاعدة السلسلة

Let y=f(t) and x=g(t) , the chain rule may be written as :

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

EXAM :

$If\ \ y = t^3 + 6 \quad , \quad x = 2t + 4$

$Find \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}$

Solution :

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dt} = 3t^2 \quad , \frac{dx}{dt} = 2 \Rightarrow \frac{dt}{dx} = \frac{1}{2}$$

$$\frac{dy}{dx} = 3t^2.\frac{1}{2} \quad = \boxed{\frac{3}{2}t^2} \Leftarrow A$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}A = \frac{dA}{dx} = \frac{dA}{dt}.\frac{dt}{dx}$$

$$= 3t.\frac{1}{2} = \frac{3t}{2}$$

EXAM :

$If\ \ y = t^2 + 1 \quad , \quad x = t^2 - 1$

$Find \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}$

Solution :

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\frac{dy}{dt} = 2t \quad , \frac{dx}{dt} = 2t \Rightarrow \frac{dt}{dx} = \frac{1}{2t}$

$\frac{dy}{dx} = 2t \cdot \frac{1}{2t} \quad = 1 \Leftarrow A$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}A = \frac{dA}{dx} = \frac{dA}{dt} \cdot \frac{dt}{dx}$

$= 0 \cdot \frac{1}{2t} = 0$

EXAM :

$If\ \ y = Cost \quad , \quad x = \sin t$

$Find \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}$

Solution :

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$\frac{dy}{dt} = -\sin t \quad , \frac{dx}{dt} = Cost \Rightarrow \frac{dt}{dx} = \frac{1}{Cost}$

$\frac{dy}{dx} = -\sin t \cdot \frac{1}{Cost} \quad = \boxed{-\tan t} \Leftarrow A$

$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}A = \frac{dA}{dx} = \frac{dA}{dt} \cdot \frac{dt}{dx}$

$= -Sec^2 t \cdot \frac{1}{Cost} = -Sec^3 t$

Partial Derivative :

DEF:

Let f be a function with two variables x, y , then the partial derivative for f respect to x is the function f_x or $\frac{\partial f}{\partial x}$ and its value at any point (x,y) in domain f is :

$$f_x(x,y)=\frac{\partial}{\partial x}f(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

DEF:

Let f be a function with two variables x, y , then the partial derivative for f respect to x is the function f_x or $\frac{\partial f}{\partial y}$ and its value at any point (x,y) in domain f is :

$$f_y(x,y)=\frac{\partial}{\partial y}f(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

DEF:

Let f be a function with three variables x, y ,z then the partial derivative for f :

$$f_x(x,y,z)=\frac{\partial}{\partial x}f(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}$$

$$f_y(x,y,z)=\frac{\partial}{\partial y}f(x,y,z)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y}$$

$$f_z(x,y,z)=\frac{\partial}{\partial z}f(x,y,z)=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}$$

Note :

$$\frac{\partial^2 f}{\partial x^2}=f_{xx} \qquad \frac{\partial^2 f}{\partial xy}=f_{xy}$$

$$\frac{\partial^2 f}{\partial yx}=f_{yx} \qquad \frac{\partial^2 f}{\partial y^2}=f_{yy}$$

$$f_{xy}=f_{yx}$$

and in three variables

$$\frac{\partial^3 f}{\partial x\,\partial y^2}=\frac{\partial}{\partial x}(\frac{\partial^2 f}{\partial y^2})=f_{xyy}\cdots\cdots\cdots etc.$$

EXAM :

$$f(x,y)=x^2-5xy+y^3$$

$$\frac{\partial f}{\partial x}=2x-5y$$

$$\frac{\partial f}{\partial y}=0-5x+3y^2=-5x+3y^2$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})=6y$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})=2$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=-5=\frac{\partial^2 f}{\partial y\,\partial x}$$

EXAM :

$$f(x,y)=(3x+5y)^5$$

$$\frac{\partial f}{\partial x}=5(3x+5y)^4.3=15(3x+5y)^4$$

$$\frac{\partial f}{\partial y}=5(3x+5y)^4.5=25(3x+5y)^4$$

$$\frac{\partial^2 f}{\partial x^2}=60(3x+5y)^3.3=180(3x+5y)^3$$

$$\frac{\partial^2 f}{\partial y^2}=100(3x+5y)^3.5=500(3x+5y)^3$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=60(3x+5y)^3.5=300(3x+5y)^3=\frac{\partial^2 f}{\partial y\,\partial x}$$

EXAM :

$$f(x,y)=xy^3-5x^2yz^4$$

$$\frac{\partial f}{\partial x}=y^3-10xyz^4$$

$$\frac{\partial f}{\partial y}=3y^2x-5x^2z^4$$

$$\frac{\partial f}{\partial z}=-20x^2yz^3$$

$$\frac{\partial^2 f}{\partial x^2}=-10yz^4$$

$$\frac{\partial^2 f}{\partial y^2}=6xy$$

$$\frac{\partial^2 f}{\partial z^2}=-60x^2yz^2$$

$$\frac{\partial^3 f}{\partial x^3}=0$$

$$\frac{\partial^3 f}{\partial y^3}=6x$$

$$\frac{\partial^3 f}{\partial z^3}=-120x^2yz$$

$$\frac{\partial^2 f}{\partial x\,\partial y}=3y^2-10xz^4$$

$$\frac{\partial^2 f}{\partial x\,\partial z}=-40xyz^3$$

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial z}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}(\frac{\partial f}{\partial z}))$$

$$\frac{\partial^3 f}{\partial z\,\partial y^2}=\frac{\partial}{\partial z}(\frac{\partial^2 f}{\partial y^2})=0$$

DEF :

Let u(x,y) , v(x,y) be a functions then we say that u & v satisfy the Cauchy Riemann Equation (C.R.E) iff

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \& \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

EXAM :

$$u=x^2-y^2 \quad , \quad v=2xy$$

$$\frac{\partial u}{\partial x}=2x \quad , \quad \frac{\partial v}{\partial y}=2x \quad \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=-2y \quad , \quad \frac{\partial v}{\partial x}=2y \quad \Rightarrow \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\therefore \quad u \,\&\, v \quad satisfy\ C.R.E$$

DEF :

When the function f(x,y) satisfy Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

then f(x,y) is said to be harmonic .

EXAM :

$$f(x,y)=x^3-3xy^2$$

$$\frac{\partial f}{\partial x}=3x^2-3y^2 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2}=6x$$

$$\frac{\partial f}{\partial y}=-6xy \quad \Rightarrow \quad \frac{\partial^2 f}{\partial y^2}=-6x$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=6x-6x=0$$

$$\therefore \quad f \quad harmonic$$

HOME WORK

1 By using the definition of the derivative find to $f'(x)$

$3x^3$, x^2+1 , x^5

$\frac{1}{x+1}$, $\frac{1}{x^2}$, $\sqrt{x+1}$

2

Find $\frac{d^2y}{dx^2}$ & $\frac{d^2x}{dy^2}$ *to the* :

$x^2y^2 - y = 7$

$2x^2y - 4y^3 + 1 = 0$

$x^{2/3} + y^{2/3} = a^{2/3}$, *a is cons.*

$x^{1/3} + y^{1/3} = a^{1/3}$, *a is cons.*

$x^3 - 4y^2 + 3 = 0$

3

If $y = t^3 + 3t^2 + 1$

$x = 4t^4 + 2t^2$

then Find $\frac{d^2y}{dx^2}$

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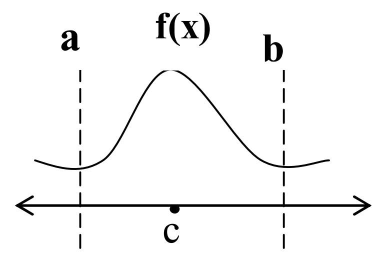
2013 -2014

PROPERTIES OF INTEGRAL : -



$$\int_a^b dx = b - a \Rightarrow \int_a^b c.dx = c(b-a)$$

$$\int_a^b c.f(x)dx = c.\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \quad ,c \in [a,b]$$

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C \quad ,n \neq -1$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$if \;\; f(x) \geq 0 \Rightarrow \int_a^b f(x)dx \geq 0$$

$$if \;\; f(x) \leq g(x) \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad ,n \neq -1$$

EXAM : -

$$\int_6^{10} dx = 10 - 6 = 4$$

$$\int_6^{10} 5.dx = 5(10-6) = 5.4 = 20$$

$$\int_1^3 x.dx = \left.\frac{x^2}{2}\right|_1^3 = \frac{1}{2}(9-1) = 4$$

$$\int_2^5 (x+5)dx = \frac{x^2}{2} + 5x \Big|_2^5 = 25.5$$

$$\int_0^1 (x^2+3x)dx = \frac{x^3}{3} + \frac{3}{2}x^2 \Big|_0^1 = \frac{11}{6}$$

$$\int_1^3 (2x^2+5x)^2(4x+5)dx = \frac{(2x^2+5x)^3}{3}\Big|_1^3 = ?$$

$$\int_1^3 (2x^2+6x)^2(2x+3)dx = \frac{1}{2}\int_1^3 (2x^2+6x)^2(4x+6)dx =$$

$$= \frac{(2x^2+6x)^3}{3}\Big|_1^3 = ?$$

$$\int_0^1 (x^3+1)^2 dx = \int_0^1 (x^6+2x^3+1)dx = \frac{x^7}{7} + \frac{x^4}{2} + x\Big|_0^1 = ?$$

EQUATION OF STRAIGHT LINE : معادلـــة الخط المستـقيـم

The general form of straight line equation is :-

$$ax + by + c = 0$$

OR

$$y = mx + b$$

EXAM : - Find the equation of the curve whose slope at any point p(x,y) is 2x+1 and passing through the point (1,3) .

$$m = \frac{dy}{dx} = 2x + 1$$

$$y = \int \frac{dy}{dx}dx = \int (2x+1)dx \Rightarrow \boxed{y = x^2 + x + C} \Leftarrow \textit{general curves}$$

$$(1,3) \in \textit{curves} \Rightarrow 3 = 1^2 + 1 + C \Rightarrow C = 1$$

$$\boxed{y = x^2 + x + 1} \Leftarrow \textit{special curve}$$

EXAM : - Find the equation of the curve whose slope at any point p(x,y) is

$$4x^3 + 18x^2 + 8x + 3$$

and passing through the point (1,11) .

$$m = \frac{dy}{dx} = 4x^3 + 18x^2 + 8x + 3$$

$$y = \int (4x^3 + 18x^2 + 8x + 3)dx$$

$$y = x^4 + 6x^3 + 4x^2 + 3x + C$$

$$11 = 1 + 6 + 4 + 3 + C \Rightarrow C = -3$$

$$y = x^4 + 6x^3 + 4x^2 + 3x - 3$$

DOUBLE INTEGRATION :

EXAM :

$$\int_0^1 \int_0^x (3 - x - y)dydx$$

$$= \int_0^1 \left. 3y - xy - \frac{1}{2}y^2 \right|_0^x dx = \int_0^1 (3x - x^2 - \frac{1}{2}x^2)dx = \int_0^1 (3x - \frac{3}{2}x^2)dx$$

$$= \left. \frac{3}{2}x^2 - \frac{1}{2}x^3 \right|_0^1 = 1$$

EXAM :

$$\int_1^2 \int_y^{y^2} dxdy$$

$$= \int_1^2 x \Big|_y^{y^2} dy = \int_1^2 (y^2 - y)dy = \left. \frac{1}{3}y^3 - \frac{1}{2}y^2 \right|_1^2 = \frac{5}{6}$$

EXAM :

$$\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} ydxdy$$

$$= \int_0^{\sqrt{2}} yx \Big|_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dx = \int_0^{\sqrt{2}} y\sqrt{4 - 2y^2} + y\sqrt{4 - 2y^2} dx = 2\int_0^{\sqrt{2}} y\sqrt{4 - 2y^2} dx$$

$$\left. \frac{2}{-4} \frac{(4 - 2y^2)^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^{\sqrt{2}} = \frac{8}{3}$$

TRIPLE INTEGRATION :

EXAM :

$$\int_0^1\int_0^x\int_{-y^2}^{x^2} (x+1)dzdydx$$

$$=\int_0^1\int_0^x (x+1)z\Big|_{-y^2}^{x^2} dydx = \int_0^1\int_0^x (x+1)x^2+(x+1)y^2\, dydx$$

$$=\int_0^1\int_0^x x^3+x^2+xy^2+y^2\, dydx = \int_0^1 x^3y+x^2y+\frac{1}{3}xy^3+\frac{1}{3}y^3\Big|_0^x dx$$

$$=\int_0^1 x^4+x^3+\frac{1}{3}x^4+\frac{1}{3}x^3\, dx = \int_0^1 \frac{4}{3}x^4+\frac{4}{3}x^3\, dx = \frac{4}{15}x^5+\frac{1}{3}x^4\Big|_0^1 = \frac{9}{15}$$

EXAM :

$$\int_0^1\int_0^x\int_{-y^2}^{x^2} dzdydx$$

$$=\int_0^1\int_0^x z\Big|_{-y^2}^{x^2} dydx = \int_0^1\int_0^x x^2+y^2\, dydx$$

$$=\int_0^1 x^2y+\frac{1}{3}y^3\Big|_0^x dx = \int_0^1 x^3+\frac{1}{3}x^3\, dx$$

$$=\int_0^1 \frac{4}{3}x^3\, dx = \frac{1}{3}x^4\Big|_0^1 = \frac{1}{3}$$

$$=\int_0^1 x^4+x^3+\frac{1}{3}x^4+\frac{1}{3}x^3\, dx = \int_0^1 \frac{4}{3}x^4+\frac{4}{3}x^3\, dx = \frac{4}{15}x^5+\frac{1}{3}x^4\Big|_0^1 = \frac{9}{15}$$

HOME WORK

1) Find the equation of the curve whose slope at any point p(x,y) is

$m = (x^4 + 16x + 4)^2 (x^3 + 4)$

and passing through the point (2,1) .

2) Find the equation of the curve whose slope at any point p(x,y) is

$m = x(x + 5)^2$

and passing through the point (2,1) .

3) Find :

$$\int_0^1 \int_0^3 x \sqrt{x^2 + y}\, dy dx$$

$$\int_0^1 \int_{-1}^{\sqrt{y}} y\, dx dy$$

$$\int_0^1 \int_x^{x^2} \int_{x-y}^{x+y} (x + 2y + 4z)\, dz dy dx$$

$$\int_0^1 \int_0^2 \int_0^3 (z^3 y^2 x)\, dx dy dz$$

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2013 -2014

1 Area

a) Area between f(x) and the axis

$$A=\int_a^b f(x)dx=\int_a^b y\,dx \qquad , \quad area\ respected\ to\ x-axis$$

$$A=\int_c^d g(y)dy=\int_a^b x\,dy \qquad , \quad area\ respected\ to\ y-axis$$

b) Area between two curves

$y_1=f(x_1)$, $y_2=f(x_2)$, $x_1=g(y_1)$, $x_2=g(y_2)$

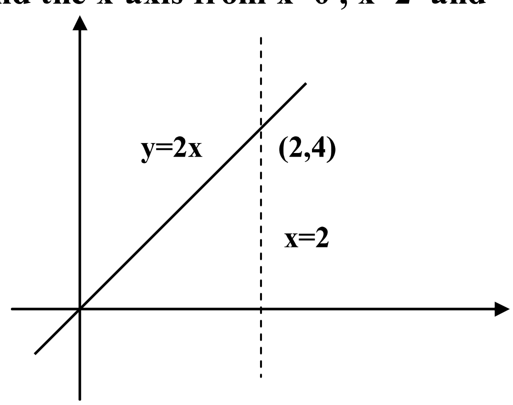
$$A=\int_a^b (y_1-y_2)dx \quad ,x-axis \quad ,y_1\ge y_2$$

$$A=\int_c^d (x_1-x_2)dy \quad ,y-axis \quad ,x_1\ge x_2$$

ملاحظات هامة :

1) اذا اعطانا دالة وحدود تكامل فالحل يكون مباشر .
2) اذا اعطانا دالتين بدون حدود تكامل فنقاطع الدالتين .
3) اذا اعطانا دالة فقط بدون حدود تكامل ، فالحل يكون بالاعتماد على المحور ، فاذا اعطى المساحة بالنسبة للمحور x نضيف معادلة y=0 واذا اعطى المساحة بالنسبة للمحور y نضيف معادلة x=0 ، ثم نقاطع الدالتين .
4) اذا طلب المساحة بالنسبة للمحور x وحدود التكامل معطاة بالنسبة للمحور y فيجب تحويل حدود التكامل بدلالة x والعكس صحيح .

<u>EXAM</u> :- Find the area bounded by the line y=2x and the x-axis from x=0 , x=2 and check the result by geometrically.



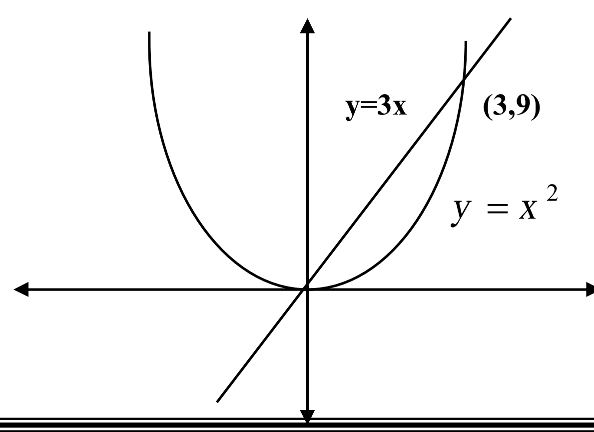
$$A=\int_a^b y\,dx=\int_0^2 2x\,dx=x^2\Big|_0^2=4\ unit^2$$

$$In\ Geometry$$

$$\triangle A=\frac{1}{2}b.h=\frac{1}{2}(2)(4)=4$$

<u>EXAM</u> :- Find the area between the curves :

$y=x^2$ *and the line* $y=3x$



Solution :

$$y=x^2 \quad \ldots\ldots\ldots(1)$$

$$y=3x \quad \ldots\ldots\ldots(2)$$

$$x^2=3x \Rightarrow x^2-3x=0$$

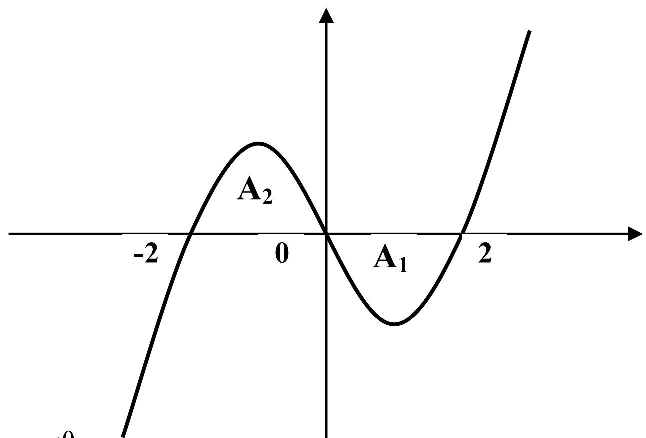
$x(x-3)=0 \Rightarrow x=0 , x=3$

$$A=\int_a^b (y_1-y_2)dx \quad , y_1 \ge y_2$$

$$=\int_0^3 (3x-x^2)dx = \frac{3}{2}x^2-\frac{1}{3}x^3\Big|_0^3 = 4.5\, unit^2$$

EXAM **:- Find the area between the curves :**

$y=x(x^2-4)$ **and the x-axis**



Solution :

$y=x(x^2-4)$

$y=0$

$\therefore x(x^2-4)=0 \Rightarrow x=0, x=-2, x=2$

$A=|A_1|+|A_2|$

$$A_1=\int_{-2}^{0} x(x^2-4)dx = \int_{-2}^{0}(x^3-4x)dx = \frac{1}{4}x^4-2x^2\Big|_{-2}^{0} = 4$$

$$A_2=\int_{0}^{2} x(x^2-4)dx = \int_{0}^{2}(x^3-4x)dx = \frac{1}{4}x^4-2x^2\Big|_{0}^{2} = -4$$

$A=|4|+|-4|=8$

EXAM **:- Find the area between the line x-y=10 and the**

a) x-axis b)y-axis

from x=0 , x=5

$a) \quad A=\int_a^b y\, dx \qquad , \quad x-y=10 \Rightarrow y=x-10$

$$=\int_0^5 (x-10)\,dx = \frac{x^2}{2}-10x\Big|_0^5 = |-37.5| = 37.5$$

$b) \quad A=\int_c^d x\, dy \qquad , \quad x-y=10 \Rightarrow x=y+10$

$$A=\int_{-10}^{-5} (y+10)\,dy \qquad x=0 \to y=-10 \quad \& x=5 \to y=-5$$

$$=\frac{1}{2}y^2+10y\Big|_{-10}^{-5} = ?$$

2 Volumes

DEF: if f(x) ≥ 0 cont. on [a,b] if the area bounded by f(x) and the x-axis from x=a to x=b rotated about the x-axis then :

$$V_x = \int_a^b \pi[f(x)]^2 dx = \int_a^b \pi y^2 dx$$

DEF: if g(y) ≥ 0 cont. on [c,d] if the area bounded by g(y) and the y-axis fromy=c to y=d rotated about the y-axis then :

$$V_y = \int_c^d \pi[g(y)]^2 dy = \int_c^d \pi x^2 dy$$

DEF : if the area bounded between two curves is rotated about x-axis

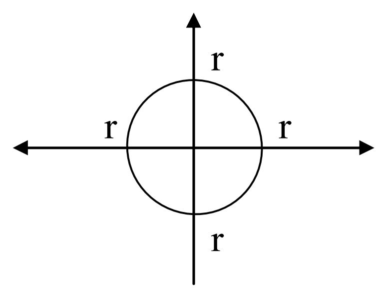
$$V_x = \int_a^b \pi(y_1^2 - y_2^2) dx \quad , y_1 \ge y_2$$

DEF : if the area bounded between two curves is rotated about y-axis

$$V_y = \int_c^d \pi(x_1^2 - x_2^2) dy \quad , x_1 \ge x_2$$

<u>EXAM</u> :- The area bounded by the circle with radius r and center is the origin is rotated about the x-axis , find the volume generated and check the result by geom.

$$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$$



$$V_x = \int_a^b \pi y^2 dx = \int_{-r}^{r} \pi(r^2 - x^2) dx$$

$$= 2\pi \int_0^r (r^2 - x^2) dx = 2\pi(r^2 x - \frac{1}{3}x^3)\Big|_0^r$$

$$= \frac{4}{3}\pi r^3 \ unit^3$$

In Geometry

the volume of sphare is $\frac{4}{3}\pi r^3 \ unit^3$

<u>EXAM</u> :- The area bounded by the function $x^2 y^2 = 1$ is rotated about the y-axis, find the volume generated from y=1 , y=2 .

$$V_y = \pi \int_c^d x^2 dy \qquad , \quad x^2 y^2 = 1 \rightarrow x^2 = y^{-2}$$

$$= \pi \int_c^d y^{-2} dy = \frac{-1}{y}\Big|_1^2 = \frac{1}{2}\pi \ unit^3$$

EXAM :- The area bounded by the line y=x-1
a) is rotated about the x-axis b) is rotated about the y-axis
find the volume generated from those rotation from x=0,1

$$a) V_x = \pi \int_a^b y^2 dx \quad , \quad y = x - 1 \Rightarrow y^2 = (x-1)^2$$

$$= \pi \int_0^1 (x-1)^2 dx = \frac{\pi}{3}(x-1)^3 \Big|_0^1 = \frac{\pi}{3} unit^3$$

$$b) V_y = \pi \int_c^d x^2 dy \quad , \quad y = x - 1 \Rightarrow x = y + 1 \Rightarrow x^2 = (y+1)^2$$

$$x = 0 \to y = -1 \quad \& \, x = 1 \to y = 0$$

$$= \pi \int_{-1}^{0} (y+1)^2 dx = \frac{\pi}{3}(y+1)^3 \Big|_{-1}^{0} = \frac{\pi}{3} unit^3$$

3 Arc Length

If y=f(x) is continous with continous derivative at each point of the curve from (a,f(a)) to (b,f(b)) then :

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx \quad , if \; \frac{dy}{dx} \; is \; cont.$$

$$S = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2} dy \quad , if \; \frac{dy}{dx} \; is \; not \; cont.$$

EXAM :- Find the length of the segment of curve $y = \frac{1}{3}(x^2+2)^{3/2} \quad , from \; x = 0,3$

$$\frac{dy}{dx} = \frac{1}{3}\frac{3}{2}(x^2+2)^{1/2}(2x) = x(x^2+2)^{1/2} \; cont.on[0,3]$$

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^3 \sqrt{1 + (x(x^2+2)^{1/2})^2} dx$$

$$= \int_0^3 \sqrt{1 + x^2(x^2+2)} dx = \int_0^3 \sqrt{x^4 + 2x^2 + 1} dx$$

$$= \int_0^3 \sqrt{(x^2+1)^2} dx = \int_0^3 (x^2+1) dx$$

$$= \frac{x^3}{3} + x \Big|_0^3 = ?$$

EXAM :- Find the length of the segment of curve $y = x^{2/3}$, from $x = -1, 8$

$$\frac{dy}{dx} = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{-1/3}} \quad not\,cont.\,on\,[-1,8]$$

$$\frac{dx}{dy} = \frac{3}{2} x^{1/3} \Rightarrow\Rightarrow x = -1 \to y = 1 \,\&\, x = 8 \to y = 4$$

$$S = \int_1^4 \sqrt{1 + (\frac{3}{2} x^{1/3})^2}\, dy = \int_1^4 \sqrt{1 + \frac{9}{4} x^{2/3}}\, dy$$

$$S = \int_1^4 \sqrt{1 + \frac{9}{4} y}\, dy = \frac{4}{9}\frac{2}{3}(1 + \frac{9}{4} y)^{3/2}\Big|_1^4$$

$$= \frac{8}{27}[(10)^{3/2} - (\frac{13}{4})^{3/2}] = ?$$

HOME WORK

1- Find the area bounded by the line 2x-5y=10 from x=0 , 10 about the x&y – axis.

2- Find the area bounded by the curves :

$y = x^2 \quad , \quad y = \sqrt{x} \quad , x = 1,2 \quad , x - axis$

$y = \sec^2 x \quad , \quad y = x \quad , x = -45,45 \quad , x - axis$

$y = x^2 = y \quad , \quad x = y - 2 \quad , x - axis$

$y = x^3 - 2x^2 \quad , \quad y = 2x^2 - 3x \quad , x - axis$

3- Find the volume generated from rotation between curves :

$y = \sec x \quad , \quad y = 0 \quad , x = 45,60 \quad , x - axis$

$y = \sqrt{25 - x^2} \quad , \quad y = 3 \quad , x - axis$

$y = c\sc x \quad , \quad y = 2 \quad , x = 45,60 \quad , x - axis$

$x = 1 - y^2 \quad , \quad x = 2 + y^2 \quad , y = -1,1 \quad , y - axis$

$y = \sin x \quad , \quad y = \cos x \quad , x = 0,45 \quad , x - axis$

$y = \tan x \quad , \quad y = -1 \quad , x = 45,60 \quad , x - axis$

4- Find the horizontal line y=k that divides the area between $y = x^2$ & $y = 9$ **into two equal parts .**

5- Find the vertical line x=k that divides the area between $x = \sqrt{y}$ & $x = 2$ **into two equal parts .**

6- Find the volume of the solid that result when the reigon above the x-axis and below $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , a,b > 0$ **to the ellipse**

7- Let V be the volume of the solid that result when the reigon enclosed by

$y = \frac{1}{x} \quad , y = 0 \quad , x = 2 \quad , x = b \quad (0 < b < 2)$

is revolved about the x-axis , find the value of " b " for which V=3.

8- Find the exact arc length of the curve over the stated interval : -

$y = 3x^{3/2} - 1 \qquad x = 0,1$

$24xy = y^4 + 48 \qquad , y = 2,4$

$x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2} \qquad y = 1,4$

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