8

Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with

$$\int x^{10}\,dx$$

we realize immediately that the derivative of x^{11} will supply an x^{10}: $(x^{11})' = 11x^{10}$. We don't want the "11", but constants are easy to alter, because differentiation "ignores" them in certain circumstances, so

$$\frac{d}{dx}\frac{1}{11}x^{11} = \frac{1}{11}11x^{10} = x^{10}.$$

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

8.1 Substitution

Needless to say, most problems we encounter will not be so simple. Here's a slightly more complicated example: find

$$\int 2x\cos(x^2)\,dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\frac{d}{dx}x^2 = 2x\cos(x^2),$$

so

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Even when the chain rule has "produced" a certain derivative, it is not always easy to see. Consider this problem:

$$\int x^3\sqrt{1-x^2}\,dx.$$

There are two factors in this expression, x^3 and $\sqrt{1-x^2}$, but it is not apparent that the chain rule is involved. Some clever rearrangement reveals that it is:

$$\int x^3\sqrt{1-x^2}\,dx = \int (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\,dx.$$

This looks messy, but we do now have something that looks like the result of the chain rule: the function $1-x^2$ has been substituted into $-(1/2)(1-x)\sqrt{x}$, and the derivative of $1-x^2$, $-2x$, multiplied on the outside. If we can find a function $F(x)$ whose derivative is $-(1/2)(1-x)\sqrt{x}$ we'll be done, since then

$$\begin{aligned}\frac{d}{dx}F(1-x^2) = -2xF'(1-x^2) &= (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2} \\ &= x^3\sqrt{1-x^2}\end{aligned}$$

But this isn't hard:

$$\begin{aligned}\int -\frac{1}{2}(1-x)\sqrt{x}\,dx &= \int -\frac{1}{2}(x^{1/2}-x^{3/2})\,dx \qquad (8.1.1)\\ &= -\frac{1}{2}\left(\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right)+C \\ &= \left(\frac{1}{5}x-\frac{1}{3}\right)x^{3/2}+C.\end{aligned}$$

So finally we have

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2)-\frac{1}{3}\right)(1-x^2)^{3/2}+C.$$

So we succeeded, but it required a clever first step, rewriting the original function so that it looked like the result of using the chain rule. Fortunately, there is a technique that makes such problems simpler, without requiring cleverness to rewrite a function in just the right way. It sometimes does not work, or may require more than one attempt, but the idea is simple: guess at the most likely candidate for the "inside function", then do some algebra to see what this requires the rest of the function to look like.

One frequently good guess is any complicated expression inside a square root, so we start by trying $u = 1-x^2$, using a new variable, u, for convenience in the manipulations that follow. Now we know that the chain rule will multiply by the derivative of this inner function:

$$\frac{du}{dx} = -2x,$$

so we need to rewrite the original function to include this:

$$\int x^3\sqrt{1-x^2} = \int x^3\sqrt{u}\frac{-2x}{-2x}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx.$$

Recall that one benefit of the Leibniz notation is that it often turns out that what looks like ordinary arithmetic gives the correct answer, even if something more complicated is going on. For example, in Leibniz notation the chain rule is

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The same is true of our current expression:

$$\int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,du.$$

Now we're almost there: since $u = 1 - x^2$, $x^2 = 1 - u$ and the integral is

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du.$$

It's no coincidence that this is exactly the integral we computed in (8.1.1), we have simply renamed the variable u to make the calculations less confusing. Just as before:

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du = \left(\frac{1}{5}u - \frac{1}{3}\right)u^{3/2} + C.$$

Then since $u = 1 - x^2$:

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2) - \frac{1}{3}\right)(1-x^2)^{3/2} + C.$$

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let u denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of u, with no x remaining in the expression. If we can integrate this new function of u, then the antiderivative of the original function is obtained by replacing u by the equivalent expression in x.

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

$$\int 2x\cos(x^2)\,dx.$$

Let $u = x^2$, then $du/dx = 2x$ or $du = 2x\,dx$. Since we have exactly $2x\,dx$ in the original integral, we can replace it by du:

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since $du/dx = 2x$, $dx = du/2x$, and then the integral becomes

$$\int 2x\cos(x^2)\,dx = \int 2x\cos u\,\frac{du}{2x} = \int \cos u\,du.$$

The important thing to remember is that you must eliminate all instances of the original variable x.

EXAMPLE 8.1.1 Evaluate $\int (ax+b)^n\,dx$, assuming that a and b are constants, $a \neq 0$, and n is a positive integer. We let $u = ax+b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int (ax+b)^n\,dx = \int \frac{1}{a}u^n\,du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

□

EXAMPLE 8.1.2 Evaluate $\int \sin(ax+b)\,dx$, assuming that a and b are constants and $a \neq 0$. Again we let $u = ax+b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int \sin(ax+b)\,dx = \int \frac{1}{a}\sin u\,du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

□

EXAMPLE 8.1.3 Evaluate $\int_2^4 x\sin(x^2)\,dx$. First we compute the antiderivative, then evaluate the definite integral. Let $u = x^2$ so $du = 2x\,dx$ or $x\,dx = du/2$. Then

$$\int x\sin(x^2)\,dx = \int \frac{1}{2}\sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Now

$$\int_2^4 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

A somewhat neater alternative to this method is to change the original limits to match the variable u. Since $u = x^2$, when $x = 2$, $u = 4$, and when $x = 4$, $u = 16$. So we can do this:

$$\int_2^4 x\sin(x^2)\,dx = \int_4^{16} \frac{1}{2}\sin u\,du = -\frac{1}{2}(\cos u)\Big|_4^{16} = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

An incorrect, and dangerous, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_2^4 \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_2^4 = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

This is incorrect because $\int_2^4 \frac{1}{2}\sin u\,du$ means that u takes on values between 2 and 4, which is wrong. It is dangerous, because it is very easy to get to the point $-\frac{1}{2}\cos(u)\Big|_2^4$ and forget to substitute x^2 back in for u, thus getting the incorrect answer $-\frac{1}{2}\cos(4)+\frac{1}{2}\cos(2)$. A somewhat clumsy, but acceptable, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_{x=2}^{x=4} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_{x=2}^{x=4} = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{\cos(16)}{2}+\frac{\cos(4)}{2}.$$

□

EXAMPLE 8.1.4 Evaluate $\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt$. Let $u=\sin(\pi t)$ so $du=\pi\cos(\pi t)\,dt$ or $du/\pi=\cos(\pi t)\,dt$. We change the limits to $\sin(\pi/4)=\sqrt{2}/2$ and $\sin(\pi/2)=1$. Then

$$\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}\frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}u^{-2}\,du = \frac{1}{\pi}\frac{u^{-1}}{-1}\Big|_{\sqrt{2}/2}^{1} = -\frac{1}{\pi}+\frac{\sqrt{2}}{\pi}.$$

□

Exercises 8.1.

Find the antiderivatives or evaluate the definite integral in each problem.

1. $\int (1-t)^9\,dt \Rightarrow$

2. $\int (x^2+1)^2\,dx \Rightarrow$

3. $\int x(x^2+1)^{100}\,dx \Rightarrow$

4. $\int \frac{1}{\sqrt[3]{1-5t}}\,dt \Rightarrow$

5. $\int \sin^3 x\cos x\,dx \Rightarrow$

6. $\int x\sqrt{100-x^2}\,dx \Rightarrow$

7. $\int \frac{x^2}{\sqrt{1-x^3}}\,dx \Rightarrow$

8. $\int \cos(\pi t)\cos\big(\sin(\pi t)\big)\,dt \Rightarrow$

9. $\int \frac{\sin x}{\cos^3 x}\,dx \Rightarrow$

10. $\int \tan x\,dx \Rightarrow$

11. $\int_0^{\pi} \sin^5(3x)\cos(3x)\,dx \Rightarrow$

12. $\int \sec^2 x\tan x\,dx \Rightarrow$

13. $\int_0^{\sqrt{\pi}/2} x\sec^2(x^2)\tan(x^2)\,dx \Rightarrow$

14. $\int \frac{\sin(\tan x)}{\cos^2 x}\,dx \Rightarrow$

15. $\int_3^4 \frac{1}{(3x-7)^2}\,dx \Rightarrow$

16. $\int_0^{\pi/6} (\cos^2 x-\sin^2 x)\,dx \Rightarrow$

17. $\int \frac{6x}{(x^2-7)^{1/9}}\,dx \Rightarrow$

18. $\int_{-1}^{1} (2x^3-1)(x^4-2x)^6\,dx \Rightarrow$

19. $\int_{-1}^{1} \sin^7 x\,dx \Rightarrow$

20. $\int f(x)f'(x)\,dx \Rightarrow$

8.2 Powers of sine and cosine

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

EXAMPLE 8.2.1 Evaluate $\int \sin^5 x\,dx$. Rewrite the function:

$$\int \sin^5 x\,dx = \int \sin x \sin^4 x\,dx = \int \sin x(\sin^2 x)^2\,dx = \int \sin x(1-\cos^2 x)^2\,dx.$$

Now use $u = \cos x$, $du = -\sin x\,dx$:

$$\begin{aligned}\int \sin x(1-\cos^2 x)^2\,dx &= \int -(1-u^2)^2\,du \\ &= \int -(1-2u^2+u^4)\,du \\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

□

EXAMPLE 8.2.2 Evaluate $\int \sin^6 x\,dx$. Use $\sin^2 x = (1-\cos(2x))/2$ to rewrite the function:

$$\begin{aligned}\int \sin^6 x\,dx = \int (\sin^2 x)^3\,dx &= \int \frac{(1-\cos 2x)^3}{8}\,dx \\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\,dx.\end{aligned}$$

Now we have four integrals to evaluate:

$$\int 1\,dx = x$$

and

$$\int -3\cos 2x\,dx = -\frac{3}{2}\sin 2x$$

are easy. The $\cos^3 2x$ integral is like the previous example:

$$\begin{aligned}\int -\cos^3 2x\,dx &= \int -\cos 2x \cos^2 2x\,dx \\ &= \int -\cos 2x(1-\sin^2 2x)\,dx \\ &= \int -\frac{1}{2}(1-u^2)\,du \\ &= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) \\ &= -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right).\end{aligned}$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right).$$

So at long last we get

$$\int \sin^6 x\,dx = \frac{x}{8} - \frac{3}{16}\sin 2x - \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + \frac{3}{16}\left(x + \frac{\sin 4x}{4}\right) + C.$$

□

EXAMPLE 8.2.3 Evaluate $\int \sin^2 x \cos^2 x\,dx$. Use the formulas $\sin^2 x = (1-\cos(2x))/2$ and $\cos^2 x = (1+\cos(2x))/2$ to get:

$$\int \sin^2 x \cos^2 x\,dx = \int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,dx.$$

The remainder is left as an exercise. □

Exercises 8.2.

Find the antiderivatives.

1. $\int \sin^2 x\,dx \Rightarrow$

2. $\int \sin^3 x\,dx \Rightarrow$

3. $\int \sin^4 x\,dx \Rightarrow$

4. $\int \cos^2 x \sin^3 x\,dx \Rightarrow$

5. $\int \cos^3 x\,dx \Rightarrow$

6. $\int \sin^2 x \cos^2 x\,dx \Rightarrow$

7. $\int \cos^3 x \sin^2 x\,dx \Rightarrow$

8. $\int \sin x(\cos x)^{3/2}\,dx \Rightarrow$

9. $\int \sec^2 x \csc^2 x\,dx \Rightarrow$

10. $\int \tan^3 x \sec x\,dx \Rightarrow$

8.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

EXAMPLE 8.3.1 Evaluate $\int \sqrt{1-x^2}\,dx$. Let $x = \sin u$ so $dx = \cos u\,du$. Then

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\cos u\,du = \int \sqrt{\cos^2 u}\cos u\,du.$$

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x = \sin u$. We could just as well think of this as $u = \arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2 \le u \le \pi/2$, so $\cos u \ge 0$. Then we continue:

$$\begin{aligned}\int \sqrt{\cos^2 u}\cos u\,du &= \int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4} + C\\ &= \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C.\end{aligned}$$

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x\cos x$, we can write $\sin 2u = 2\sin u\cos u = 2\sin(\arcsin x)\sqrt{1-\sin^2 u} = 2x\sqrt{1-\sin^2(\arcsin x)} = 2x\sqrt{1-x^2}$. Then the full antiderivative is

$$\frac{\arcsin x}{2} + \frac{2x\sqrt{1-x^2}}{4} = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

□

This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity $\sin^2 x + \cos^2 x = 1$ in one of three forms:

$$\cos^2 x = 1 - \sin^2 x \qquad \sec^2 x = 1 + \tan^2 x \qquad \tan^2 x = \sec^2 x - 1.$$

If your function contains $1-x^2$, as in the example above, try $x = \sin u$; if it contains $1+x^2$ try $x = \tan u$; and if it contains x^2-1, try $x = \sec u$. Sometimes you will need to try something a bit different to handle constants other than one.

EXAMPLE 8.3.2 Evaluate $\int \sqrt{4-9x^2}\,dx$. We start by rewriting this so that it looks more like the previous example:

$$\int \sqrt{4-9x^2}\,dx = \int \sqrt{4(1-(3x/2)^2)}\,dx = \int 2\sqrt{1-(3x/2)^2}\,dx.$$

Now let $3x/2 = \sin u$ so $(3/2)\,dx = \cos u\,du$ or $dx = (2/3)\cos u\,du$. Then

$$\begin{aligned}
\int 2\sqrt{1-(3x/2)^2}\,dx &= \int 2\sqrt{1-\sin^2 u}\,(2/3)\cos u\,du = \frac{4}{3}\int \cos^2 u\,du \\
&= \frac{4u}{6} + \frac{4\sin 2u}{12} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{2\sin u\cos u}{3} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{2\sin(\arcsin(3x/2))\cos(\arcsin(3x/2))}{3} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{2(3x/2)\sqrt{1-(3x/2)^2}}{3} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{x\sqrt{4-9x^2}}{2} + C,
\end{aligned}$$

using some of the work from example 8.3.1. □

EXAMPLE 8.3.3 Evaluate $\int \sqrt{1+x^2}\,dx$. Let $x = \tan u$, $dx = \sec^2 u\,du$, so

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+\tan^2 u}\sec^2 u\,du = \int \sqrt{\sec^2 u}\sec^2 u\,du.$$

Since $u = \arctan(x)$, $-\pi/2 \le u \le \pi/2$ and $\sec u \ge 0$, so $\sqrt{\sec^2 u} = \sec u$. Then

$$\int \sqrt{\sec^2 u}\sec^2 u\,du = \int \sec^3 u\,du.$$

In problems of this type, two integrals come up frequently: $\int \sec^3 u\,du$ and $\int \sec u\,du$. Both have relatively nice expressions but they are a bit tricky to discover.

First we do $\int \sec u\,du$, which we will need to compute $\displaystyle\int \sec^3 u\,du$:

$$\begin{aligned}\int \sec u\,du &= \int \sec u\,\frac{\sec u+\tan u}{\sec u+\tan u}\,du\\ &= \int \frac{\sec^2 u+\sec u\tan u}{\sec u+\tan u}\,du.\end{aligned}$$

Now let $w = \sec u + \tan u$, $dw = \sec u\tan u + \sec^2 u\,du$, exactly the numerator of the function we are integrating. Thus

$$\begin{aligned}\int \sec u\,du = \int \frac{\sec^2 u+\sec u\tan u}{\sec u+\tan u}\,du = \int \frac{1}{w}\,dw &= \ln|w| + C\\ &= \ln|\sec u+\tan u| + C.\end{aligned}$$

Now for $\displaystyle\int \sec^3 u\,du$:

$$\begin{aligned}\sec^3 u &= \frac{\sec^3 u}{2} + \frac{\sec^3 u}{2} = \frac{\sec^3 u}{2} + \frac{(\tan^2 u+1)\sec u}{2}\\ &= \frac{\sec^3 u}{2} + \frac{\sec u\tan^2 u}{2} + \frac{\sec u}{2} = \frac{\sec^3 u + \sec u\tan^2 u}{2} + \frac{\sec u}{2}.\end{aligned}$$

We already know how to integrate $\sec u$, so we just need the first quotient. This is "simply" a matter of recognizing the product rule in action:

$$\int \sec^3 u + \sec u\tan^2 u\,du = \sec u\tan u.$$

So putting these together we get

$$\int \sec^3 u\,du = \frac{\sec u\tan u}{2} + \frac{\ln|\sec u+\tan u|}{2} + C,$$

and reverting to the original variable x:

$$\begin{aligned}\int \sqrt{1+x^2}\,dx &= \frac{\sec u\tan u}{2} + \frac{\ln|\sec u+\tan u|}{2} + C\\ &= \frac{\sec(\arctan x)\tan(\arctan x)}{2} + \frac{\ln|\sec(\arctan x)+\tan(\arctan x)|}{2} + C\\ &= \frac{x\sqrt{1+x^2}}{2} + \frac{\ln|\sqrt{1+x^2}+x|}{2} + C,\end{aligned}$$

using $\tan(\arctan x) = x$ and $\sec(\arctan x) = \sqrt{1+\tan^2(\arctan x)} = \sqrt{1+x^2}$. □

Exercises 8.3.

Find the antiderivatives.

1. $\displaystyle\int \csc x\,dx \Rightarrow$

2. $\displaystyle\int \csc^3 x\,dx \Rightarrow$

3. $\displaystyle\int \sqrt{x^2-1}\,dx \Rightarrow$

4. $\displaystyle\int \sqrt{9+4x^2}\,dx \Rightarrow$

5. $\displaystyle\int x\sqrt{1-x^2}\,dx \Rightarrow$

6. $\displaystyle\int x^2\sqrt{1-x^2}\,dx \Rightarrow$

7. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx \Rightarrow$

8. $\displaystyle\int \sqrt{x^2+2x}\,dx \Rightarrow$

9. $\displaystyle\int \frac{1}{x^2(1+x^2)}\,dx \Rightarrow$

10. $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx \Rightarrow$

11. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx \Rightarrow$

12. $\displaystyle\int \frac{x^3}{\sqrt{4x^2-1}}\,dx \Rightarrow$

13. Compute $\displaystyle\int \sqrt{x^2+1}\,dx$. (Hint: make the substitution $x = \sinh(u)$ and then use exercise 6 in section 4.11.)

14. Fix $t > 0$. The shaded region in the left-hand graph in figure 4.11.2 is bounded by $y = x\tanh t$, $y = 0$, and $x^2 - y^2 = 1$. Prove that twice the area of this region is t, as claimed in section 4.11.

8.4 Integration by Parts

We have already seen that recognizing the product rule can be useful, when we noticed that

$$\int \sec^3 u + \sec u \tan^2 u\,du = \sec u \tan u.$$

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

We can rewrite this as

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx,$$

and then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

$$\int f(x)g'(x)\,dx$$

but that

$$\int f'(x)g(x)\,dx$$

is easier. This technique for turning one integral into another is called **integration by parts**, and is usually written in more compact form. If we let $u = f(x)$ and $v = g(x)$ then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$ and

$$\int u\,dv = uv - \int v\,du.$$

To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)\,dx$.

EXAMPLE 8.4.1 Evaluate $\int x\ln x\,dx$. Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x\ln x\,dx = \frac{x^2\ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2\ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C.$$

□

EXAMPLE 8.4.2 Evaluate $\int x\sin x\,dx$. Let $u = x$ so $du = dx$. Then we must let $dv = \sin x\,dx$ so $v = -\cos x$ and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

□

EXAMPLE 8.4.3 Evaluate $\int \sec^3 x\,dx$. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x\tan x\,dx$ and $v = \tan x$

and

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \tan^2 x \sec x\,dx\\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx\\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

At first this looks useless—we're right back to $\int \sec^3 x\,dx$. But looking more closely:

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ \int \sec^3 x\,dx + \int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx\\ 2\int \sec^3 x\,dx &= \sec x \tan x + \int \sec x\,dx\\ \int \sec^3 x\,dx &= \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx\\ &= \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C.\end{aligned}$$

□

EXAMPLE 8.4.4 Evaluate $\int x^2 \sin x\,dx$. Let $u = x^2$, $dv = \sin x\,dx$; then $du = 2x\,dx$ and $v = -\cos x$. Now $\int x^2 \sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$. This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x\,dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned}\int x^2 \sin x\,dx &= -x^2\cos x + \int 2x\cos x\,dx\\ &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx\\ &= -x^2\cos x + 2x\sin x + 2\cos x + C.\end{aligned}$$

□

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

sign	u	dv
	x^2	$\sin x$
$-$	$2x$	$-\cos x$
	2	$-\sin x$
$-$	0	$\cos x$

or

u	dv
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

To form the first table, we start with u at the top of the second column and repeatedly compute the derivative; starting with dv at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a "$-$" in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a "$-$" to every second row.

To compute with this second table we begin at the top. Multiply the first entry in column u by the second entry in column dv to get $-x^2\cos x$, and add this to the integral of the product of the second entry in column u and second entry in column dv. This gives:

$$-x^2\cos x + \int 2x\cos x\,dx,$$

or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, $(x^2)(-\cos x)$ and $(-2x)(-\sin x)$ and then once straight across, $(2)(-\sin x)$, and combine these as

$$-x^2\cos x + 2x\sin x - \int 2\sin x\,dx,$$

giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get $(x^2)(-\cos x)$, $(-2x)(-\sin x)$, and $(2)(\cos x)$, and once straight across, $(0)(\cos x)$. We combine these as before to get

$$-x^2\cos x + 2x\sin x + 2\cos x + \int 0\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Typically we would fill in the table one line at a time, until the "straight across" multiplication gives an easy integral. If we can see that the u column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the "$+C$", as above.

Exercises 8.4.

Find the antiderivatives.

1. $\int x\cos x\,dx \Rightarrow$

2. $\int x^2\cos x\,dx \Rightarrow$

3. $\int xe^x\,dx \Rightarrow$

4. $\int xe^{x^2}\,dx \Rightarrow$

5. $\int \sin^2 x\,dx \Rightarrow$

6. $\int \ln x\,dx \Rightarrow$

7. $\int x\arctan x\,dx \Rightarrow$

8. $\int x^3\sin x\,dx \Rightarrow$

9. $\int x^3\cos x\,dx \Rightarrow$

10. $\int x\sin^2 x\,dx \Rightarrow$

11. $\int x\sin x\cos x\,dx \Rightarrow$

12. $\int \arctan(\sqrt{x})\,dx \Rightarrow$

13. $\int \sin(\sqrt{x})\,dx \Rightarrow$

14. $\int \sec^2 x\csc^2 x\,dx \Rightarrow$

8.5 Rational Functions

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2+x-6}, \qquad \frac{1}{(x-3)^2}, \qquad \frac{x^2+1}{x^2-1},$$

are all rational functions of x. There is a general technique called "partial fractions" that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial ax^2+bx+c.

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax+b)^n$, the substitution $u=ax+b$ will always work. The denominator becomes u^n, and each x in the numerator is replaced by $(u-b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.

EXAMPLE 8.5.1 Find $\int \frac{x^3}{(3-2x)^5}\,dx$. Using the substitution $u = 3-2x$ we get

$$\begin{aligned}
\int \frac{x^3}{(3-2x)^5}\,dx &= \frac{1}{-2}\int \frac{\left(\frac{u-3}{-2}\right)^3}{u^5}\,du = \frac{1}{16}\int \frac{u^3-9u^2+27u-27}{u^5}\,du \\
&= \frac{1}{16}\int u^{-2} - 9u^{-3} + 27u^{-4} - 27u^{-5}\,du \\
&= \frac{1}{16}\left(\frac{u^{-1}}{-1} - \frac{9u^{-2}}{-2} + \frac{27u^{-3}}{-3} - \frac{27u^{-4}}{-4}\right) + C \\
&= \frac{1}{16}\left(\frac{(3-2x)^{-1}}{-1} - \frac{9(3-2x)^{-2}}{-2} + \frac{27(3-2x)^{-3}}{-3} - \frac{27(3-2x)^{-4}}{-4}\right) + C \\
&= -\frac{1}{16(3-2x)} + \frac{9}{32(3-2x)^2} - \frac{9}{16(3-2x)^3} + \frac{27}{64(3-2x)^4} + C
\end{aligned}$$

□

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of x^2 and put it outside the integral, so we can assume that the denominator has the form x^2+bx+c. There are three possible cases, depending on how the quadratic factors: either $x^2+bx+c = (x-r)(x-s)$, $x^2+bx+c = (x-r)^2$, or it doesn't factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

EXAMPLE 8.5.2 Determine whether x^2+x+1 factors, and factor it if possible. The quadratic formula tells us that $x^2+x+1=0$ when

$$x = \frac{-1 \pm \sqrt{1-4}}{2}.$$

Since there is no square root of -3, this quadratic does not factor. □

EXAMPLE 8.5.3 Determine whether x^2-x-1 factors, and factor it if possible. The quadratic formula tells us that $x^2-x-1=0$ when

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right).$$

□

If $x^2+bx+c=(x-r)^2$ then we have the special case we have already seen, that can be handled with a substitution. The other two cases require different approaches.

If $x^2+bx+c=(x-r)(x-s)$, we have an integral of the form

$$\int \frac{p(x)}{(x-r)(x-s)}\,dx$$

where $p(x)$ is a polynomial. The first step is to make sure that $p(x)$ has degree less than 2.

EXAMPLE 8.5.4 Rewrite $\int \frac{x^3}{(x-2)(x+3)}\,dx$ in terms of an integral with a numerator that has degree less than 2. To do this we use long division of polynomials to discover that

$$\frac{x^3}{(x-2)(x+3)} = \frac{x^3}{x^2+x-6} = x-1+\frac{7x-6}{x^2+x-6} = x-1+\frac{7x-6}{(x-2)(x+3)},$$

so

$$\int \frac{x^3}{(x-2)(x+3)}\,dx = \int x-1\,dx + \int \frac{7x-6}{(x-2)(x+3)}\,dx.$$

The first integral is easy, so only the second requires some work. □

Now consider the following simple algebra of fractions:

$$\frac{A}{x-r}+\frac{B}{x-s} = \frac{A(x-s)+B(x-r)}{(x-r)(x-s)} = \frac{(A+B)x-As-Br}{(x-r)(x-s)}.$$

That is, adding two fractions with constant numerator and denominators $(x-r)$ and $(x-s)$ produces a fraction with denominator $(x-r)(x-s)$ and a polynomial of degree less than 2 for the numerator. We want to reverse this process: starting with a single fraction, we want to write it as a sum of two simpler fractions. An example should make it clear how to proceed.

EXAMPLE 8.5.5 Evaluate $\int \frac{x^3}{(x-2)(x+3)}\,dx$. We start by writing $\frac{7x-6}{(x-2)(x+3)}$ as the sum of two fractions. We want to end up with

$$\frac{7x-6}{(x-2)(x+3)} = \frac{A}{x-2}+\frac{B}{x+3}.$$

If we go ahead and add the fractions on the right hand side we get

$$\frac{7x-6}{(x-2)(x+3)} = \frac{(A+B)x+3A-2B}{(x-2)(x+3)}.$$

So all we need to do is find A and B so that $7x-6=(A+B)x+3A-2B$, which is to say, we need $7=A+B$ and $-6=3A-2B$. This is a problem you've seen before: solve a system of two equations in two unknowns. There are many ways to proceed; here's one: If $7 = A+B$ then $B = 7-A$ and so $-6 = 3A-2B = 3A-2(7-A) = 3A-14+2A = 5A-14$. This is easy to solve for A: $A = 8/5$, and then $B = 7 - A = 7 - 8/5 = 27/5$. Thus

$$\int \frac{7x-6}{(x-2)(x+3)}\,dx = \int \frac{8}{5}\frac{1}{x-2} + \frac{27}{5}\frac{1}{x+3}\,dx = \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.$$

The answer to the original problem is now

$$\begin{aligned}\int \frac{x^3}{(x-2)(x+3)}\,dx &= \int x-1\,dx + \int \frac{7x-6}{(x-2)(x+3)}\,dx \\ &= \frac{x^2}{2} - x + \frac{8}{5}\ln|x-2| + \frac{27}{5}\ln|x+3| + C.\end{aligned}$$

□

Now suppose that x^2+bx+c doesn't factor. Again we can use long division to ensure that the numerator has degree less than 2, then we complete the square.

EXAMPLE 8.5.6 Evaluate $\displaystyle\int \frac{x+1}{x^2+4x+8}\,dx$. The quadratic denominator does not factor. We could complete the square and use a trigonometric substitution, but it is simpler to rearrange the integrand:

$$\int \frac{x+1}{x^2+4x+8}\,dx = \int \frac{x+2}{x^2+4x+8}\,dx - \int \frac{1}{x^2+4x+8}\,dx.$$

The first integral is an easy substitution problem, using $u = x^2+4x+8$:

$$\int \frac{x+2}{x^2+4x+8}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2+4x+8|.$$

For the second integral we complete the square:

$$x^2+4x+8 = (x+2)^2+4 = 4\left(\left(\frac{x+2}{2}\right)^2+1\right),$$

making the integral

$$\frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx.$$

Using $u = \dfrac{x+2}{2}$ we get

$$\frac{1}{4}\int \frac{1}{\left(\frac{x+2}{2}\right)^2+1}\,dx = \frac{1}{4}\int \frac{2}{u^2+1}\,du = \frac{1}{2}\arctan\left(\frac{x+2}{2}\right).$$

The final answer is now

$$\int \frac{x+1}{x^2+4x+8}\,dx = \frac{1}{2}\ln|x^2+4x+8| - \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C.$$

□

Exercises 8.5.

Find the antiderivatives.

1. $\displaystyle\int \frac{1}{4-x^2}\,dx \Rightarrow$

2. $\displaystyle\int \frac{x^4}{4-x^2}\,dx \Rightarrow$

3. $\displaystyle\int \frac{1}{x^2+10x+25}\,dx \Rightarrow$

4. $\displaystyle\int \frac{x^2}{4-x^2}\,dx \Rightarrow$

5. $\displaystyle\int \frac{x^4}{4+x^2}\,dx \Rightarrow$

6. $\displaystyle\int \frac{1}{x^2+10x+29}\,dx \Rightarrow$

7. $\displaystyle\int \frac{x^3}{4+x^2}\,dx \Rightarrow$

8. $\displaystyle\int \frac{1}{x^2+10x+21}\,dx \Rightarrow$

9. $\displaystyle\int \frac{1}{2x^2-x-3}\,dx \Rightarrow$

10. $\displaystyle\int \frac{1}{x^2+3x}\,dx \Rightarrow$

8.6 Numerical Integration

We have now seen some of the most generally useful methods for discovering antiderivatives, and there are others. Unfortunately, some functions have no simple antiderivatives; in such cases if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

Of course, we already know one way to approximate an integral: if we think of the integral as computing an area, we can add up the areas of some rectangles. While this is quite simple, it is usually the case that a large number of rectangles is needed to get acceptable accuracy. A similar approach is much better: we approximate the area under a curve over a small interval as the area of a trapezoid. In figure 8.6.1 we see an area under a curve approximated by rectangles and by trapezoids; it is apparent that the trapezoids give a substantially better approximation on each subinterval.

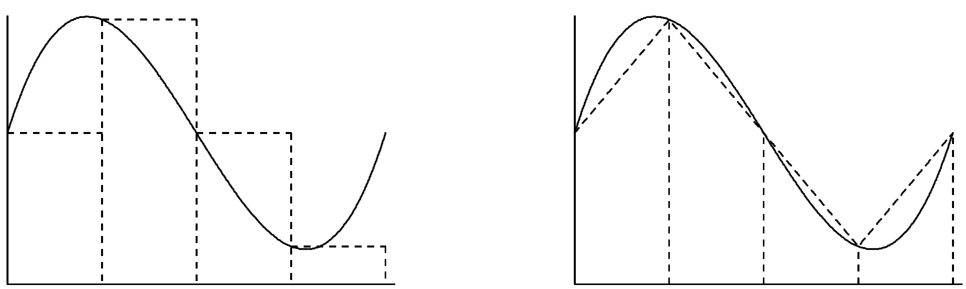


Figure 8.6.1 Approximating an area with rectangles and with trapezoids. (AP)

As with rectangles, we divide the interval into n equal subintervals of length Δx. A typical trapezoid is pictured in figure 8.6.2; it has area $\displaystyle\frac{f(x_i)+f(x_{i+1})}{2}\Delta x$. If we add up the areas of all trapezoids we get

$$\frac{f(x_0)+f(x_1)}{2}\Delta x+\frac{f(x_1)+f(x_2)}{2}\Delta x+\cdots+\frac{f(x_{n-1})+f(x_n)}{2}\Delta x=$$
$$\left(\frac{f(x_0)}{2}+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{f(x_n)}{2}\right)\Delta x.$$

This is usually known as the **Trapezoid Rule**. For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

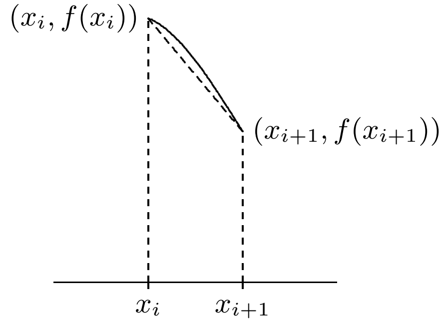


Figure 8.6.2 A single trapezoid.

In practice, an approximation is useful only if we know how accurate it is; for example, we might need a particular value accurate to three decimal places. When we compute a particular approximation to an integral, the error is the difference between the approximation and the true value of the integral. For any approximation technique, we need an **error estimate**, a value that is guaranteed to be larger than the actual error. If A is an approximation and E is the associated error estimate, then we know that the true value of the integral is between $A - E$ and $A + E$. In the case of our approximation of the integral, we want $E = E(\Delta x)$ to be a function of Δx that gets small rapidly as Δx gets small. Fortunately, for many functions, there is such an error estimate associated with the trapezoid approximation.

THEOREM 8.6.1 Suppose f has a second derivative f'' everywhere on the interval $[a, b]$, and $|f''(x)| \le M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error estimate for the trapezoid approximation is

$$E(\Delta x)=\frac{b-a}{12}M(\Delta x)^2=\frac{(b-a)^3}{12n^2}M.$$

■

Let's see how we can use this.

EXAMPLE 8.6.2 Approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places. The second derivative of $f = e^{-x^2}$ is $(4x^2-2)e^{-x^2}$, and it is not hard to see that on $[0,1]$, $|(4x^2-2)e^{-x^2}| \le 2$. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$ or

$$\begin{aligned}\frac{1}{12}(2)\frac{1}{n^2} &< 0.005\\ \frac{1}{6}(200) &< n^2\\ 5.77 \approx \sqrt{\frac{100}{3}} &< n\end{aligned}$$

With $n = 6$, the error estimate is thus $1/6^3 < 0.0047$. We compute the trapezoid approximation for six intervals:

$$\left(\frac{f(0)}{2} + f(1/6) + f(2/6) + \cdots + f(5/6) + \frac{f(1)}{2}\right)\frac{1}{6} \approx 0.74512.$$

So the true value of the integral is between $0.74512 - 0.0047 = 0.74042$ and $0.74512 + 0.0047 = 0.74982$. Unfortunately, the first rounds to 0.74 and the second rounds to 0.75, so we can't be sure of the correct value in the second decimal place; we need to pick a larger n. As it turns out, we need to go to $n = 12$ to get two bounds that both round to the same value, which turns out to be 0.75. For comparison, using 12 rectangles to approximate the area gives 0.7727, which is considerably less accurate than the approximation using six trapezoids.

In practice it generally pays to start by requiring better than the maximum possible error; for example, we might have initially required $E(\Delta x) < 0.001$, or

$$\begin{aligned}\frac{1}{12}(2)\frac{1}{n^2} &< 0.001\\ \frac{1}{6}(1000) &< n^2\\ 12.91 \approx \sqrt{\frac{500}{3}} &< n\end{aligned}$$

Had we immediately tried $n = 13$ this would have given us the desired answer. □

The trapezoid approximation works well, especially compared to rectangles, because the tops of the trapezoids form a reasonably good approximation to the curve when Δx is fairly small. We can extend this idea: what if we try to approximate the curve more closely, by using something other than a straight line? The obvious candidate is a parabola: if we can approximate a short piece of the curve with a parabola with equation $y = ax^2 + bx + c$, we can easily compute the area under the parabola.

There are an infinite number of parabolas through any two given points, but only one through three given points. If we find a parabola through three consecutive points $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$ on the curve, it should be quite close to the curve over the whole interval $[x_i, x_{i+2}]$, as in figure 8.6.3. If we divide the interval $[a, b]$ into an even number of subintervals, we can then approximate the curve by a sequence of parabolas, each covering two of the subintervals. For this to be practical, we would like a simple formula for the area under one parabola, namely, the parabola through $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, and $(x_{i+2}, f(x_{i+2}))$. That is, we should attempt to write down the parabola $y = ax^2 + bx + c$ through these points and then integrate it, and hope that the result is fairly simple. Although the algebra involved is messy, this turns out to be possible. The algebra is well within the capability of a good computer algebra system like Sage, so we will present the result without all of the algebra; you can see how to do it in this Sage worksheet.

To find the parabola, we solve these three equations for a, b, and c:

$$\begin{aligned} f(x_i) &= a(x_{i+1} - \Delta x)^2 + b(x_{i+1} - \Delta x) + c \\ f(x_{i+1}) &= a(x_{i+1})^2 + b(x_{i+1}) + c \\ f(x_{i+2}) &= a(x_{i+1} + \Delta x)^2 + b(x_{i+1} + \Delta x) + c \end{aligned}$$

Not surprisingly, the solutions turn out to be quite messy. Nevertheless, Sage can easily compute and simplify the integral to get

$$\int_{x_{i+1}-\Delta x}^{x_{i+1}+\Delta x} ax^2 + bx + c\,dx = \frac{\Delta x}{3}(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})).$$

Now the sum of the areas under all parabolas is

$$\frac{\Delta x}{3}(f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \cdots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) =$$
$$\frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).$$

This is just slightly more complicated than the formula for trapezoids; we need to remember the alternating 2 and 4 coefficients; note that n must be even for this to make sense. This approximation technique is referred to as **Simpson's Rule**.

As with the trapezoid method, this is useful only with an error estimate:

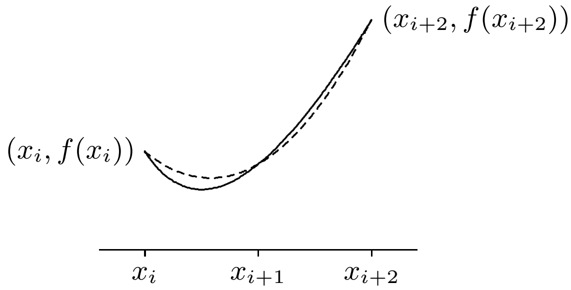


Figure 8.6.3 A parabola (dashed) approximating a curve (solid). (AP)

THEOREM 8.6.3 Suppose f has a fourth derivative $f^{(4)}$ everywhere on the interval $[a,b]$, and $|f^{(4)}(x)| \le M$ for all x in the interval. With $\Delta x = (b-a)/n$, an error estimate for Simpson's approximation is

$$E(\Delta x) = \frac{b-a}{180} M(\Delta x)^4 = \frac{(b-a)^5}{180n^4} M.$$

■

EXAMPLE 8.6.4 Let us again approximate $\int_0^1 e^{-x^2}\,dx$ to two decimal places. The fourth derivative of $f = e^{-x^2}$ is $(16x^2 - 48x^2 + 12)e^{-x^2}$; on $[0,1]$ this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$, but taking a cue from our earlier example, let's require $E(\Delta x) < 0.001$:

$$\begin{aligned} \frac{1}{180}(12)\frac{1}{n^4} &< 0.001 \\ \frac{200}{3} &< n^4 \\ 2.86 \approx \sqrt[4]{\frac{200}{3}} &< n \end{aligned}$$

So we try $n = 4$, since we need an even number of subintervals. Then the error estimate is $12/180/4^4 < 0.0003$ and the approximation is

$$(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1))\frac{1}{3\cdot 4} \approx 0.746855.$$

So the true value of the integral is between $0.746855 - 0.0003 = 0.746555$ and $0.746855 + 0.0003 = 0.7471555$, both of which round to 0.75. □

Exercises 8.6.

In the following problems, compute the trapezoid and Simpson approximations using 4 subintervals, and compute the error estimate for each. (Finding the maximum values of the second and fourth derivatives can be challenging for some of these; you may use a graphing calculator or computer software to estimate the maximum values.) If you have access to Sage or similar software, approximate each integral to two decimal places. You can use this Sage worksheet to get started.

1. $\int_1^3 x\,dx \Rightarrow$

2. $\int_0^3 x^2\,dx \Rightarrow$

3. $\int_2^4 x^3\,dx \Rightarrow$

4. $\int_1^3 \frac{1}{x}\,dx \Rightarrow$

5. $\int_1^2 \frac{1}{1+x^2}\,dx \Rightarrow$

6. $\int_0^1 x\sqrt{1+x}\,dx \Rightarrow$

7. $\int_1^5 \frac{x}{1+x}\,dx \Rightarrow$

8. $\int_0^1 \sqrt{x^3+1}\,dx \Rightarrow$

9. $\int_0^1 \sqrt{x^4+1}\,dx \Rightarrow$

10. $\int_1^4 \sqrt{1+1/x}\,dx \Rightarrow$

11. Using Simpson's rule on a parabola $f(x)$, even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate f will be f itself. Remarkably, Simpson's rule also computes the integral of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ exactly. Show this is true by showing that

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{x_2 - x_0}{3\cdot 2}(f(x_0) + 4f((x_0+x_2)/2) + f(x_2)).$$

Note that the right hand side of this equation is exactly the Simpson approximation for the cubic. This does require a bit of messy algebra, so you may prefer to use Sage.

8.7 Additional exercises

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

1. $\int (t+4)^3\,dt \Rightarrow$

2. $\int t(t^2-9)^{3/2}\,dt \Rightarrow$

3. $\int (e^{t^2}+16)te^{t^2}\,dt \Rightarrow$

4. $\int \sin t\cos 2t\,dt \Rightarrow$

5. $\int \tan t\sec^2 t\,dt \Rightarrow$

6. $\int \frac{2t+1}{t^2+t+3}\,dt \Rightarrow$

7. $\int \frac{1}{t(t^2-4)}\,dt \Rightarrow$

8. $\int \frac{1}{(25-t^2)^{3/2}}\,dt \Rightarrow$

9. $\int \frac{\cos 3t}{\sqrt{\sin 3t}}\,dt \Rightarrow$

10. $\int t\sec^2 t\,dt \Rightarrow$

11. $\int \frac{e^t}{\sqrt{e^t+1}}\,dt \Rightarrow$

12. $\int \cos^4 t\,dt \Rightarrow$

13. $\displaystyle\int \frac{1}{t^2+3t}\,dt \Rightarrow$

14. $\displaystyle\int \frac{1}{t^2\sqrt{1+t^2}}\,dt \Rightarrow$

15. $\displaystyle\int \frac{\sec^2 t}{(1+\tan t)^3}\,dt \Rightarrow$

16. $\displaystyle\int t^3\sqrt{t^2+1}\,dt \Rightarrow$

17. $\displaystyle\int e^t \sin t\,dt \Rightarrow$

18. $\displaystyle\int (t^{3/2}+47)^3\sqrt{t}\,dt \Rightarrow$

19. $\displaystyle\int \frac{t^3}{(2-t^2)^{5/2}}\,dt \Rightarrow$

20. $\displaystyle\int \frac{1}{t(9+4t^2)}\,dt \Rightarrow$

21. $\displaystyle\int \frac{\arctan 2t}{1+4t^2}\,dt \Rightarrow$

22. $\displaystyle\int \frac{t}{t^2+2t-3}\,dt \Rightarrow$

23. $\displaystyle\int \sin^3 t\cos^4 t\,dt \Rightarrow$

24. $\displaystyle\int \frac{1}{t^2-6t+9}\,dt \Rightarrow$

25. $\displaystyle\int \frac{1}{t(\ln t)^2}\,dt \Rightarrow$

26. $\displaystyle\int t(\ln t)^2\,dt \Rightarrow$

27. $\displaystyle\int t^3 e^t\,dt \Rightarrow$

28. $\displaystyle\int \frac{t+1}{t^2+t-1}\,dt \Rightarrow$

**Functions and Their Graphs

Functions are a tool for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description.

**DEFINITION Function

A function $f(x)$ from a set D to a set Y is a rule that assigns a unique value $f(x)$ in Y to each x in D. In symbols:

$$y = f(x) \qquad (\text{“}y\text{ equals } f \text{ of } x\text{”})$$

Where: the symbol (f) represents the function, and (x) is the independent variable, which represent the input value of f. Also, (y) is the dependent variable, and represents the corresponding output value of f at x.

The set D of all possible input values is called the domain of the function. The set of all values of $f(x)$ as x varies throughout D is called the range of the function. The range may not include every element in the set Y.

EXAMPLE 1 Identifying Domain and Range

Verify the domains and ranges of these functions.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

(a) $y = x^2$

The formula $y = x^2$ gives a real y-value for any real number x, so the **domain** is $(-\infty, \infty)$. The **range** of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root.

(b) The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, **we cannot divide any number by zero**. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers

(c) $y = \sqrt{x}$

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$.
The **domain** of $y = \sqrt{x}$ is $[0, \infty)$
The **range** of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root.

(d) $y = \sqrt{4 - x}$

The quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 0$.
The formula gives real y-values for all $x \leq 4$. The **domain** is $(-\infty, 4]$
The **range** of function is $[0, \infty)$, the set of all nonnegative numbers.

(e) $y = \sqrt{1 - x^2}$

The **domain** is $[-1, 1]$
The **range** is $[0, 1]$

Graph of Functions

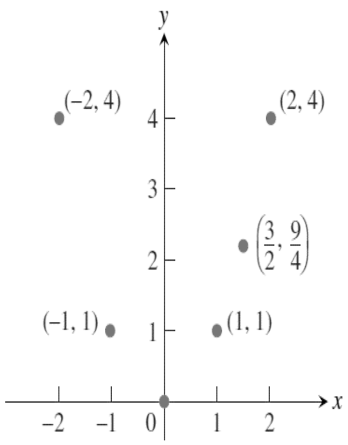
The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $y = f(x)$ is the height of the graph above (or below) the point x.

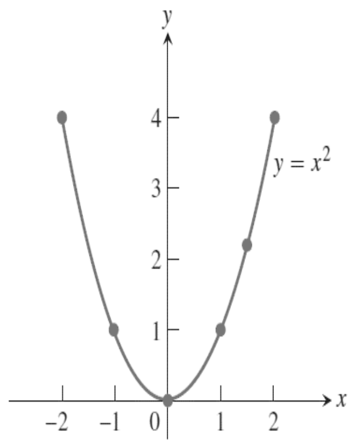
Example: Plot the function $y = x^2$ over the interval [-2, 2]

Solution:

- Make a table of x y-pairs that satisfy the equation $y = x^2$
- Plot the points (x, y) whose coordinates appear in the table
- Draw a smooth curve (labeled with its equation) through the plotted points

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

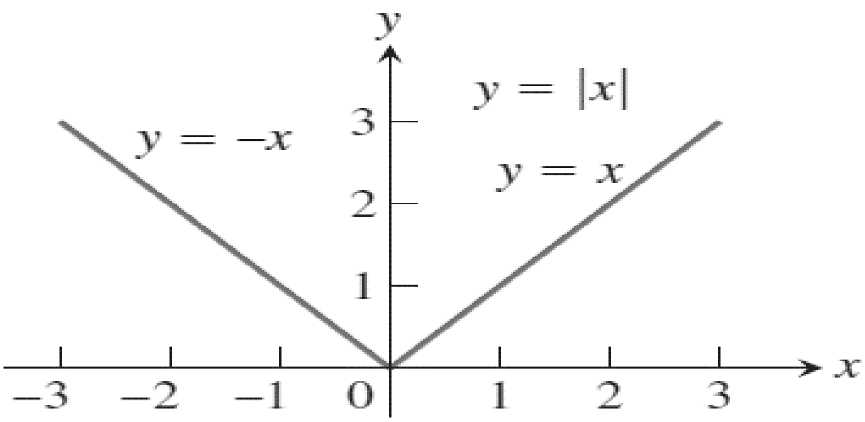




**Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the absolute value function.

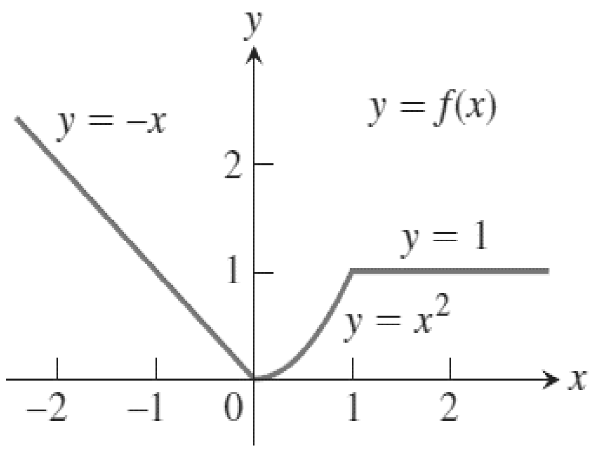
whose graph is given in Figure . The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x < 0$.



The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

The values of f are given by:
$y = -x$ when $x < 0$,
$y = x^2$ when $0 \leq x \leq 1$ and
$y = 1$ when $x > 1$



Real Numbers and the Real Line

Calculus is based on the real number system. Real numbers are numbers that can be expressed as decimals.

We distinguish three special subsets of real numbers:

1. The **natural numbers**, namely 1, 2, 3, 4,…

2. The **integers**, namely 0, ±1, ±2, ±3,…

3. The **rational numbers**, which are ratios of integers. These numbers can be expressed in the form of a function *m*/*n*, where *m* and *n* are integers and $n \neq 0$. Examples are:

$$\frac{2}{3}, \quad \frac{-4}{9}, \quad \frac{2}{1}$$

The rational numbers are real numbers with decimal expressions that are:

· Terminating (ending in an infinite string), for example:

$\frac{3}{4} = 0.75000\cdots = 0.75$

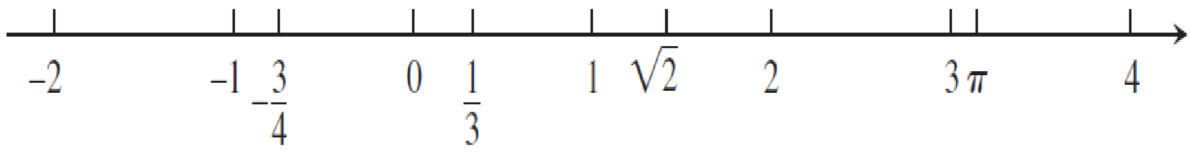
· Repeating (ending with a block of digits that repeats over and over), for example:

$\frac{23}{11} = 2.090909\cdots = 2.202292$

4. the **irrational numbers**

$$\sqrt{2} \quad , \sqrt[3]{5} \quad , \pi$$

The real numbers can be represented geometrically as points on a number line called the real line.



Rules of Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Longrightarrow a + c < b + c$
2. $a < b \Longrightarrow a - c < b - c$
3. $a < b \; and \; c > 0 \Longrightarrow ac < bc$
4. $a < b \; and \; c < 0 \Longrightarrow bc < ac$

    Special case: $a < b \qquad -b < -a$

5. $a > 0 \Longrightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Longrightarrow \frac{1}{b} < \frac{1}{a}$

Intervals

The interval is a subset of the real line. It contains at least two numbers, and contains all the real numbers lying between any two of its elements. A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint, but not the other, and **open** if it contains neither end points.

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \le x \le b\}$	Closed	
	$[a, b)$	$\{x \mid a \le x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \le b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \ge a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \le b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

EXAMPLE 1 Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\dfrac{x}{3} < 2x + 1$ **(c)** $\dfrac{6}{x - 1} \ge 5$

Solution

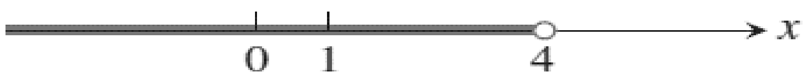
(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \qquad \text{Add 1 to both sides.}$$

$$x < 4 \qquad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).



(b) $$-\frac{x}{3} < 2x + 1$$

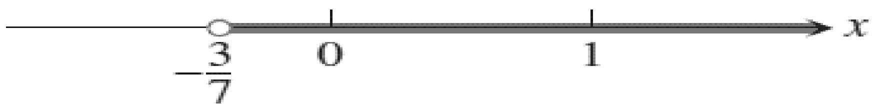
$$-x < 6x + 3$$ Multiply both sides by 3.

$$0 < 7x + 3$$ Add x to both sides.

$$-3 < 7x$$ Subtract 3 from both sides.

$$-\frac{3}{7} < x$$ Divide by 7.

The solution set is the open interval (- 3/7 , ∞)



(c) The inequality can hold only if x>1, because otherwise 6/(x-1) is undefined or negative.

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5$$ Multiply both sides by $(x - 1)$.

$$11 \geq 5x$$ Add 5 to both sides.

$$\frac{11}{5} \geq x.$$ Or $x \leq \frac{11}{5}$.

The solution set is the half-open interval (1, 11/5] (Figure 1.1c).

Absolute Value

The absolute value of a number x, denoted by $|X|$ is defined by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Absolute Value Properties

1.	$\lvert -a \rvert = \lvert a \rvert$	A number and its additive inverse or negative have the same absolute value.
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3.	$\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4.	$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

Illustrating the Triangle Inequality

$$|-3 + 5| = |2| = 2 < |-3| + |5| = 8$$

$$|3 + 5| = |8| = |3| + |5|$$

$$|-3 - 5| = |-8| = 8 = |-3| + |-5|$$

Absolute Values and Intervals

If a is any positive number, then

5. $|x| = a$ if and only if $x = \pm a$

6. $|x| < a$ if and only if $-a < x < a$

7. $|x| > a$ if and only if $x > a$ or $x < -a$

8. $|x| \leq a$ if and only if $-a \leq x \leq a$

9. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$

EXAMPLE 4 Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$2x - 3 = 7 \qquad 2x - 3 = -7 \qquad \text{Equivalent equations without absolute values}$$

$$2x = 10 \qquad 2x = -4 \qquad \text{Solve as usual.}$$

$$x = 5 \qquad x = -2$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$.

EXAMPLE 5 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

Solution We have

$$\left|5 - \frac{2}{x}\right| < 1 \Leftrightarrow -1 < 5 - \frac{2}{x} < 1 \qquad \text{Property 6}$$

$$\Leftrightarrow -6 < -\frac{2}{x} < -4 \qquad \text{Subtract 5.}$$

$$\Leftrightarrow 3 > \frac{1}{x} > 2 \qquad \text{Multiply by } -\tfrac{1}{2}.$$

$$\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}. \qquad \text{Take reciprocals.}$$

Notice how the various rules for inequalities were used here. Multiplying by a negative number reverses the inequality. So does taking reciprocals in an inequality in which both sides are positive. The original inequality holds if and only if $(1/3) < x < (1/2)$. The solution set is the open interval $(1/3, 1/2)$. ■

EXAMPLE 6 Solve the inequality and show the solution set on the real line:

(a) $|2x - 3| \leq 1$ **(b)** $|2x - 3| \geq 1$

Solution

(a)

$$|2x - 3| \leq 1$$

$$-1 \leq 2x - 3 \leq 1 \qquad \text{Property 8}$$

$$2 \leq 2x \leq 4 \qquad \text{Add 3.}$$

$$1 \leq x \leq 2 \qquad \text{Divide by 2.}$$

The solution set is the closed interval [1, 2] (Figure 1.4a).

(b)

$$|2x - 3| \geq 1$$

$$2x - 3 \geq 1 \quad \text{or} \quad 2x - 3 \leq -1 \qquad \text{Property 9}$$

$$x - \frac{3}{2} \geq \frac{1}{2} \quad \text{or} \quad x - \frac{3}{2} \leq -\frac{1}{2} \qquad \text{Divide by 2.}$$

$$x \geq 2 \quad \text{or} \quad x \leq 1 \qquad \text{Add } \frac{3}{2}.$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Figure 1.4b).

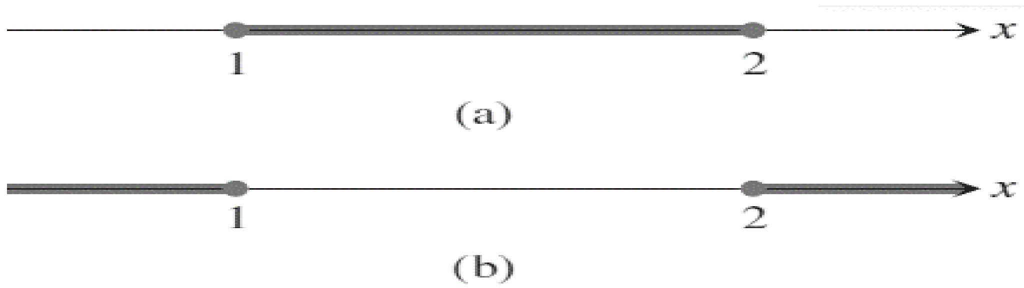


FIGURE 1.4 The solution sets (a) [1, 2] and (b) $(-\infty, 1] \cup [2, \infty)$ in Example 6.

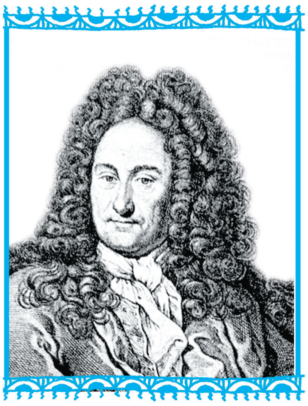
Chapter 7

INTEGRALS

❖ *Just as a mountaineer climbs a mountain – because it is there, so a good mathematics student studies new material because it is there.* — *JAMES B. BRISTOL* ❖

7.1 Introduction

Differential Calculus is centred on the concept of the derivative. The original motivation for the derivative was the problem of defining tangent lines to the graphs of functions and calculating the slope of such lines. Integral Calculus is motivated by the problem of defining and calculating the area of the region bounded by the graph of the functions.



G .W. Leibnitz (1646 -1716)

If a function f is differentiable in an interval I, i.e., its derivative f' exists at each point of I, then a natural question arises that given f' at each point of I, can we determine the function? The functions that could possibly have given function as a derivative are called anti derivatives (or primitive) of the function. Further, the formula that gives all these anti derivatives is called the ***indefinite integral*** of the function and such process of finding anti derivatives is called integration. Such type of problems arise in many practical situations. For instance, if we know the instantaneous velocity of an object at any instant, then there arises a natural question, i.e., can we determine the position of the object at any instant? There are several such practical and theoretical situations where the process of integration is involved. The development of integral calculus arises out of the efforts of solving the problems of the following types:

(a) the problem of finding a function whenever its derivative is given,

(b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the ***Integral Calculus***.

There is a connection, known as the ***Fundamental Theorem of Calculus***, between indefinite integral and definite integral which makes the definite integral as a practical tool for science and engineering. The definite integral is also used to solve many interesting problems from various disciplines like economics, finance and probability.

In this Chapter, we shall confine ourselves to the study of indefinite and definite integrals and their elementary properties including some techniques of integration.

7.2 Integration as an Inverse Process of Differentiation

Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, i.e., the original function. Such a process is called *integration* or *anti differentiation*.
Let us consider the following examples:

We know that $$\frac{d}{dx}(\sin x) = \cos x \quad \text{... (1)}$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2 \quad \text{... (2)}$$

and $$\frac{d}{dx}(e^x) = e^x \quad \text{... (3)}$$

We observe that in (1), the function cos x is the derived function of sin x. We say that sin x is an anti derivative (or an integral) of cos x. Similarly, in (2) and (3), $\frac{x^3}{3}$ and e^x are the anti derivatives (or integrals) of x^2 and e^x, respectively. Again, we note that for any real number C, treated as constant function, its derivative is zero and hence, we can write (1), (2) and (3) as follows :

$$\frac{d}{dx}(\sin x + \text{C}) = \cos x\,, \quad \frac{d}{dx}\left(\frac{x^3}{3} + \text{C}\right) = x^2 \text{ and } \frac{d}{dx}(e^x + \text{C}) = e^x$$

Thus, anti derivatives (or integrals) of the above cited functions are not unique. Actually, there exist infinitely many anti derivatives of each of these functions which can be obtained by choosing C arbitrarily from the set of real numbers. For this reason C is customarily referred to as ***arbitrary constant***. In fact, C is the ***parameter*** by varying which one gets different anti derivatives (or integrals) of the given function. More generally, if there is a function F such that $\frac{d}{dx}\text{F}(x) = f(x)$, $\forall x \in$ I (interval), then for any arbitrary real number C, (also called *constant of integration*)

$$\frac{d}{dx}[\text{F}(x) + \text{C}] = f(x),\ x \in \text{I}$$

Thus, $\{F + C, C \in \mathbf{R}\}$ denotes a family of anti derivatives of f.

Remark Functions with same derivatives differ by a constant. To show this, let g and h be two functions having the same derivatives on an interval I.

Consider the function $f = g - h$ defined by $f(x) = g(x) - h(x)$, $\forall\, x \in \mathrm{I}$

Then $$\frac{df}{dx} = f' = g' - h' \text{ giving } f'(x) = g'(x) - h'(x) \;\forall\, x \in \mathrm{I}$$

or $f'(x) = 0$, $\forall\, x \in$ I by hypothesis,

i.e., the rate of change of f with respect to x is zero on I and hence f is constant.

In view of the above remark, it is justified to infer that the family $\{F + C, C \in \mathbf{R}\}$ provides all possible anti derivatives of f.

We introduce a new symbol, namely, $\int f(x)\, dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x.

Symbolically, we write $\int f(x)\, dx = F(x) + C$.

Notation Given that $\frac{dy}{dx} = f(x)$, we write $y = \int f(x)\, dx$.

For the sake of convenience, we mention below the following symbols/terms/phrases with their meanings as given in the Table (7.1).

Table 7.1

Symbols/Terms/Phrases	**Meaning**
$\int f(x)\, dx$	Integral of f with respect to x
$f(x)$ in $\int f(x)\, dx$	Integrand
x in $\int f(x)\, dx$	Variable of integration
Integrate	Find the integral
An integral of f	A function F such that $F'(x) = f(x)$
Integration	The process of finding the integral
Constant of Integration	Any real number C, considered as constant function

We already know the formulae for the derivatives of many important functions. From these formulae, we can write down immediately the corresponding formulae (referred to as standard formulae) for the integrals of these functions, as listed below which will be used to find integrals of other functions.

Derivatives	**Integrals (Anti derivatives)**
(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n$;	$\int x^n\, dx=\frac{x^{n+1}}{n+1}+C$, $n \neq -1$
Particularly, we note that	
$\frac{d}{dx}(x)=1$;	$\int dx = x+C$
(ii) $\frac{d}{dx}(\sin x)=\cos x$;	$\int \cos x\, dx=\sin x+C$
(iii) $\frac{d}{dx}(-\cos x)=\sin x$;	$\int \sin x\, dx=-\cos x+C$
(iv) $\frac{d}{dx}(\tan x)=\sec^2 x$;	$\int \sec^2 x\, dx=\tan x+C$
(v) $\frac{d}{dx}(-\cot x)=\text{cosec}^2 x$;	$\int \text{cosec}^2 x\, dx=-\cot x+C$
(vi) $\frac{d}{dx}(\sec x)=\sec x \tan x$;	$\int \sec x \tan x\, dx=\sec x+C$
(vii) $\frac{d}{dx}(-\text{cosec}\, x)=\text{cosec}\, x \cot x$;	$\int \text{cosec}\, x \cot x\, dx=-\text{cosec}\, x+C$
(viii) $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=\sin^{-1} x+C$
(ix) $\frac{d}{dx}(-\cos^{-1} x)=\frac{1}{\sqrt{1-x^2}}$;	$\int \frac{dx}{\sqrt{1-x^2}}=-\cos^{-1} x+C$
(x) $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=\tan^{-1} x+C$
(xi) $\frac{d}{dx}(-\cot^{-1} x)=\frac{1}{1+x^2}$;	$\int \frac{dx}{1+x^2}=-\cot^{-1} x+C$

(xii) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}}$; $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C$

(xiii) $\frac{d}{dx}\left(-\operatorname{cosec}^{-1} x\right) = \frac{1}{x\sqrt{x^2 - 1}}$; $\int \frac{dx}{x\sqrt{x^2 - 1}} = -\operatorname{cosec}^{-1} x + C$

(xiv) $\frac{d}{dx}(e^x) = e^x$; $\int e^x dx = e^x + C$

(xv) $\frac{d}{dx}(\log |x|) = \frac{1}{x}$; $\int \frac{1}{x} dx = \log |x| + C$

(xvi) $\frac{d}{dx}\left(\frac{a^x}{log\ a}\right) = a^x$; $\int a^x dx = \frac{a^x}{log\ a} + C$

Note In practice, we normally do not mention the interval over which the various functions are defined. However, in any specific problem one has to keep it in mind.

7.2.1 *Geometrical interpretation of indefinite integral*

Let $f(x) = 2x$. Then $\int f(x)\, dx = x^2 + C$. For different values of C, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + C$, where C is arbitrary constant, represents a family of integrals. By assigning different values to C, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

Clearly, for C = 0, we obtain $y = x^2$, a parabola with its vertex on the origin. The curve $y = x^2 + 1$ for C = 1 is obtained by shifting the parabola $y = x^2$ one unit along y-axis in positive direction. For C = – 1, $y = x^2 - 1$ is obtained by shifting the parabola $y = x^2$ one unit along y-axis in the negative direction. Thus, for each positive value of C, each parabola of the family has its vertex on the positive side of the y-axis and for negative values of C, each has its vertex along the negative side of the y-axis. Some of these have been shown in the Fig 7.1.

Let us consider the intersection of all these parabolas by a line $x = a$. In the Fig 7.1, we have taken $a > 0$. The same is true when $a < 0$. If the line $x = a$ intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals $2a$. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\, dx = x^2 + C = F_C(x)$ (say), implies that

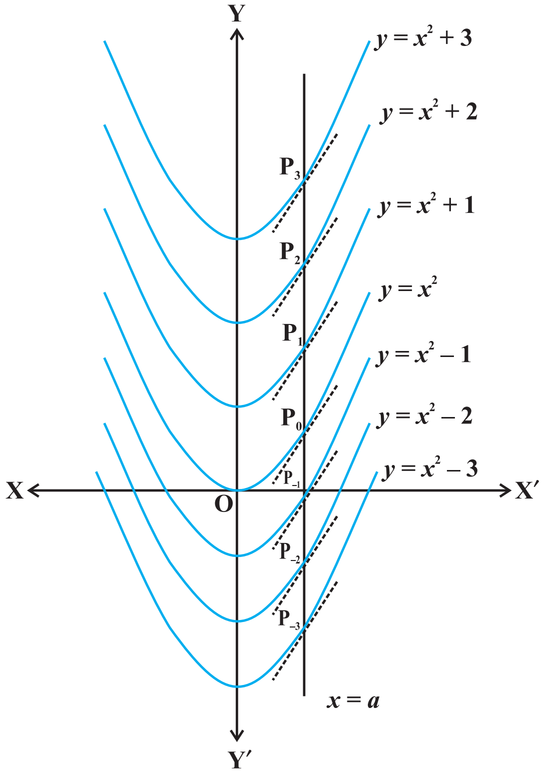


Fig 7.1

the tangents to all the curves $y = F_C(x)$, C ∈ **R**, at the points of intersection of the curves by the line $x = a$, (a ∈ **R**), are parallel.

Further, the following equation (statement) $\int f(x)\, dx = F(x) + C = y$ (say), represents a family of curves. The different values of C will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

7.2.2 *Some properties of indefinite integral*

In this sub section, we shall derive some properties of indefinite integrals.

(I) The process of differentiation and integration are inverses of each other in the sense of the following results :

$$\frac{d}{dx}\int f(x)\, dx = f(x)$$

and $$\int f'(x)\, dx = f(x) + C, \text{ where C is any arbitrary constant.}$$

Proof Let F be any anti derivative of f, i.e.,

$$\frac{d}{dx}\mathrm{F}(x) = f(x)$$

Then $$\int f(x)\,dx = \mathrm{F}(x) + \mathrm{C}$$

Therefore $$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\left(\mathrm{F}(x)+\mathrm{C}\right)$$

$$= \frac{d}{dx}\mathrm{F}(x) = f(x)$$

Similarly, we note that

$$f'(x) = \frac{d}{dx}f(x)$$

and hence $$\int f'(x)\,dx = f(x) + \mathrm{C}$$

where C is arbitrary constant called constant of integration.

(II) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

Proof Let f and g be two functions such that

$$\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)\,dx$$

or $$\frac{d}{dx}\left[\int f(x)\,dx - \int g(x)\,dx\right] = 0$$

Hence $\int f(x)\,dx - \int g(x)\,dx =$ C, where C is any real number (Why?)

or $$\int f(x)\,dx = \int g(x)\,dx + \mathrm{C}$$

So the families of curves $\left\{\int f(x)\,dx + \mathrm{C}_1, \mathrm{C}_1 \in \mathrm{R}\right\}$

and $\left\{\int g(x)\,dx + \mathrm{C}_2, \mathrm{C}_2 \in \mathrm{R}\right\}$ are identical.

Hence, in this sense, $\int f(x)\,dx$ and $\int g(x)\,dx$ are equivalent.

Note The equivalence of the families $\left\{\int f(x)\,dx + C_1, C_1 \in \mathbf{R}\right\}$ and $\left\{\int g(x)\,dx + C_2, C_2 \in \mathbf{R}\right\}$ is customarily expressed by writing $\int f(x)\,dx = \int g(x)\,dx$, without mentioning the parameter.

(III) $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Proof By Property (I), we have

$$\frac{d}{dx}\left[\int [f(x) + g(x)]\,dx\right] = f(x) + g(x) \qquad \text{... (1)}$$

On the otherhand, we find that

$$\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$= f(x) + g(x) \qquad \text{... (2)}$$

Thus, in view of Property (II), it follows by (1) and (2) that

$$\int \left(f(x) + g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx.$$

(IV) For any real number k, $\int k\,f(x)\,dx = k\int f(x)\,dx$

Proof By the Property (I), $\frac{d}{dx}\int k\,f(x)\,dx = k\,f(x)$.

Also $\quad \frac{d}{dx}\left[k\int f(x)\,dx\right] = k\frac{d}{dx}\int f(x)\,dx = k\,f(x)$

Therefore, using the Property (II), we have $\int k\,f(x)\,dx = k\int f(x)\,dx$.

(V) Properties (III) and (IV) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int [k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]\,dx$$

$$= k_1\int f_1(x)\,dx + k_2\int f_2(x)\,dx + ... + k_n\int f_n(x)\,dx.$$

To find an anti derivative of a given function, we search intuitively for a function whose derivative is the given function. The search for the requisite function for finding an anti derivative is known as integration by the method of inspection. We illustrate it through some examples.

Example 1 Write an anti derivative for each of the following functions using the method of inspection:

(i) $\cos 2x$ (ii) $3x^2 + 4x^3$ (iii) $\frac{1}{x}, x \neq 0$

Solution

(i) We look for a function whose derivative is $\cos 2x$. Recall that

$$\frac{d}{dx} \sin 2x = 2 \cos 2x$$

or $$\cos 2x = \frac{1}{2}\frac{d}{dx}(\sin 2x) = \frac{d}{dx}\left(\frac{1}{2}\sin 2x\right)$$

Therefore, an anti derivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$.

(ii) We look for a function whose derivative is $3x^2 + 4x^3$. Note that

$$\frac{d}{dx}\left(x^3 + x^4\right) = 3x^2 + 4x^3.$$

Therefore, an anti derivative of $3x^2 + 4x^3$ is $x^3 + x^4$.

(iii) We know that

$$\frac{d}{dx}(\log x) = \frac{1}{x}, x > 0 \text{ and } \frac{d}{dx}[\log(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}, x < 0$$

Combining above, we get $\frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}, x \neq 0$

Therefore, $\int \frac{1}{x}\, dx = \log|x|$ is one of the anti derivatives of $\frac{1}{x}$.

Example 2 Find the following integrals:

(i) $\int \frac{x^3 - 1}{x^2}\, dx$ (ii) $\int (x^{\frac{2}{3}} + 1)\, dx$ (iii) $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\, dx$

Solution

(i) We have

$$\int \frac{x^3 - 1}{x^2}\, dx = \int x\, dx - \int x^{-2}\, dx \qquad \text{(by Property V)}$$

$$= \left(\frac{x^{1+1}}{1+1} + C_1\right) - \left(\frac{x^{-2+1}}{-2+1} + C_2\right); \ C_1, C_2 \text{ are constants of integration}$$

$$= \frac{x^2}{2} + C_1 - \frac{x^{-1}}{-1} - C_2 = \frac{x^2}{2} + \frac{1}{x} + C_1 - C_2$$

$$= \frac{x^2}{2} + \frac{1}{x} + C, \text{ where } C = C_1 - C_2 \text{ is another constant of integration.}$$

Note From now onwards, we shall write only one constant of integration in the final answer.

(ii) We have

$$\int (x^{\frac{2}{3}} + 1)\, dx = \int x^{\frac{2}{3}}\, dx + \int dx$$

$$= \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + x + C = \frac{3}{5} x^{\frac{5}{3}} + x + C$$

(iii) We have $\int (x^{\frac{3}{2}} + 2e^x - \frac{1}{x})\, dx = \int x^{\frac{3}{2}}\, dx + \int 2e^x\, dx - \int \frac{1}{x} dx$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 2e^x - \log|x| + C$$

$$= \frac{2}{5} x^{\frac{5}{2}} + 2e^x - \log|x| + C$$

Example 3 Find the following integrals:

(i) $\int (\sin x + \cos x)\, dx$ (ii) $\int \operatorname{cosec} x\, (\operatorname{cosec} x + \cot x)\, dx$

(iii) $\int \frac{1 - \sin x}{\cos^2 x}\, dx$

Solution

(i) We have

$$\int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx$$

$$= -\cos x + \sin x + C$$

(ii) We have

$$\int (\operatorname{cosec} x(\operatorname{cosec} x + \cot x)\, dx = \int \operatorname{cosec}^2 x\, dx + \int \operatorname{cosec} x \cot x\, dx$$

$$= -\cot x - \operatorname{cosec} x + C$$

(iii) We have

$$\int \frac{1-\sin x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx - \int \frac{\sin x}{\cos^2 x}\, dx$$

$$= \int \sec^2 x\, dx - \int \tan x \sec x\, dx$$

$$= \tan x - \sec x + C$$

Example 4 Find the anti derivative F of f defined by $f(x) = 4x^3 - 6$, where F (0) = 3

Solution One anti derivative of $f(x)$ is $x^4 - 6x$ since

$$\frac{d}{dx}(x^4 - 6x) = 4x^3 - 6$$

Therefore, the anti derivative F is given by

$$F(x) = x^4 - 6x + C, \text{ where C is constant.}$$

Given that $F(0) = 3$, which gives,

$$3 = 0 - 6 \times 0 + C \quad \text{or} \quad C = 3$$

Hence, the required anti derivative is the unique function F defined by

$$F(x) = x^4 - 6x + 3.$$

Remarks

(i) We see that if F is an anti derivative of f, then so is F + C, where C is any constant. Thus, if we know one anti derivative F of a function f, we can write down an infinite number of anti derivatives of f by adding any constant to F expressed by $F(x) + C, C \in \mathbf{R}$. In applications, it is often necessary to satisfy an additional condition which then determines a specific value of C giving unique anti derivative of the given function.

(ii) Sometimes, F is not expressible in terms of elementary functions viz., polynomial, logarithmic, exponential, trigonometric functions and their inverses etc. We are therefore blocked for finding $\int f(x)\, dx$. For example, it is not possible to find $\int e^{-x^2} dx$ by inspection since we can not find a function whose derivative is e^{-x^2}

(iii) When the variable of integration is denoted by a variable other than x, the integral formulae are modified accordingly. For instance

$$\int y^4\, dy = \frac{y^{4+1}}{4+1} + C = \frac{1}{5} y^5 + C$$

7.2.3 *Comparison between differentiation and integration*

1. Both are operations on functions.
2. Both satisfy the property of linearity, i.e.,

   (i) $\frac{d}{dx}\left[k_1 f_1(x) + k_2 f_2(x)\right] = k_1 \frac{d}{dx} f_1(x) + k_2 \frac{d}{dx} f_2(x)$

   (ii) $\int \left[k_1 f_1(x) + k_2 f_2(x)\right] dx = k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx$

   Here k_1 and k_2 are constants.
3. We have already seen that all functions are not differentiable. Similarly, all functions are not integrable. We will learn more about nondifferentiable functions and nonintegrable functions in higher classes.
4. The derivative of a function, when it exists, is a unique function. The integral of a function is not so. However, they are unique upto an additive constant, i.e., any two integrals of a function differ by a constant.
5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.
6. We can speak of the derivative at a point. We never speak of the integral at a point, we speak of the integral of a function over an interval on which the integral is defined as will be seen in Section 7.7.
7. The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at a point. Similarly, the indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal (perpendicular) to the axis representing the variable of integration.
8. The derivative is used for finding some physical quantities like the velocity of a moving particle, when the distance traversed at any time t is known. Similarly, the integral is used in calculating the distance traversed when the velocity at time t is known.
9. Differentiation is a process involving limits. **So is integration,** as will be seen in Section 7.7.

10. The process of differentiation and integration are inverses of each other as discussed in Section 7.2.2 (i).

EXERCISE 7.1

Find an anti derivative (or integral) of the following functions by the method of inspection.

1. $\sin 2x$ **2.** $\cos 3x$ **3.** e^{2x}

4. $(ax + b)^2$ **5.** $\sin 2x - 4\, e^{3x}$

Find the following integrals in Exercises 6 to 20:

6. $\int (4\, e^{3x} + 1)\, dx$ **7.** $\int x^2 \left(1 - \frac{1}{x^2}\right) dx$ **8.** $\int (ax^2 + bx + c)\, dx$

9. $\int (2x^2 + e^x)\, dx$ **10.** $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ **11.** $\int \frac{x^3 + 5x^2 - 4}{x^2}\, dx$

12. $\int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$ **13.** $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$ **14.** $\int (1 - x)\sqrt{x}\, dx$

15. $\int \sqrt{x}\,(3x^2 + 2x + 3)\, dx$ **16.** $\int (2x - 3\cos x + e^x)\, dx$

17. $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$ **18.** $\int \sec x\,(\sec x + \tan x)\, dx$

19. $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx$ **20.** $\int \frac{2 - 3\sin x}{\cos^2 x}\, dx.$

Choose the correct answer in Exercises 21 and 22.

21. The anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ equals

(A) $\frac{1}{3}x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ (B) $\frac{2}{3}x^{\frac{2}{3}} + \frac{1}{2}x^2 + C$

(C) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ (D) $\frac{3}{2}x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{1}{2}} + C$

22. If $\frac{d}{dx} f(x) = 4x^3 - \frac{3}{x^4}$ such that $f(2) = 0$. Then $f(x)$ is

(A) $x^4 + \frac{1}{x^3} - \frac{129}{8}$ (B) $x^3 + \frac{1}{x^4} + \frac{129}{8}$

(C) $x^4 + \frac{1}{x^3} + \frac{129}{8}$ (D) $x^3 + \frac{1}{x^4} - \frac{129}{8}$

7.3 Methods of Integration

In previous section, we discussed integrals of those functions which were readily obtainable from derivatives of some functions. It was based on inspection, i.e., on the search of a function F whose derivative is *f* which led us to the integral of *f*. However, this method, which depends on inspection, is not very suitable for many functions. Hence, we need to develop additional techniques or methods for finding the integrals by reducing them into standard forms. Prominent among them are methods based on:

1. Integration by Substitution
2. Integration using Partial Fractions
3. Integration by Parts

7.3.1 *Integration by substitution*

In this section, we consider the method of integration by substitution.

The given integral $\int f(x)\, dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$.

Consider $$I = \int f(x)\, dx$$

Put $x = g(t)$ so that $\frac{dx}{dt} = g'(t)$.

We write $$dx = g'(t)\, dt$$

Thus $$I = \int f(x)\, dx = \int f(g(t))\, g'(t)\, dt$$

This change of variable formula is one of the important tools available to us in the name of integration by substitution. It is often important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated in the following examples.

Example 5 Integrate the following functions w.r.t. x:

(i) $\sin mx$ (ii) $2x \sin(x^2 + 1)$

(iii) $\frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}}$ (iv) $\frac{\sin(\tan^{-1} x)}{1+x^2}$

Solution

(i) We know that derivative of mx is m. Thus, we make the substitution $mx = t$ so that $mdx = dt$.

Therefore, $$\int \sin mx\, dx = \frac{1}{m}\int \sin t\, dt = -\frac{1}{m}\cos t + C = -\frac{1}{m}\cos mx + C$$

(ii) Derivative of $x^2 + 1$ is $2x$. Thus, we use the substitution $x^2 + 1 = t$ so that $2x\ dx = dt$.

Therefore, $\int 2x \sin (x^2 + 1)\ dx = \int \sin t\ dt = -\cos t + C = -\cos (x^2 + 1) + C$

(iii) Derivative of $\sqrt{x}$ is $\frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. Thus, we use the substitution

$\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}} dx = dt$ giving $dx = 2t\ dt$.

Thus, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \int \frac{2t \tan^4 t \sec^2 t\ dt}{t} = 2 \int \tan^4 t\ \sec^2 t\ dt$$

Again, we make another substitution $\tan t = u$ so that $\sec^2 t\ dt = du$

Therefore, $$2 \int \tan^4 t \sec^2 t\ dt = 2 \int u^4\ du = 2 \frac{u^5}{5} + C$$

$$= \frac{2}{5} \tan^5 t + C \text{ (since } u = \tan t)$$

$$= \frac{2}{5} \tan^5 \sqrt{x} + C \text{ (since } t = \sqrt{x})$$

Hence, $$\int \frac{\tan^4 \sqrt{x} \sec^2 \sqrt{x}}{\sqrt{x}} dx = \frac{2}{5} \tan^5 \sqrt{x} + C$$

Alternatively, make the substitution $\tan \sqrt{x} = t$

(iv) Derivative of $\tan^{-1} x = \frac{1}{1+x^2}$. Thus, we use the substitution

$$\tan^{-1} x = t \text{ so that } \frac{dx}{1+x^2} = dt.$$

Therefore, $$\int \frac{\sin (\tan^{-1} x)}{1+x^2} dx = \int \sin t\ dt = -\cos t + C = -\cos (\tan^{-1} x) + C$$

Now, we discuss some important integrals involving trigonometric functions and their standard integrals using substitution technique. These will be used later without reference.

(i) $\int \tan x\ dx = \log |\sec x| + C$

We have

$$\int \tan x\ dx = \int \frac{\sin x}{\cos x} dx$$

Put $\cos x = t$ so that $\sin x\, dx = -dt$

Then $$\int \tan x\, dx = -\int \frac{dt}{t} = -\log|t| + C = -\log|\cos x| + C$$

or $$\int \tan x\, dx = \log|\sec x| + C$$

(ii) $\int \cot x\, dx = \log|\sin x| + C$

We have $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

Put $\sin x = t$ so that $\cos x\, dx = dt$

Then $$\int \cot x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sin x| + C$$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

We have

$$\int \sec x\, dx = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\, dx$$

Put $\sec x + \tan x = t$ so that $\sec x\,(\tan x + \sec x)\, dx = dt$

Therefore, $\int \sec x\, dx = \int \frac{dt}{t} = \log|t| + C = \log|\sec x + \tan x| + C$

(iv) $\int \text{cosec}\, x\, dx = \log|\text{cosec}\, x - \cot x| + C$

We have

$$\int \text{cosec}\, x\, dx = \int \frac{\text{cosec}\, x\,(\text{cosec}\, x + \cot x)}{(\text{cosec}\, x + \cot x)}\, dx$$

Put $\text{cosec}\, x + \cot x = t$ so that $-\text{cosec}\, x\,(\text{cosec}\, x + \cot x)\, dx = dt$

So $$\int \text{cosec}\, x\, dx = -\int \frac{dt}{t} = -\log|t| = -\log|\text{cosec}\, x + \cot x| + C$$

$$= -\log\left|\frac{\text{cosec}^2 x - \cot^2 x}{\text{cosec}\, x - \cot x}\right| + C$$

$$= \log|\text{cosec}\, x - \cot x| + C$$

Example 6 Find the following integrals:

(i) $\int \sin^3 x \cos^2 x\, dx$ (ii) $\int \frac{\sin x}{\sin(x + a)}\, dx$ (iii) $\int \frac{1}{1 + \tan x}\, dx$

Solution

(i) We have

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x\, (\sin x)\, dx$$

$$= \int (1-\cos^2 x) \cos^2 x\, (\sin x)\, dx$$

Put $t = \cos x$ so that $dt = -\sin x\, dx$

Therefore, $\int \sin^2 x \cos^2 x\, (\sin x)\, dx = -\int (1-t^2)\, t^2\, dt$

$$= -\int (t^2 - t^4)\, dt = -\left(\frac{t^3}{3} - \frac{t^5}{5}\right) + C$$

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C$$

(ii) Put $x + a = t$. Then $dx = dt$. Therefore

$$\int \frac{\sin x}{\sin (x+a)}\, dx = \int \frac{\sin (t-a)}{\sin t}\, dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t}\, dt$$

$$= \cos a \int dt - \sin a \int \cot t\, dt$$

$$= (\cos a)\, t - (\sin a)\left[\log |\sin t| + C_1\right]$$

$$= (\cos a)\, (x+a) - (\sin a)\left[\log |\sin (x+a)| + C_1\right]$$

$$= x \cos a + a \cos a - (\sin a) \log |\sin (x+a)| - C_1 \sin a$$

Hence, $\int \frac{\sin x}{\sin (x+a)}\, dx = x \cos a - \sin a \log |\sin (x+a)| + C,$

where, $C = -C_1 \sin a + a \cos a$, is another arbitrary constant.

(iii) $\int \frac{dx}{1+\tan x} = \int \frac{\cos x\, dx}{\cos x + \sin x}$

$$= \frac{1}{2}\int \frac{(\cos x + \sin x + \cos x - \sin x)\, dx}{\cos x + \sin x}$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$$

$$= \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx \qquad \ldots (1)$$

Now, consider $I = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx$

Put $\cos x + \sin x = t$ so that $(\cos x - \sin x)\,dx = dt$

Therefore $\quad I = \int \frac{dt}{t} = \log|t| + C_2 = \log|\cos x + \sin x| + C_2$

Putting it in (1), we get

$$\int \frac{dx}{1+\tan x} = \frac{x}{2} + \frac{C_1}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + \frac{C_1}{2} + \frac{C_2}{2}$$

$$= \frac{x}{2} + \frac{1}{2}\log|\cos x + \sin x| + C, \left(C = \frac{C_1}{2} + \frac{C_2}{2}\right)$$

EXERCISE 7.2

Integrate the functions in Exercises 1 to 37:

1. $\frac{2x}{1+x^2}$ **2.** $\frac{(\log x)^2}{x}$ **3.** $\frac{1}{x + x\log x}$

4. $\sin x \sin(\cos x)$ **5.** $\sin(ax+b)\cos(ax+b)$

6. $\sqrt{ax+b}$ **7.** $x\sqrt{x+2}$ **8.** $x\sqrt{1+2x^2}$

9. $(4x+2)\sqrt{x^2+x+1}$ **10.** $\frac{1}{x-\sqrt{x}}$ **11.** $\frac{x}{\sqrt{x+4}}, x>0$

12. $(x^3-1)^{\frac{1}{3}}x^5$ **13.** $\frac{x^2}{(2+3x^3)^3}$ **14.** $\frac{1}{x(\log x)^m}, x>0, m\neq 1$

15. $\frac{x}{9-4x^2}$ **16.** e^{2x+3} **17.** $\frac{x}{e^{x^2}}$

18. $\dfrac{e^{\tan^{-1}x}}{1+x^2}$ **19.** $\dfrac{e^{2x}-1}{e^{2x}+1}$ **20.** $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

21. $\tan^2(2x-3)$ **22.** $\sec^2(7-4x)$ **23.** $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

24. $\dfrac{2\cos x-3\sin x}{6\cos x+4\sin x}$ **25.** $\dfrac{1}{\cos^2 x(1-\tan x)^2}$ **26.** $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$

27. $\sqrt{\sin 2x}\cos 2x$ **28.** $\dfrac{\cos x}{\sqrt{1+\sin x}}$ **29.** $\cot x \log \sin x$

30. $\dfrac{\sin x}{1+\cos x}$ **31.** $\dfrac{\sin x}{(1+\cos x)^2}$ **32.** $\dfrac{1}{1+\cot x}$

33. $\dfrac{1}{1-\tan x}$ **34.** $\dfrac{\sqrt{\tan x}}{\sin x\cos x}$ **35.** $\dfrac{(1+\log x)^2}{x}$

36. $\dfrac{(x+1)(x+\log x)^2}{x}$ **37.** $\dfrac{x^3\sin(\tan^{-1}x^4)}{1+x^8}$

Choose the correct answer in Exercises 38 and 39.

38. $\int \dfrac{10x^9+10^x\log_{e^{10}} dx}{x^{10}+10^x}$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

39. $\int \dfrac{dx}{\sin^2 x\cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

7.3.2 *Integration using trigonometric identities*

When the integrand involves some trigonometric functions, we use some known identities to find the integral as illustrated through the following example.

Example 7 Find (i) $\int \cos^2 x\, dx$ (ii) $\int \sin 2x \cos 3x\, dx$ (iii) $\int \sin^3 x\, dx$

Solution

(i) Recall the identity $\cos 2x = 2\cos^2 x - 1$, which gives

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Therefore, $$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C$$

(ii) Recall the identity $\sin x \cos y = \frac{1}{2}[\sin (x+y) + \sin (x-y)]$ (Why?)

Then $$\int \sin 2x \cos 3x\,dx = \frac{1}{2}\left[\int \sin 5x\,dx \bullet \int \sin x\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right] + C$$

$$= -\frac{1}{10}\cos 5x + \frac{1}{2}\cos x + C$$

(iii) From the identity $\sin 3x = 3\sin x - 4\sin^3 x$, we find that

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Therefore, $$\int \sin^3 x\,dx = \frac{3}{4}\int \sin x\,dx - \frac{1}{4}\int \sin 3x\,dx$$

$$= -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$$

Alternatively, $\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$

Put $\cos x = t$ so that $-\sin x\,dx = dt$

Therefore, $$\int \sin^3 x\,dx = -\int \left(1-t^2\right) dt = -\int dt + \int t^2\,dt = -t + \frac{t^3}{3} + C$$

$$= -\cos x + \frac{1}{3}\cos^3 x + C$$

Remark It can be shown using trigonometric identities that both answers are equivalent.

EXERCISE 7.3

Find the integrals of the functions in Exercises 1 to 22:

1. $\sin^2 (2x + 5)$ **2.** $\sin 3x \cos 4x$ **3.** $\cos 2x \cos 4x \cos 6x$

4. $\sin^3 (2x + 1)$ **5.** $\sin^3 x \cos^3 x$ **6.** $\sin x \sin 2x \sin 3x$

7. $\sin 4x \sin 8x$ **8.** $\dfrac{1-\cos x}{1+\cos x}$ **9.** $\dfrac{\cos x}{1+\cos x}$

10. $\sin^4 x$ **11.** $\cos^4 2x$ **12.** $\dfrac{\sin^2 x}{1+\cos x}$

13. $\dfrac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}$ **14.** $\dfrac{\cos x - \sin x}{1+\sin 2x}$ **15.** $\tan^3 2x \sec 2x$

16. $\tan^4 x$ **17.** $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$ **18.** $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

19. $\dfrac{1}{\sin x \cos^3 x}$ **20.** $\dfrac{\cos 2x}{(\cos x + \sin x)^2}$ **21.** $\sin^{-1} (\cos x)$

22. $\dfrac{1}{\cos (x-a) \cos (x-b)}$

Choose the correct answer in Exercises 23 and 24.

23. $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\, dx$ is equal to

(A) $\tan x + \cot x + C$ (B) $\tan x + \operatorname{cosec} x + C$

(C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

24. $\displaystyle\int \frac{e^x (1+x)}{\cos^2 (e^x x)}\, dx$ equals

(A) $-\cot (ex^x) + C$ (B) $\tan (xe^x) + C$

(C) $\tan (e^x) + C$ (D) $\cot (e^x) + C$

7.4 Integrals of Some Particular Functions

In this section, we mention below some important formulae of integrals and apply them for integrating many other related standard integrals:

(1) $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

(3) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

(4) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$$

(5) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$$

(6) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log \left| x + \sqrt{x^2 + a^2} \right| + C$$

We now prove the above results:

(1) We have $\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)}$

$$= \frac{1}{2a}\left[\frac{(x+a)-(x-a)}{(x-a)(x+a)}\right] = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

Therefore, $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$$

$$= \frac{1}{2a}\left[\log|(x-a)| - \log|(x+a)|\right] + C$$

$$= \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

(2) In view of (1) above, we have

$$\frac{1}{a^2 - x^2} = \frac{1}{2a}\left[\frac{(a+x)+(a-x)}{(a+x)(a-x)}\right] = \frac{1}{2a}\left[\frac{1}{a-x} + \frac{1}{a+x}\right]$$

Therefore, $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\left[\int \frac{dx}{a-x} + \int \frac{dx}{a+x}\right]$

$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|] + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

Note The technique used in (1) will be explained in Section 7.5.

(3) Put $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

Therefore, $\int \frac{dx}{x^2 + a^2} = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2}$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) Let $x = a\sec\theta$. Then $dx = a\sec\theta\tan\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$$= \int \sec\theta\, d\theta = \log|\sec\theta + \tan\theta| + C_1$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| - \log|a| + C_1$$

$$= \log\left|x + \sqrt{x^2 - a^2}\right| + C, \text{ where } C = C_1 - \log|a|$$

(5) Let $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$

$$= \int d\theta = \theta + C = \sin^{-1}\frac{x}{a} + C$$

(6) Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\, d\theta$.

Therefore, $\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta + a^2}}$

$$= \int \sec\theta\, d\theta = \log|(\sec\theta + \tan\theta)| + C_1$$

$$= \log \left| \frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1} \right| + C_1$$

$$= \log \left| x + \sqrt{x^2 + a^2} \right| - \log |a| + C_1$$

$$= \log \left| x + \sqrt{x^2 + a^2} \right| + C \text{, where } C = C_1 - \log |a|$$

Applying these standard formulae, we now obtain some more formulae which are useful from applications point of view and can be applied directly to evaluate other integrals.

(7) To find the integral $\int \frac{dx}{ax^2 + bx + c}$, we write

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

Now, put $x + \frac{b}{2a} = t$ so that $dx = dt$ and writing $\frac{c}{a} - \frac{b^2}{4a^2} = \pm k^2$. We find the integral reduced to the form $\frac{1}{a}\int \frac{dt}{t^2 \pm k^2}$ depending upon the sign of $\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)$ and hence can be evaluated.

(8) To find the integral of the type $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, proceeding as in (7), we obtain the integral using the standard formulae.

(9) To find the integral of the type $\int \frac{px + q}{ax^2 + bx + c}\, dx$, where p, q, a, b, c are constants, we are to find real numbers A, B such that

$$px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A\,(2ax + b) + B$$

To determine A and B, we equate from both sides the coefficients of x and the constant terms. A and B are thus obtained and hence the integral is reduced to one of the known forms.

(10) For the evaluation of the integral of the type $\int \frac{(px+q)\,dx}{\sqrt{ax^2+bx+c}}$, we proceed as in (9) and transform the integral into known standard forms.

Let us illustrate the above methods by some examples.

Example 8 Find the following integrals:

(i) $\int \frac{dx}{x^2-16}$ (ii) $\int \frac{dx}{\sqrt{2x-x^2}}$

Solution

(i) We have $\int \frac{dx}{x^2-16} = \int \frac{dx}{x^2-4^2} = \frac{1}{8}\log\left|\frac{x-4}{x+4}\right| + C$ [by 7.4 (1)]

(ii) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

Put $x - 1 = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}(t) + C$ [by 7.4 (5)]

$$= \sin^{-1}(x-1) + C$$

Example 9 Find the following integrals :

(i) $\int \frac{dx}{x^2-6x+13}$ (ii) $\int \frac{dx}{3x^2+13x-10}$ (iii) $\int \frac{dx}{\sqrt{5x^2-2x}}$

Solution

(i) We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2-6x+13} = \int \frac{1}{(x-3)^2+2^2}\,dx$

Let $x - 3 = t$. Then $dx = dt$

Therefore, $\int \frac{dx}{x^2-6x+13} = \int \frac{dt}{t^2+2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C$ [by 7.4 (3)]

$$= \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$$

(ii) The given integral is of the form 7.4 (7). We write the denominator of the integrand,

$$3x^2 + 13x - 10 = 3\left(x^2 + \frac{13x}{3} - \frac{10}{3}\right)$$

$$= 3\left[\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2\right] \text{ (completing the square)}$$

Thus $\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dx}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}$

Put $x + \frac{13}{6} = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{3x^2 + 13x - 10} = \frac{1}{3}\int \frac{dt}{t^2 - \left(\frac{17}{6}\right)^2}$

$$= \frac{1}{3 \times 2 \times \frac{17}{6}} \log \left|\frac{t - \frac{17}{6}}{t + \frac{17}{6}}\right| + C_1 \qquad \text{[by 7.4 (i)]}$$

$$= \frac{1}{17} \log \left|\frac{x + \frac{13}{6} - \frac{17}{6}}{x + \frac{13}{6} + \frac{17}{6}}\right| + C_1$$

$$= \frac{1}{17} \log \left|\frac{6x - 4}{6x + 30}\right| + C_1$$

$$= \frac{1}{17} \log \left|\frac{3x - 2}{x + 5}\right| + C_1 + \frac{1}{17} \log \frac{1}{3}$$

$$= \frac{1}{17} \log \left|\frac{3x - 2}{x + 5}\right| + C, \text{ where } C = C_1 + \frac{1}{17} \log \frac{1}{3}$$

(iii) We have $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \int \frac{dx}{\sqrt{5\left(x^2 - \frac{2x}{5}\right)}}$

$$= \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{\left(x - \frac{1}{5}\right)^2 - \left(\frac{1}{5}\right)^2}} \quad \text{(completing the square)}$$

Put $x - \frac{1}{5} = t$. Then $dx = dt$.

Therefore, $\int \frac{dx}{\sqrt{5x^2 - 2x}} = \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{5}\right)^2}}$

$$= \frac{1}{\sqrt{5}} \log \left| t + \sqrt{t^2 - \left(\frac{1}{5}\right)^2} \right| + C \quad \text{[by 7.4 (4)]}$$

$$= \frac{1}{\sqrt{5}} \log \left| x - \frac{1}{5} + \sqrt{x^2 - \frac{2x}{5}} \right| + C$$

Example 10 Find the following integrals:

(i) $\int \frac{x+2}{2x^2 + 6x + 5} dx$ (ii) $\int \frac{x+3}{\sqrt{5 - 4x + x^2}} dx$

Solution

(i) Using the formula 7.4 (9), we express

$$x + 2 = A \frac{d}{dx}\left(2x^2 + 6x + 5\right) + B = A(4x + 6) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$4A = 1$ and $6A + B = 2$ or $A = \frac{1}{4}$ and $B = \frac{1}{2}$.

Therefore, $\int \frac{x+2}{2x^2 + 6x + 5} = \frac{1}{4} \int \frac{4x + 6}{2x^2 + 6x + 5} dx + \frac{1}{2} \int \frac{dx}{2x^2 + 6x + 5}$

$$= \frac{1}{4} I_1 + \frac{1}{2} I_2 \quad \text{(say)} \quad \ldots (1)$$

In I_1, put $2x^2 + 6x + 5 = t$, so that $(4x + 6)\, dx = dt$

Therefore, $$I_1 = \int \frac{dt}{t} = \log |t| + C_1$$

$$= \log |2x^2 + 6x + 5| + C_1 \qquad \text{... (2)}$$

and $$I_2 = \int \frac{dx}{2x^2 + 6x + 5} = \frac{1}{2}\int \frac{dx}{x^2 + 3x + \frac{5}{2}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

Put $x + \frac{3}{2} = t$, so that $dx = dt$, we get

$$I_2 = \frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2 \times \frac{1}{2}} \tan^{-1} 2t + C_2 \qquad \text{[by 7.4 (3)]}$$

$$= \tan^{-1} 2\left(x + \frac{3}{2}\right) + C_2 = \tan^{-1}(2x + 3) + C_2 \qquad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$\int \frac{x + 2}{2x^2 + 6x + 5}\, dx = \frac{1}{4}\log\left|2x^2 + 6x + 5\right| + \frac{1}{2}\tan^{-1}(2x + 3) + C$$

where, $$C = \frac{C_1}{4} + \frac{C_2}{2}$$

(ii) This integral is of the form given in 7.4 (10). Let us express

$$x + 3 = A\frac{d}{dx}(5 - 4x - x^2) + B = A(-4 - 2x) + B$$

Equating the coefficients of x and the constant terms from both sides, we get

$-2A = 1$ and $-4A + B = 3$, i.e., $A = -\frac{1}{2}$ and $B = 1$

Therefore, $\int \frac{x+3}{\sqrt{5-4x-x^2}}\,dx = -\frac{1}{2}\int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{5-4x-x^2}}$

$$= -\frac{1}{2}\,I_1 + I_2 \qquad \ldots (1)$$

In I_1, put $5 - 4x - x^2 = t$, so that $(-4 - 2x)\,dx = dt$.

Therefore, $I_1 = \int \frac{(-4-2x)\,dx}{\sqrt{5-4x-x^2}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + C_1$

$$= 2\sqrt{5-4x-x^2} + C_1 \qquad \ldots (2)$$

Now consider $I_2 = \int \frac{dx}{\sqrt{5-4x-x^2}} = \int \frac{dx}{\sqrt{9-(x+2)^2}}$

Put $x + 2 = t$, so that $dx = dt$.

Therefore, $I_2 = \int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1}\frac{t}{3} + C_2$ [by 7.4 (5)]

$$= \sin^{-1}\frac{x+2}{3} + C_2 \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{\sqrt{5-4x-x^2}} = -\sqrt{5-4x-x^2} + \sin^{-1}\frac{x+2}{3} + C, \text{ where } C = C_2 - \frac{C_1}{2}$$

EXERCISE 7.4

Integrate the functions in Exercises 1 to 23.

1. $\frac{3x^2}{x^6+1}$ **2.** $\frac{1}{\sqrt{1+4x^2}}$ **3.** $\frac{1}{\sqrt{(2-x)^2+1}}$

4. $\frac{1}{\sqrt{9-25x^2}}$ **5.** $\frac{3x}{1+2x^4}$ **6.** $\frac{x^2}{1-x^6}$

7. $\frac{x-1}{\sqrt{x^2-1}}$ **8.** $\frac{x^2}{\sqrt{x^6+a^6}}$ **9.** $\frac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

10. $\frac{1}{\sqrt{x^2+2x+2}}$ **11.** $\frac{1}{9x^2+6x+5}$ **12.** $\frac{1}{\sqrt{7-6x-x^2}}$

13. $\frac{1}{\sqrt{(x-1)(x-2)}}$ **14.** $\frac{1}{\sqrt{8+3x-x^2}}$ **15.** $\frac{1}{\sqrt{(x-a)(x-b)}}$

16. $\frac{4x+1}{\sqrt{2x^2+x-3}}$ **17.** $\frac{x+2}{\sqrt{x^2-1}}$ **18.** $\frac{5x-2}{1+2x+3x^2}$

19. $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$ **20.** $\frac{x+2}{\sqrt{4x-x^2}}$ **21.** $\frac{x+2}{\sqrt{x^2+2x+3}}$

22. $\frac{x+3}{x^2-2x-5}$ **23.** $\frac{5x+3}{\sqrt{x^2+4x+10}}$.

Choose the correct answer in Exercises 24 and 25.

24. $\int \frac{dx}{x^2+2x+2}$ equals

(A) $x \tan^{-1}(x+1) + C$ (B) $\tan^{-1}(x+1) + C$

(C) $(x+1)\tan^{-1}x + C$ (D) $\tan^{-1}x + C$

25. $\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

7.5 Integration by Partial Fractions

Recall that a rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P (x) and Q(x) are polynomials in x and Q$(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational functions can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$,

where $T(x)$ is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is a proper rational function. As we know how to integrate polynomials, the integration of any rational function is reduced to the integration of a proper rational function. The rational functions which we shall consider here for integration purposes will be those whose denominators can be factorised into linear and quadratic factors. Assume that we want to evaluate $\int \frac{P(x)}{Q(x)} dx$, where $\frac{P(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods. The following Table 7.2 indicates the types of simpler partial fractions that are to be associated with various kind of rational functions.

Table 7.2

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where $x^2 + bx + c$ cannot be factorised further	$\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$,

In the above table, A, B and C are real numbers to be determined suitably.

Example 11 Find $\int \frac{dx}{(x+1)(x+2)}$

Solution The integrand is a proper rational function. Therefore, by using the form of partial fraction [Table 7.2 (i)], we write

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \quad \text{... (1)}$$

where, real numbers A and B are to be determined suitably. This gives

$$1 = A(x+2) + B(x+1).$$

Equating the coefficients of x and the constant term, we get

$$A + B = 0$$

and

$$2A + B = 1$$

Solving these equations, we get A =1 and B = – 1.

Thus, the integrand is given by

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{-1}{x+2}$$

Therefore,

$$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log|x+1| - \log|x+2| + C$$

$$= \log\left|\frac{x+1}{x+2}\right| + C$$

Remark The equation (1) above is an identity, i.e. a statement true for all (permissible) values of x. Some authors use the symbol '≡' to indicate that the statement is an identity and use the symbol '=' to indicate that the statement is an equation, i.e., to indicate that the statement is true only for certain values of x.

Example 12 Find $\int \frac{x^2+1}{x^2-5x+6} dx$

Solution Here the integrand $\frac{x^2+1}{x^2-5x+6}$ is not proper rational function, so we divide $x^2 + 1$ by $x^2 - 5x + 6$ and find that

$$\frac{x^2+1}{x^2-5x+6} = 1+\frac{5x-5}{x^2-5x+6} = 1+\frac{5x-5}{(x-2)\,(x-3)}$$

Let $$\frac{5x-5}{(x-2)\,(x-3)} = \frac{A}{x-2}+\frac{B}{x-3}$$

So that $$5x-5 = A\,(x-3)+B\,(x-2)$$

Equating the coefficients of x and constant terms on both sides, we get A + B = 5 and 3A + 2B = 5. Solving these equations, we get A = – 5 and B = 10

Thus, $$\frac{x^2+1}{x^2-5x+6} = 1-\frac{5}{x-2}+\frac{10}{x-3}$$

Therefore, $$\int \frac{x^2+1}{x^2-5x+6}\,dx = \int dx - 5\int \frac{1}{x-2}\,dx + 10\int \frac{dx}{x-3}$$

$$= x - 5\log|x-2| + 10\log|x-3| + C.$$

Example 13 Find $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$

Solution The integrand is of the type as given in Table 7.2 (4). We write

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$$

So that $$3x-2 = A\,(x+1)\,(x+3)+B\,(x+3)+C\,(x+1)^2$$

$$= A\,(x^2+4x+3)+B\,(x+3)+C\,(x^2+2x+1\,)$$

Comparing coefficient of x^2, x and constant term on both sides, we get A + C = 0, 4A + B + 2C = 3 and 3A + 3B + C = – 2. Solving these equations, we get

$A=\frac{11}{4}, B=\frac{-5}{2}$ and $C=\frac{-11}{4}$. Thus the integrand is given by

$$\frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4\,(x+1)}-\frac{5}{2\,(x+1)^2}-\frac{11}{4\,(x+3)}$$

Therefore, $$\int \frac{3x-2}{(x+1)^2(x+3)} = \frac{11}{4}\int \frac{dx}{x+1}-\frac{5}{2}\int \frac{dx}{(x+1)^2}-\frac{11}{4}\int \frac{dx}{x+3}$$

$$= \frac{11}{4}\log|x+1|+\frac{5}{2\,(x+1)}-\frac{11}{4}\log|x+3|+C$$

$$= \frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2\,(x+1)}+C$$

Example 14 Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Solution Consider $\frac{x^2}{(x^2+1)(x^2+4)}$ and put $x^2 = y$.

Then $$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

Write $$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$$

So that $$y = A(y+4) + B(y+1)$$

Comparing coefficients of y and constant terms on both sides, we get A + B = 1 and 4A + B = 0, which give

$$A = -\frac{1}{3} \quad \text{and} \quad B = \frac{4}{3}$$

Thus, $$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

Therefore, $$\int \frac{x^2 dx}{(x^2+1)(x^2+4)} = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

In the above example, the substitution was made only for the partial fraction part and not for the integration part. Now, we consider an example, where the integration involves a combination of the substitution method and the partial fraction method.

Example 15 Find $\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi} d\phi$

Solution Let $y = \sin\phi$

Then $$dy = \cos\phi \; d\phi$$

Therefore, $$\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi}\,d\phi = \int \frac{(3y-2)\,dy}{5-(1-y^2)-4y}$$

$$= \int \frac{3y-2}{y^2-4y+4}\,dy$$

$$= \int \frac{3y-2}{(y-2)^2} = \text{I (say)}$$

Now, we write $$\frac{3y-2}{(y-2)^2} = \frac{\text{A}}{y-2} + \frac{\text{B}}{(y-2)^2}$$ [by Table 7.2 (2)]

Therefore, $3y - 2 = \text{A}\,(y-2) + \text{B}$

Comparing the coefficients of y and constant term, we get A = 3 and B – 2A = – 2, which gives A = 3 and B = 4.

Therefore, the required integral is given by

$$\text{I} = \int [\frac{3}{y-2} + \frac{4}{(y-2)^2}]\,dy = 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2}$$

$$= 3\log|y-2| + 4\left(-\frac{1}{y-2}\right) + \text{C}$$

$$= 3\log|\sin\phi - 2| + \frac{4}{2-\sin\phi} + \text{C}$$

$$= 3\log(2-\sin\phi) + \frac{4}{2-\sin\phi} + \text{C}$$ (since, $2 - \sin\phi$ is always positive)

Example 16 Find $\int \frac{x^2+x+1\,dx}{(x+2)\,(x^2+1)}$

Solution The integrand is a proper rational function. Decompose the rational function into partial fraction [Table 2.2(5)]. Write

$$\frac{x^2+x+1}{(x^2+1)\,(x+2)} = \frac{\text{A}}{x+2} + \frac{\text{B}x+\text{C}}{(x^2+1)}$$

Therefore, $x^2 + x + 1 = \text{A}\,(x^2+1) + (\text{B}x + \text{C})\,(x+2)$

Equating the coefficients of x^2, x and of constant term of both sides, we get A + B =1, 2B + C = 1 and A + 2C = 1. Solving these equations, we get $A=\frac{3}{5}, B=\frac{2}{5}$ and $C=\frac{1}{5}$

Thus, the integrand is given by

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{3}{5(x+2)}+\frac{\frac{2}{5}x+\frac{1}{5}}{x^2+1} = \frac{3}{5(x+2)}+\frac{1}{5}\left(\frac{2x+1}{x^2+1}\right)$$

Therefore,
$$\int\frac{x^2+x+1}{(x^2+1)(x+2)}dx = \frac{3}{5}\int\frac{dx}{x+2}+\frac{1}{5}\int\frac{2x}{x^2+1}dx+\frac{1}{5}\int\frac{1}{x^2+1}dx$$

$$= \frac{3}{5}\log|x+2|+\frac{1}{5}\log\left|x^2+1\right|+\frac{1}{5}\tan^{-1}x+C$$

EXERCISE 7.5

Integrate the rational functions in Exercises 1 to 21.

1. $\frac{x}{(x+1)(x+2)}$ **2.** $\frac{1}{x^2-9}$ **3.** $\frac{3x-1}{(x-1)(x-2)(x-3)}$

4. $\frac{x}{(x-1)(x-2)(x-3)}$ **5.** $\frac{2x}{x^2+3x+2}$ **6.** $\frac{1-x^2}{x(1-2x)}$

7. $\frac{x}{(x^2+1)(x-1)}$ **8.** $\frac{x}{(x-1)^2(x+2)}$ **9.** $\frac{3x+5}{x^3-x^2-x+1}$

10. $\frac{2x-3}{(x^2-1)(2x+3)}$ **11.** $\frac{5x}{(x+1)(x^2-4)}$ **12.** $\frac{x^3+x+1}{x^2-1}$

13. $\frac{2}{(1-x)(1+x^2)}$ **14.** $\frac{3x-1}{(x+2)^2}$ **15.** $\frac{1}{x^4-1}$

16. $\frac{1}{x(x^n+1)}$ [Hint: multiply numerator and denominator by x^{n-1} and put $x^n=t$]

17. $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ [Hint : Put $\sin x = t$]

18. $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ **19.** $\frac{2x}{(x^2+1)(x^2+3)}$ **20.** $\frac{1}{x(x^4-1)}$

21. $\frac{1}{(e^x-1)}$ [Hint : Put $e^x = t$]

Choose the correct answer in each of the Exercises 22 and 23.

22. $\int \frac{x\,dx}{(x-1)(x-2)}$ equals

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$ (B) $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$ (D) $\log|(x-1)(x-2)| + C$

23. $\int \frac{dx}{x(x^2+1)}$ equals

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$ (B) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$ (D) $\frac{1}{2}\log|x| + \log(x^2+1) + C$

7.6 Integration by Parts

In this section, we describe one more method of integration, that is found quite useful in integrating products of functions.

If u and v are any two differentiable functions of a single variable x (say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides, we get

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

or $$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \quad \text{... (1)}$$

Let $$u = f(x) \text{ and } \frac{dv}{dx} = g(x). \text{ Then}$$

$$\frac{du}{dx} = f'(x) \text{ and } v = \int g(x)\,dx$$

Therefore, expression (1) can be rewritten as

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [\int g(x)\, dx]\, f'(x)\, dx$$

i.e.,

$$\int f(x)\, g(x)\, dx = f(x)\int g(x)\, dx - \int [f'(x) \int g(x)\, dx]\, dx$$

If we take f as the first function and g as the second function, then this formula may be stated as follows:

"The integral of the product of two functions = (first function) × (integral of the second function) – Integral of [(differential coefficient of the first function) × (integral of the second function)]"

Example 17 Find $\int x \cos x\, dx$

Solution Put $f(x) = x$ (first function) and $g(x) = \cos x$ (second function).
Then, integration by parts gives

$$\int x \cos x\, dx = x \int \cos x\, dx - \int [\frac{d}{dx}(x) \int \cos x\, dx]\, dx$$

$$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + C$$

Suppose, we take $f(x) = \cos x$ and $g(x) = x$. Then

$$\int x \cos x\, dx = \cos x \int x\, dx - \int [\frac{d}{dx}(\cos x) \int x\, dx]\, dx$$

$$= (\cos x)\frac{x^2}{2} + \int \sin x\, \frac{x^2}{2} dx$$

Thus, it shows that the integral $\int x \cos x\, dx$ is reduced to the comparatively more complicated integral having more power of x. Therefore, the proper choice of the first function and the second function is significant.

Remarks

(i) It is worth mentioning that integration by parts is not applicable to product of functions in all cases. For instance, the method does not work for $\int \sqrt{x} \sin x\, dx$. The reason is that there does not exist any function whose derivative is $\sqrt{x}$ sin x.

(ii) Observe that while finding the integral of the second function, we did not add any constant of integration. If we write the integral of the second function cos x

as $\sin x + k$, where k is any constant, then

$$\int x \cos x\, dx = x(\sin x + k) - \int (\sin x + k)\, dx$$

$$= x(\sin x + k) - \int (\sin x\, dx - \int k\, dx$$

$$= x(\sin x + k) - \cos x - kx + C = x \sin x + \cos x + C$$

This shows that adding a constant to the integral of the second function is superfluous so far as the final result is concerned while applying the method of integration by parts.

(iii) Usually, if any function is a power of x or a polynomial in x, then we take it as the first function. However, in cases where other function is inverse trigonometric function or logarithmic function, then we take them as first function.

Example 18 Find $\int \log x\, dx$

Solution To start with, we are unable to guess a function whose derivative is $\log x$. We take $\log x$ as the first function and the constant function 1 as the second function. Then, the integral of the second function is x.

Hence,
$$\int (\log x . 1)\, dx = \log x \int 1\, dx - \int [\frac{d}{dx}(\log x) \int 1\, dx]\, dx$$

$$= (\log x) \cdot x - \int \frac{1}{x} x\, dx = x \log x - x + C.$$

Example 19 Find $\int x e^x dx$

Solution Take first function as x and second function as e^x. The integral of the second function is e^x.

Therefore,
$$\int x e^x dx = x e^x - \int 1 \cdot e^x dx = xe^x - e^x + C.$$

Example 20 Find $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution Let first function be $\sin^{-1} x$ and second function be $\frac{x}{\sqrt{1-x^2}}$.

First we find the integral of the second function, i.e., $\int \frac{x\, dx}{\sqrt{1-x^2}}$.

Put $t = 1 - x^2$. Then $dt = -2x\, dx$

Therefore, $$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

Hence, $$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx = (\sin^{-1}x)\left(-\sqrt{1-x^2}\right) - \int \frac{1}{\sqrt{1-x^2}}(-\sqrt{1-x^2})\,dx$$
$$= -\sqrt{1-x^2}\,\sin^{-1}x + x + C = x - \sqrt{1-x^2}\,\sin^{-1}x + C$$

Alternatively, this integral can also be worked out by making substitution $\sin^{-1}x = \theta$ and then integrating by parts.

Example 21 Find $\int e^x \sin x\,dx$

Solution Take e^x as the first function and $\sin x$ as second function. Then, integrating by parts, we have

$$I = \int e^x \sin x\,dx = e^x(-\cos x) + \int e^x \cos x\,dx$$
$$= -e^x \cos x + I_1 \text{ (say)} \qquad \dots (1)$$

Taking e^x and $\cos x$ as the first and second functions, respectively, in I_1, we get

$$I_1 = e^x \sin x - \int e^x \sin x\,dx$$

Substituting the value of I_1 in (1), we get

$$I = -e^x \cos x + e^x \sin x - I \quad \text{or} \quad 2I = e^x(\sin x - \cos x)$$

Hence, $$I = \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C$$

Alternatively, above integral can also be determined by taking $\sin x$ as the first function and e^x the second function.

7.6.1 *Integral of the type* $\int e^x[f(x) + f'(x)]\,dx$

We have $$I = \int e^x[f(x) + f'(x)]\,dx = \int e^x f(x)\,dx + \int e^x f'(x)\,dx$$
$$= I_1 + \int e^x f'(x)\,dx, \text{ where } I_1 = \int e^x f(x)\,dx \qquad \dots (1)$$

Taking $f(x)$ and e^x as the first function and second function, respectively, in I_1 and integrating it by parts, we have $I_1 = f(x)\,e^x - \int f'(x)\,e^x dx + C$

Substituting I_1 in (1), we get

$$I = e^x f(x) - \int f'(x)\,e^x dx + \int e^x f'(x)\,dx + C = e^x f(x) + C$$

Thus, $$\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$$

Example 22 Find (i) $\int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx$ (ii) $\int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx$

Solution

(i) We have $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx$

Consider $f(x) = \tan^{-1} x$, then $f'(x) = \frac{1}{1+x^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $I = \int e^x (\tan^{-1} x + \frac{1}{1+x^2})\, dx = e^x \tan^{-1} x + C$

(ii) We have $I = \int \frac{(x^2+1)\, e^x}{(x+1)^2}\, dx = \int e^x [\frac{x^2 - 1 + 1 + 1)}{(x+1)^2}]\, dx$

$$= \int e^x [\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2}]\, dx = \int e^x [\frac{x-1}{x+1} + \frac{2}{(x+1)^2}]\, dx$$

Consider $f(x) = \frac{x-1}{x+1}$, then $f'(x) = \frac{2}{(x+1)^2}$

Thus, the given integrand is of the form $e^x [f(x) + f'(x)]$.

Therefore, $$\int \frac{x^2+1}{(x+1)^2} e^x\, dx = \frac{x-1}{x+1} e^x + C$$

EXERCISE 7.6

Integrate the functions in Exercises 1 to 22.

1. $x \sin x$ **2.** $x \sin 3x$ **3.** $x^2 e^x$ **4.** $x \log x$

5. $x \log 2x$ **6.** $x^2 \log x$ **7.** $x \sin^{-1} x$ **8.** $x \tan^{-1} x$

9. $x \cos^{-1} x$ **10.** $(\sin^{-1} x)^2$ **11.** $\frac{x \cos^{-1} x}{\sqrt{1-x^2}}$ **12.** $x \sec^2 x$

13. $\tan^{-1} x$ **14.** $x (\log x)^2$ **15.** $(x^2 + 1) \log x$

16. $e^x(\sin x + \cos x)$ **17.** $\dfrac{x\,e^x}{(1+x)^2}$ **18.** $e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)$

19. $e^x\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right)$ **20.** $\dfrac{(x-3)\,e^x}{(x-1)^3}$ **21.** $e^{2x}\sin x$

22. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Choose the correct answer in Exercises 23 and 24.

23. $\int x^2 e^{x^3}\,dx$ equals

(A) $\dfrac{1}{3}e^{x^3}+C$ (B) $\dfrac{1}{3}e^{x^2}+C$

(C) $\dfrac{1}{2}e^{x^3}+C$ (D) $\dfrac{1}{2}e^{x^2}+C$

24. $\int e^x \sec x\,(1+\tan x)\,dx$ equals

(A) $e^x\cos x + C$ (B) $e^x\sec x + C$

(C) $e^x\sin x + C$ (D) $e^x\tan x + C$

7.6.2 *Integrals of some more types*

Here, we discuss some special types of standard integrals based on the technique of integration by parts :

(i) $\int\sqrt{x^2-a^2}\,dx$ (ii) $\int\sqrt{x^2+a^2}\,dx$ (iii) $\int\sqrt{a^2-x^2}\,dx$

(i) Let $I=\int\sqrt{x^2-a^2}\,dx$

Taking constant function 1 as the second function and integrating by parts, we have

$$I = x\sqrt{x^2-a^2} - \int\frac{1}{2}\frac{2x}{\sqrt{x^2-a^2}}\,x\,dx$$

$$= x\sqrt{x^2-a^2} - \int\frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$= x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= x\sqrt{x^2-a^2} - \text{I} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$2\text{I} = x\sqrt{x^2-a^2} - a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

or $$\mathbf{I = \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C}$$

Similarly, integrating other two integrals by parts, taking constant function 1 as the second function, we get

(ii) $$\mathbf{\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C}$$

(iii) $$\mathbf{\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C}$$

Alternatively, integrals (i), (ii) and (iii) can also be found by making trigonometric substitution $x = a\sec\theta$ in (i), $x = a\tan\theta$ in (ii) and $x = a\sin\theta$ in (iii) respectively.

Example 23 Find $\int \sqrt{x^2+2x+5}\,dx$

Solution Note that

$$\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{(x+1)^2+4}\,dx$$

Put $x + 1 = y$, so that $dx = dy$. Then

$$\int \sqrt{x^2+2x+5}\,dx = \int \sqrt{y^2+2^2}\,dy$$

$$= \frac{1}{2}y\sqrt{y^2+4} + \frac{4}{2}\log\left|y+\sqrt{y^2+4}\right| + C \quad \text{[using 7.6.2 (ii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left|x+1+\sqrt{x^2+2x+5}\right| + C$$

Example 24 Find $\int \sqrt{3-2x-x^2}\,dx$

Solution Note that $\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx$

Put $x + 1 = y$ so that $dx = dy$.

Thus $$\int\sqrt{3-2x-x^2}\,dx = \int\sqrt{4-y^2}\,dy$$

$$= \frac{1}{2}y\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} + C \qquad \text{[using 7.6.2 (iii)]}$$

$$= \frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$$

EXERCISE 7.7

Integrate the functions in Exercises 1 to 9.

1. $\sqrt{4-x^2}$ **2.** $\sqrt{1-4x^2}$ **3.** $\sqrt{x^2+4x+6}$

4. $\sqrt{x^2+4x+1}$ **5.** $\sqrt{1-4x-x^2}$ **6.** $\sqrt{x^2+4x-5}$

7. $\sqrt{1+3x-x^2}$ **8.** $\sqrt{x^2+3x}$ **9.** $\sqrt{1+\frac{x^2}{9}}$

Choose the correct answer in Exercises 10 to 11.

10. $\int\sqrt{1+x^2}\,dx$ is equal to

(A) $\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\log\left|\left(x+\sqrt{1+x^2}\right)\right| + C$

(B) $\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$ (C) $\frac{2}{3}x(1+x^2)^{\frac{3}{2}} + C$

(D) $\frac{x^2}{2}\sqrt{1+x^2} + \frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right| + C$

11. $\int\sqrt{x^2-8x+7}\,dx$ is equal to

(A) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} + 9\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(B) $\frac{1}{2}(x+4)\sqrt{x^2-8x+7} + 9\log\left|x+4+\sqrt{x^2-8x+7}\right| + C$

(C) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - 3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

(D) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7} - \frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right| + C$

7.7 Definite Integral

In the previous sections, we have studied about the indefinite integrals and discussed few methods of finding them including integrals of some special functions. In this section, we shall study what is called definite integral of a function. The definite integral has a unique value. A definite integral is denoted by $\int_a^b f(x)\, dx$, where a is called the lower limit of the integral and b is called the upper limit of the integral. The definite integral is introduced either as the limit of a sum or if it has an anti derivative F in the interval $[a, b]$, then its value is the difference between the values of F at the end points, i.e., $F(b) - F(a)$. Here, we shall consider these two cases separately as discussed below:

7.7.1 *Definite integral as the limit of a sum*

Let f be a continuous function defined on close interval $[a, b]$. Assume that all the values taken by the function are non negative, so the graph of the function is a curve above the x-axis.

The definite integral $\int_a^b f(x)\, dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis. To evaluate this area, consider the region PRSQP between this curve, x-axis and the ordinates $x = a$ and $x = b$ (Fig 7.2).

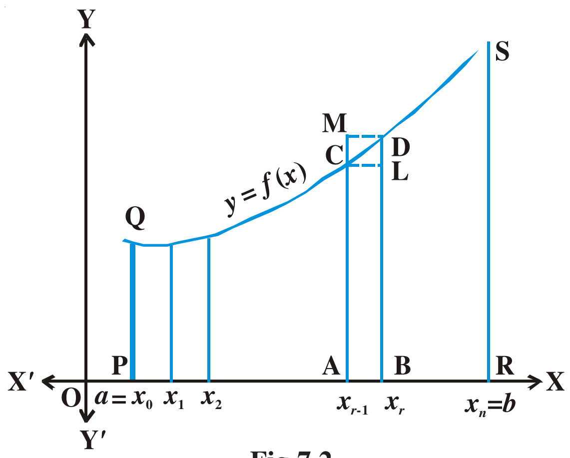


Fig 7.2

Divide the interval $[a, b]$ into n equal subintervals denoted by $[x_0, x_1]$, $[x_1, x_2]$,...., $[x_{r-1}, x_r]$, ..., $[x_{n-1}, x_n]$, where $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... , $x_r = a + rh$ and $x_n = b = a + nh$ or $n = \dfrac{b-a}{h}$. We note that as $n \to \infty$, $h \to 0$.

The region PRSQP under consideration is the sum of n subregions, where each subregion is defined on subintervals $[x_{r-1}, x_r]$, $r = 1, 2, 3, \ldots, n$.

From Fig 7.2, we have

area of the rectangle (ABLC) < area of the region (ABDCA) < area of the rectangle (ABDM) ... (1)

Evidently as $x_r - x_{r-1} \to 0$, i.e., $h \to 0$ all the three areas shown in (1) become nearly equal to each other. Now we form the following sums.

$$s_n = h\,[f(x_0) + \ldots + f(x_{n-1})] = h\sum_{r=0}^{n-1} f(x_r) \quad \ldots (2)$$

and

$$S_n = h\,[f(x_1) + f(x_2) + \ldots + f(x_n)] = h\sum_{r=1}^{n} f(x_r) \quad \ldots (3)$$

Here, s_n and S_n denote the sum of areas of all lower rectangles and upper rectangles raised over subintervals $[x_{r-1}, x_r]$ for $r = 1, 2, 3, \ldots, n$, respectively.

In view of the inequality (1) for an arbitrary subinterval $[x_{r-1}, x_r]$, we have

$$s_n < \text{area of the region PRSQP} < S_n \quad \ldots (4)$$

As $n \to \infty$ strips become narrower and narrower, it is assumed that the limiting values of (2) and (3) are the same in both cases and the common limiting value is the required area under the curve.

Symbolically, we write

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = \text{area of the region PRSQP} = \int_a^b f(x)dx \quad \ldots (5)$$

It follows that this area is also the limiting value of any area which is between that of the rectangles below the curve and that of the rectangles above the curve. For the sake of convenience, we shall take rectangles with height equal to that of the curve at the left hand edge of each subinterval. Thus, we rewrite (5) as

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h]$$

or

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty} \frac{1}{n}[f(a) + f(a+h) + \ldots + f(a+(n-1)\,h] \quad \ldots (6)$$

where

$$h = \frac{b-a}{n} \to 0 \text{ as } n \to \infty$$

The above expression (6) is known as the definition of definite integral as the *limit of sum*.

Remark The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as $\int_a^b f(t)\,dt$ or $\int_a^b f(u)\,du$ instead of $\int_a^b f(x)\,dx$. Hence, the variable of integration is called a *dummy variable.*

Example 25 Find $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Solution By definition

$$\int_a^b f(x)\,dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+...+f(a+(n-1)h],$$

where, $$h = \frac{b-a}{n}$$

In this example, $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $h = \frac{2-0}{n} = \frac{2}{n}$

Therefore,

$$\int_0^2 (x^2+1)\,dx = 2\lim_{n\to\infty}\frac{1}{n}[f(0)+f(\frac{2}{n})+f(\frac{4}{n})+...+f(\frac{2(n-1)}{n})]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[1+(\frac{2^2}{n^2}+1)+(\frac{4^2}{n^2}+1)+...+\left(\frac{(2n-2)^2}{n^2}+1\right)]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[\underbrace{(1+1+...+1)}_{n\text{-terms}}+\frac{1}{n^2}(2^2+4^2+...+(2n-2)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2^2}{n^2}(1^2+2^2+...+(n-1)^2]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{4}{n^2}\frac{(n-1)\,n\,(2n-1)}{6}]$$

$$= 2\lim_{n\to\infty}\frac{1}{n}[n+\frac{2}{3}\frac{(n-1)\,(2n-1)}{n}]$$

$$= 2\lim_{n\to\infty}[1+\frac{2}{3}(1-\frac{1}{n})\,(2-\frac{1}{n})] = 2\,[1+\frac{4}{3}] = \frac{14}{3}$$

Example 26 Evaluate $\int_0^2 e^x\,dx$ as the limit of a sum.

Solution By definition

$$\int_0^2 e^x\,dx = (2-0)\lim_{n\to\infty}\frac{1}{n}\left[e^0 + e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2n-2}{n}}\right]$$

Using the sum to n terms of a G.P., where $a = 1$, $r = e^{\frac{2}{n}}$, we have

$$\int_0^2 e^x\,dx = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$$

$$= \frac{2(e^2-1)}{\lim\limits_{n\to\infty}\left[\dfrac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right]\cdot 2} = e^2 - 1 \qquad \left[\text{using } \lim_{h\to 0}\frac{(e^h-1)}{h} = 1\right]$$

EXERCISE 7.8

Evaluate the following definite integrals as limit of sums.

1. $\int_a^b x\,dx$
2. $\int_0^5 (x+1)\,dx$
3. $\int_2^3 x^2\,dx$
4. $\int_1^4 (x^2 - x)\,dx$
5. $\int_{-1}^1 e^x\,dx$
6. $\int_0^4 (x + e^{2x})\,dx$

7.8 Fundamental Theorem of Calculus

7.8.1 *Area function*

We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the area of the light shaded region

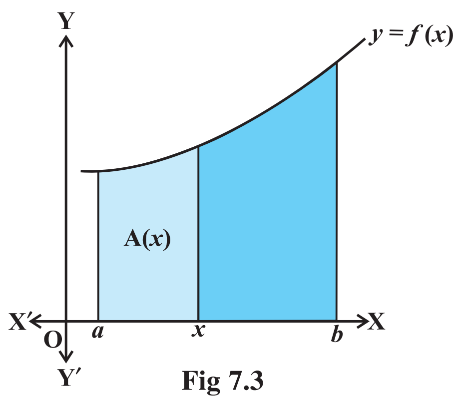


Fig 7.3

in Fig 7.3 [Here it is assumed that $f(x) > 0$ for $x \in [a, b]$, the assertion made below is equally true for other functions as well]. The area of this shaded region depends upon the value of x.

In other words, the area of this shaded region is a function of x. We denote this function of x by A(x). We call the function A(x) as *Area function* and is given by

$$\mathbf{A}(x) = \int_a^x f(x)\,dx \qquad \dots (1)$$

Based on this definition, the two basic fundamental theorems have been given. However, we only state them as their proofs are beyond the scope of this text book.

7.8.2 *First fundamental theorem of integral calculus*

Theorem 1 Let f be a continuous function on the closed interval $[a, b]$ and let A (x) be the area function. Then $\mathbf{A}'(x) = f(x)$, for all $x \in [a, b]$.

7.8.3 *Second fundamental theorem of integral calculus*

We state below an important theorem which enables us to evaluate definite integrals by making use of anti derivative.

Theorem 2 Let f be continuous function defined on the closed interval $[a, b]$ and F be an anti derivative of f. Then $\int_a^b f(x)\,dx = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$.

Remarks

(i) In words, the Theorem 2 tells us that $\int_a^b f(x)\,dx$ = (value of the anti derivative F of f at the upper limit b – value of the same anti derivative at the lower limit a).

(ii) This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum.

(iii) The crucial operation in evaluating a definite integral is that of finding a function whose derivative is equal to the integrand. This strengthens the relationship between differentiation and integration.

(iv) In $\int_a^b f(x)\,dx$, the function f needs to be well defined and continuous in $[a, b]$.

For instance, the consideration of definite integral $\int_{-2}^{3} x(x^2-1)^{\frac{1}{2}}\,dx$ is erroneous since the function f expressed by $f(x) = x(x^2-1)^{\frac{1}{2}}$ is not defined in a portion $-1 < x < 1$ of the closed interval $[-2, 3]$.

Steps for calculating $\int_a^b f(x)\,dx$.

(i) Find the indefinite integral $\int f(x)\,dx$. Let this be F(x). There is no need to keep integration constant C because if we consider F(x) + C instead of F(x), we get

$\int_a^b f(x)\,dx = [\mathrm{F}(x) + \mathrm{C}]_a^b = [\mathrm{F}(b) + \mathrm{C}] - [\mathrm{F}(a) + \mathrm{C}] = \mathrm{F}(b) - \mathrm{F}(a)$.

Thus, the arbitrary constant disappears in evaluating the value of the definite integral.

(ii) Evaluate F(b) – F(a) = $[\mathrm{F}(x)]_a^b$, which is the value of $\int_a^b f(x)\,dx$.

We now consider some examples

Example 27 Evaluate the following integrals:

(i) $\int_2^3 x^2\,dx$ (ii) $\int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$

(iii) $\int_1^2 \frac{x\,dx}{(x+1)(x+2)}$ (iv) $\int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$

Solution

(i) Let $\mathrm{I} = \int_2^3 x^2\,dx$. Since $\int x^2\,dx = \frac{x^3}{3} = \mathrm{F}(x)$,

Therefore, by the second fundamental theorem, we get

$$\mathrm{I} = \mathrm{F}(3) - \mathrm{F}(2) = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

(ii) Let $\mathrm{I} = \int_4^9 \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx$. We first find the anti derivative of the integrand.

Put $30 - x^{\frac{3}{2}} = t$. Then $-\frac{3}{2}\sqrt{x}\,dx = dt$ or $\sqrt{x}\,dx = -\frac{2}{3}dt$

Thus, $\int \frac{\sqrt{x}}{(30 - x^{\frac{3}{2}})^2}\,dx = -\frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3}\left[\frac{1}{t}\right] = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right] = \mathrm{F}(x)$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(9) - F(4) = \frac{2}{3}\left[\frac{1}{(30 - x^{\frac{3}{2}})}\right]_4^9$$

$$= \frac{2}{3}\left[\frac{1}{(30-27)} - \frac{1}{30-8}\right] = \frac{2}{3}\left[\frac{1}{3} - \frac{1}{22}\right] = \frac{19}{99}$$

(iii) Let $I = \int_1^2 \frac{x\,dx}{(x+1)(x+2)}$

Using partial fraction, we get $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$

So $$\int \frac{x\,dx}{(x+1)(x+2)} = -\log|x+1| + 2\log|x+2| = F(x)$$

Therefore, by the second fundamental theorem of calculus, we have

$$I = F(2) - F(1) = [-\log 3 + 2\log 4] - [-\log 2 + 2\log 3]$$

$$= -3\log 3 + \log 2 + 2\log 4 = \log\left(\frac{32}{27}\right)$$

(iv) Let $I = \int_0^{\frac{\pi}{4}} \sin^3 2t \cos 2t\,dt$. Consider $\int \sin^3 2t \cos 2t\,dt$

Put $\sin 2t = u$ so that $2\cos 2t\,dt = du$ or $\cos 2t\,dt = \frac{1}{2}du$

So $$\int \sin^3 2t \cos 2t\,dt = \frac{1}{2}\int u^3 du$$

$$= \frac{1}{8}[u^4] = \frac{1}{8}\sin^4 2t = F(t) \text{ say}$$

Therefore, by the second fundamental theorem of integral calculus

$$I = F\left(\frac{\pi}{4}\right) - F(0) = \frac{1}{8}\left[\sin^4\frac{\pi}{2} - \sin^4 0\right] = \frac{1}{8}$$

EXERCISE 7.9

Evaluate the definite integrals in Exercises 1 to 20.

1. $\int_{-1}^{1}(x+1)\,dx$ **2.** $\int_{2}^{3}\frac{1}{x}\,dx$ **3.** $\int_{1}^{2}(4x^3-5x^2+6x+9)\,dx$

4. $\int_{0}^{\frac{\pi}{4}}\sin 2x\,dx$ **5.** $\int_{0}^{\frac{\pi}{2}}\cos 2x\,dx$ **6.** $\int_{4}^{5}e^x dx$ **7.** $\int_{0}^{\frac{\pi}{4}}\tan x\,dx$

8. $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\text{cosec}\,x\,dx$ **9.** $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$ **10.** $\int_{0}^{1}\frac{dx}{1+x^2}$ **11.** $\int_{2}^{3}\frac{dx}{x^2-1}$

12. $\int_{0}^{\frac{\pi}{2}}\cos^2 x\,dx$ **13.** $\int_{2}^{3}\frac{x\,dx}{x^2+1}$ **14.** $\int_{0}^{1}\frac{2x+3}{5x^2+1}dx$ **15.** $\int_{0}^{1}x\,e^{x^2}dx$

16. $\int_{1}^{2}\frac{5x^2}{x^2+4x+3}$ **17.** $\int_{0}^{\frac{\pi}{4}}(2\sec^2 x+x^3+2)\,dx$ **18.** $\int_{0}^{\pi}(\sin^2\frac{x}{2}-\cos^2\frac{x}{2})\,dx$

19. $\int_{0}^{2}\frac{6x+3}{x^2+4}dx$ **20.** $\int_{0}^{1}(x\,e^x+\sin\frac{\pi x}{4})\,dx$

Choose the correct answer in Exercises 21 and 22.

21. $\int_{1}^{\sqrt{3}}\frac{dx}{1+x^2}$ equals

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

22. $\int_{0}^{\frac{2}{3}}\frac{dx}{4+9x^2}$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{12}$ (C) $\frac{\pi}{24}$ (D) $\frac{\pi}{4}$

7.9 Evaluation of Definite Integrals by Substitution

In the previous sections, we have discussed several methods for finding the indefinite integral. One of the important methods for finding the indefinite integral is the method of substitution.

To evaluate $\int_a^b f(x)\, dx$, by substitution, the steps could be as follows:

1. Consider the integral without limits and substitute, $y = f(x)$ or $x = g(y)$ to reduce the given integral to a known form.
2. Integrate the new integrand with respect to the new variable without mentioning the constant of integration.
3. Resubstitute for the new variable and write the answer in terms of the original variable.
4. Find the values of answers obtained in (3) at the given limits of integral and find the difference of the values at the upper and lower limits.

Note In order to quicken this method, we can proceed as follows: After performing steps 1, and 2, there is no need of step 3. Here, the integral will be kept in the new variable itself, and the limits of the integral will accordingly be changed, so that we can perform the last step.

Let us illustrate this by examples.

Example 28 Evaluate $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\, dx$.

Solution Put $t = x^5 + 1$, then $dt = 5x^4\, dx$.

Therefore,
$$\int 5x^4\sqrt{x^5+1}\, dx = \int \sqrt{t}\, dt = \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{3}(x^5+1)^{\frac{3}{2}}$$

Hence,
$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\, dx = \frac{2}{3}\left[(x^5+1)^{\frac{3}{2}}\right]_{-1}^{1}$$

$$= \frac{2}{3}\left[(1^5+1)^{\frac{3}{2}} - \left((-1)^5+1\right)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Alternatively, first we transform the integral and then evaluate the transformed integral with new limits.

Let $t = x^5 + 1$. Then $dt = 5x^4\,dx$.

Note that, when $x = -1,\ t = 0$ and when $x = 1,\ t = 2$

Thus, as x varies from – 1 to 1, t varies from 0 to 2

Therefore
$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt$$
$$= \frac{2}{3}\left[t^{\frac{3}{2}}\right]_0^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 0^{\frac{3}{2}}\right] = \frac{2}{3}(2\sqrt{2}) = \frac{4\sqrt{2}}{3}$$

Example 29 Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$

Solution Let $t = \tan^{-1}x$, then $dt = \dfrac{1}{1+x^2}\,dx$. The new limits are, when $x = 0, t = 0$ and when $x = 1,\ t = \dfrac{\pi}{4}$. Thus, as x varies from 0 to 1, t varies from 0 to $\dfrac{\pi}{4}$.

Therefore
$$\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} t\,dt \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\frac{\pi^2}{16} - 0\right] = \frac{\pi^2}{32}$$

EXERCISE 7.10

Evaluate the integrals in Exercises 1 to 8 using substitution.

1. $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$ **2.** $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\,\cos^5\phi\,d\phi$ **3.** $\displaystyle\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

4. $\displaystyle\int_0^2 x\sqrt{x+2}$ (Put $x + 2 = t^2$) **5.** $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\,dx$

6. $\displaystyle\int_0^2 \frac{dx}{x+4-x^2}$ **7.** $\displaystyle\int_{-1}^1 \frac{dx}{x^2+2x+5}$ **8.** $\displaystyle\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right)e^{2x}\,dx$

Choose the correct answer in Exercises 9 and 10.

9. The value of the integral $\displaystyle\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{\frac{1}{3}}}{x^4}\,dx$ is

(A) 6 (B) 0 (C) 3 (D) 4

10. If $f(x) = \displaystyle\int_0^x t\sin t\,dt$, then $f'(x)$ is

(A) $\cos x + x\sin x$ (B) $x\sin x$
(C) $x\cos x$ (D) $\sin x + x\cos x$

7.10 Some Properties of Definite Integrals

We list below some important properties of definite integrals. These will be useful in evaluating the definite integrals more easily.

$\mathbf{P_0}$: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$\mathbf{P_1}$: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. In particular, $\int_a^a f(x)\,dx = 0$

$\mathbf{P_2}$: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$\mathbf{P_3}$: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$\mathbf{P_4}$: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

(Note that P_4 is a particular case of P_3)

$\mathbf{P_5}$: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$\mathbf{P_6}$: $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$ and

0 if $f(2a-x) = -f(x)$

$\mathbf{P_7}$: (i) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, i.e., if $f(-x) = f(x)$.

(ii) $\int_{-a}^a f(x)\,dx = 0$, if f is an odd function, i.e., if $f(-x) = -f(x)$.

We give the proofs of these properties one by one.

Proof of $\mathbf{P_0}$ It follows directly by making the substitution $x = t$.

Proof of $\mathbf{P_1}$ Let F be anti derivative of f. Then, by the second fundamental theorem of calculus, we have $\int_a^b f(x)\,dx = \text{F}(b) - \text{F}(a) = -[\text{F}(a) - \text{F}(b)] = -\int_b^a f(x)\,dx$

Here, we observe that, if $a = b$, then $\int_a^a f(x)\,dx = 0$.

Proof of $\mathbf{P_2}$ Let F be anti derivative of f. Then

$$\int_a^b f(x)\,dx = \text{F}(b) - \text{F}(a) \quad \text{... (1)}$$

$$\int_a^c f(x)\,dx = \text{F}(c) - \text{F}(a) \quad \text{... (2)}$$

and

$$\int_c^b f(x)\,dx = \text{F}(b) - \text{F}(c) \quad \text{... (3)}$$

Adding (2) and (3), we get $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \mathrm{F}(b) - \mathrm{F}(a) = \int_a^b f(x)\,dx$

This proves the property P_2.

Proof of P_3 Let $t = a + b - x$. Then $dt = -\,dx$. When $x = a$, $t = b$ and when $x = b$, $t = a$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$$

$$= \int_a^b f(a+b-t)\,dt \quad (\text{by } P_1)$$

$$= \int_a^b f(a+b-x)\,dx \text{ by } P_0$$

Proof of P_4 Put $t = a - x$. Then $dt = -\,dx$. When $x = 0$, $t = a$ and when $x = a$, $t = 0$. Now proceed as in P_3.

Proof of P_5 Using P_2, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$.

Let $t = 2a - x$ in the second integral on the right hand side. Then $dt = -\,dx$. When $x = a$, $t = a$ and when $x = 2a$, $t = 0$. Also $x = 2a - t$.

Therefore, the second integral becomes

$$\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Hence $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Proof of P_6 Using P_5, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$ ... (1)

Now, if $f(2a - x) = f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

and if $f(2a - x) = -f(x)$, then (1) becomes

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx - \int_0^a f(x)\,dx = 0$$

Proof of P_7 Using P_2, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx. \text{ Then}$$

Let $t = -x$ in the first integral on the right hand side. $dt = -\,dx$. When $x = -a$, $t = a$ and when $x = 0$, $t = 0$. Also $x = -t$.

Therefore $\int_{-a}^{a} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt + \int_{0}^{a} f(x)\,dx$

$$= \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \text{(by } P_0\text{)} \quad \ldots (1)$$

(i) Now, if f is an even function, then $f(-x) = f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

(ii) If f is an odd function, then $f(-x) = -f(x)$ and so (1) becomes

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$$

Example 30 Evaluate $\int_{-1}^{2} \left| x^3 - x \right| dx$

Solution We note that $x^3 - x \geq 0$ on $[-1, 0]$ and $x^3 - x \leq 0$ on $[0, 1]$ and that $x^3 - x \geq 0$ on $[1, 2]$. So by P_2 we write

$$\int_{-1}^{2} \left| x^3 - x \right| dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} -(x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= -\left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right)$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + 2 - \frac{1}{4} + \frac{1}{2} = \frac{3}{2} - \frac{3}{4} + 2 = \frac{11}{4}$$

Example 31 Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx$

Solution We observe that $\sin^2 x$ is an even function. Therefore, by P_7 (i), we get

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\,dx = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x\,dx$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2}\,dx = \int_0^{\frac{\pi}{4}} (1-\cos 2x)\,dx$$

$$= \left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} = \left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - 0 = \frac{\pi}{4} - \frac{1}{2}$$

Example 32 Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

Solution Let I = $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$. Then, by P_4, we have

$$\text{I} = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)\,dx}{1+\cos^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x\,dx}{1+\cos^2 x} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} - \text{I}$$

or $$2\,\text{I} = \pi\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

or $$\text{I} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x = t$ so that $-\sin x\,dx = dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$. Therefore, (by P_1) we get

$$\text{I} = \frac{-\pi}{2}\int_1^{-1} \frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= \pi\int_0^{1} \frac{dt}{1+t^2} \quad \left(\text{by } P_7,\ \text{since } \frac{1}{1+t^2} \text{ is even function}\right)$$

$$= \pi\left[\tan^{-1} t\right]_0^1 = \pi\left[\tan^{-1} 1 - \tan^{-1} 0\right] = \pi\left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$$

Example 33 Evaluate $\int_{-1}^{1} \sin^5 x \cos^4 x\,dx$

Solution Let I = $\int_{-1}^{1} \sin^5 x\cos^4 x\,dx$. Let $f(x) = \sin^5 x \cos^4 x$. Then

$f(-x) = \sin^5(-x)\cos^4(-x) = -\sin^5 x\cos^4 x = -f(x)$, i.e., f is an odd function. Therefore, by P_7 (ii), I = 0

Example 34 Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$

Solution Let $I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx$ ... (1)

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 (\frac{\pi}{2} - x)}{\sin^4(\frac{\pi}{2} - x) + \cos^4(\frac{\pi}{2} - x)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} dx \quad ... (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

Hence $\qquad I = \frac{\pi}{4}$

Example 35 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}}$

Solution Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1 + \sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}}$ ... (1)

Then, by P_3

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}\, dx}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad ... (2)$$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \text{ Hence } I = \frac{\pi}{12}$$

Example 36 Evaluate $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Solution Let $I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$

Then, by P_4

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x \cos x + \log 2 - \log 2) \, dx \text{ (by adding and subtracting log 2)}$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx \quad \text{(Why?)}$$

Put $2x = t$ in the first integral. Then $2\, dx = dt$, when $x = 0, t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

Therefore
$$2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log \sin t \, dt - \frac{\pi}{2} \log 2 \quad [\text{by } P_6 \text{ as } \sin(\pi - t) = \sin t)$$

$$= \int_0^{\frac{\pi}{2}} \log \sin x \, dx - \frac{\pi}{2} \log 2 \quad \text{(by changing variable } t \text{ to } x)$$

$$= I - \frac{\pi}{2} \log 2$$

Hence
$$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \frac{-\pi}{2} \log 2.$$

EXERCISE 7.11

By using the properties of definite integrals, evaluate the integrals in Exercises 1 to 19.

1. $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$ **2.** $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$ **3.** $\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} x\, dx}{\sin^{\frac{3}{2}} x + \cos^{\frac{3}{2}} x}$

4. $\int_0^{\frac{\pi}{2}} \frac{\cos^5 x\, dx}{\sin^5 x + \cos^5 x}$ **5.** $\int_{-5}^{5} |x+2|\, dx$ **6.** $\int_2^8 |x-5|\, dx$

7. $\int_0^1 x(1-x)^n\, dx$ **8.** $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$ **9.** $\int_0^2 x\sqrt{2-x}\, dx$

10. $\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)\, dx$ **11.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

12. $\int_0^{\pi} \frac{x\, dx}{1+\sin x}$ **13.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x\, dx$ **14.** $\int_0^{2\pi} \cos^5 x\, dx$

15. $\int_0^{\frac{\pi}{2}} \frac{\sin x - \cos x}{1+\sin x\cos x}\, dx$ **16.** $\int_0^{\pi} \log(1+\cos x)\, dx$ **17.** $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}\, dx$

18. $\int_0^4 |x-1|\, dx$

19. Show that $\int_0^a f(x)g(x)\, dx = 2\int_0^a f(x)\, dx$, if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$

Choose the correct answer in Exercises 20 and 21.

20. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^3 + x\cos x + \tan^5 x + 1)\, dx$ is

(A) 0 (B) 2 (C) π (D) 1

21. The value of $\int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ is

(A) 2 (B) $\frac{3}{4}$ (C) 0 (D) –2

Miscellaneous Examples

Example 37 Find $\int \cos 6x \sqrt{1+\sin 6x}\, dx$

Solution Put $t = 1 + \sin 6x$, so that $dt = 6 \cos 6x\, dx$

Therefore
$$\int \cos 6x \sqrt{1+\sin 6x}\, dx = \frac{1}{6}\int t^{\frac{1}{2}} dt$$
$$= \frac{1}{6} \times \frac{2}{3}(t)^{\frac{3}{2}} + \text{C} = \frac{1}{9}(1+\sin 6x)^{\frac{3}{2}} + \text{C}$$

Example 38 Find $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx$

Solution We have $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \int \frac{(1 - \frac{1}{x^3})^{\frac{1}{4}}}{x^4} dx$

Put $1 - \frac{1}{x^3} = 1 - x^{-3} = t$, so that $\frac{3}{x^4} dx = dt$

Therefore $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \frac{1}{3}\int t^{\frac{1}{4}} dt = \frac{1}{3} \times \frac{4}{5} t^{\frac{5}{4}} + \text{C} = \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{\frac{5}{4}} + \text{C}$

Example 39 Find $\int \frac{x^4\, dx}{(x-1)(x^2+1)}$

Solution We have

$$\frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{x^3 - x^2 + x - 1}$$

$$= (x+1) + \frac{1}{(x-1)(x^2+1)} \quad \text{... (1)}$$

Now express
$$\frac{1}{(x-1)(x^2+1)} = \frac{\text{A}}{(x-1)} + \frac{\text{B}x + \text{C}}{(x^2+1)} \quad \text{... (2)}$$

So $$1 = A(x^2+1) + (Bx + C)(x-1)$$

$$= (A + B)x^2 + (C - B)x + A - C$$

Equating coefficients on both sides, we get A + B = 0, C – B = 0 and A – C = 1,

which give $A = \frac{1}{2}, B = C = -\frac{1}{2}$. Substituting values of A, B and C in (2), we get

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \quad \text{... (3)}$$

Again, substituting (3) in (1), we have

$$\frac{x^4}{(x-1)(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

Therefore

$$\int \frac{x^4}{(x-1)(x^2+x+1)}dx = \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

Example 40 Find $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

Solution Let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

$$= \int \log(\log x)\,dx + \int \frac{1}{(\log x)^2}dx$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$I = x\log(\log x) - \int \frac{1}{x\log x} x\,dx + \int \frac{dx}{(\log x)^2}$$

$$= x\log(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \quad \text{... (1)}$$

Again, consider $\int \frac{dx}{\log x}$, take 1 as the second function and integrate it by parts,

we have $$\int \frac{dx}{\log x} = \left[\frac{x}{\log x} - \int x\left\{-\frac{1}{(\log x)^2}\left(\frac{1}{x}\right)\right\}dx\right] \quad \text{... (2)}$$

Putting (2) in (1), we get

$$I = x\log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2} = x\log(\log x) - \frac{x}{\log x} + C$$

Example 41 Find $\int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx$

Solution We have

$$I = \int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx = \int \sqrt{\tan x}\,(1 + \cot x)\, dx$$

Put $\tan x = t^2$, so that $\sec^2 x\, dx = 2t\, dt$

or $\qquad dx = \dfrac{2t\, dt}{1+t^4}$

Then $\qquad I = \int t\left(1 + \frac{1}{t^2}\right) \frac{2t}{(1+t^4)}\, dt$

$$= 2\int \frac{(t^2+1)}{t^4+1}\, dt = 2\int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t^2+\frac{1}{t^2}\right)} = 2\int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t-\frac{1}{t}\right)^2 + 2}$$

Put $t - \dfrac{1}{t} = y$, so that $\left(1 + \dfrac{1}{t^2}\right) dt = dy$. Then

$$I = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2} = \sqrt{2}\,\tan^{-1}\frac{y}{\sqrt{2}} + C = \sqrt{2}\,\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + C$$

$$= \sqrt{2}\,\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + C = \sqrt{2}\,\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C$$

Example 42 Find $\int \dfrac{\sin 2x \cos 2x\, dx}{\sqrt{9 - \cos^4 (2x)}}$

Solution Let $I = \int \dfrac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}}\, dx$

Put $\cos^2 (2x) = t$ so that $4 \sin 2x \cos 2x\, dx = -dt$

Therefore $$\text{I} = -\frac{1}{4}\int \frac{dt}{\sqrt{9-t^2}} = -\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right) + \text{C} = -\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + \text{C}$$

Example 43 Evaluate $\int_{-1}^{\frac{3}{2}} |x \sin (\pi x)| dx$

Solution Here $f(x) = |x \sin \pi x| = \begin{cases} x \sin \pi x \text{ for } -1 \le x \le 1 \\ -x \sin \pi x \text{ for } 1 \le x \le \frac{3}{2} \end{cases}$

Therefore $$\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx = \int_{-1}^{1} x \sin \pi x\, dx + \int_{1}^{\frac{3}{2}} - x \sin \pi x\, dx$$

$$= \int_{-1}^{1} x \sin \pi x\, dx - \int_{1}^{\frac{3}{2}} x \sin \pi x\, dx$$

Integrating both integrals on righthand side, we get

$$\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\, dx = \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{-x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$

$$= \frac{2}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3}{\pi} + \frac{1}{\pi^2}$$

Example 44 Evaluate $\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Solution Let $\text{I} = \int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)}$ (using P_4)

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} - \text{I}$$

Thus $$2\text{I} = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

or $$I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \text{ (using } P_6)$$

$$= \pi\left[\int_0^{\frac{\pi}{4}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right]$$

$$= \pi\left[\int_0^{\frac{\pi}{4}} \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\text{cosec}^2 x\, dx}{a^2\cot^2 x + b^2}\right]$$

$$= \pi\left[\int_0^1 \frac{dt}{a^2 + b^2t^2} - \int_1^0 \frac{du}{a^2u^2 + b^2}\right] (\textit{put } \tan x = t \textit{ and } \cot x = u)$$

$$= \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1 - \frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0 = \frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a} + \tan^{-1}\frac{a}{b}\right] = \frac{\pi^2}{2ab}$$

Miscellaneous Exercise on Chapter 7

Integrate the functions in Exercises 1 to 24.

1. $\frac{1}{x - x^3}$

2. $\frac{1}{\sqrt{x+a} + \sqrt{x+b}}$

3. $\frac{1}{x\sqrt{ax - x^2}}$ [Hint: Put $x = \frac{a}{t}$]

4. $\frac{1}{x^2 (x^4 + 1)^{\frac{3}{4}}}$

5. $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ [Hint: $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1 + x^{\frac{1}{6}}\right)}$, put $x = t^6$]

6. $\frac{5x}{(x+1)(x^2+9)}$

7. $\frac{\sin x}{\sin (x - a)}$

8. $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

9. $\frac{\cos x}{\sqrt{4 - \sin^2 x}}$

10. $\frac{\sin^8 - \cos^8 x}{1 - 2\sin^2 x\cos^2 x}$

11. $\frac{1}{\cos (x+a)\cos (x+b)}$

12. $\frac{x^3}{\sqrt{1 - x^8}}$

13. $\frac{e^x}{(1+e^x)(2+e^x)}$

14. $\frac{1}{(x^2+1)(x^2+4)}$

15. $\cos^3 x\; e^{\log \sin x}$

16. $e^{3\log x}(x^4 + 1)^{-1}$

17. $f'(ax + b)\,[f(ax + b)]^n$

18. $\frac{1}{\sqrt{\sin^3 x \sin (x + \alpha)}}$

19. $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$, $x \in [0, 1]$

20. $\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}$ **21.** $\dfrac{2+\sin 2x}{1+\cos 2x}e^x$ **22.** $\dfrac{x^2+x+1}{(x+1)^2(x+2)}$

23. $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$ **24.** $\dfrac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}$

Evaluate the definite integrals in Exercises 25 to 33.

25. $\int_{\frac{\pi}{2}}^{\pi} e^x\left(\dfrac{1-\sin x}{1-\cos x}\right)dx$ **26.** $\int_0^{\frac{\pi}{4}} \dfrac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$ **27.** $\int_0^{\frac{\pi}{2}} \dfrac{\cos^2 x\,dx}{\cos^2 x+4\sin^2 x}$

28. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{\sin x+\cos x}{\sqrt{\sin 2x}}dx$ **29.** $\int_0^1 \dfrac{dx}{\sqrt{1+x}-\sqrt{x}}$ **30.** $\int_0^{\frac{\pi}{4}} \dfrac{\sin x+\cos x}{9+16\sin 2x}dx$

31. $\int_0^{\frac{\pi}{2}} \sin 2x\tan^{-1}(\sin x)\,dx$ **32.** $\int_0^{\pi} \dfrac{x\tan x}{\sec x+\tan x}dx$

33. $\int_1^4 [|x-1|+|x-2|+|x-3|]\,dx$

Prove the following (Exercises 34 to 39)

34. $\int_1^3 \dfrac{dx}{x^2(x+1)}=\dfrac{2}{3}+\log\dfrac{2}{3}$ **35.** $\int_0^1 x e^x dx=1$

36. $\int_{-1}^1 x^{17}\cos^4 x\,dx=0$ **37.** $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx=\dfrac{2}{3}$

38. $\int_0^{\frac{\pi}{4}} 2\tan^3 x\,dx=1-\log 2$ **39.** $\int_0^1 \sin^{-1}x\,dx=\dfrac{\pi}{2}-1$

40. Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Choose the correct answers in Exercises 41 to 44.

41. $\int \dfrac{dx}{e^x+e^{-x}}$ is equal to

(A) $\tan^{-1}(e^x)+C$ (B) $\tan^{-1}(e^{-x})+C$

(C) $\log(e^x-e^{-x})+C$ (D) $\log(e^x+e^{-x})+C$

42. $\int \dfrac{\cos 2x}{(\sin x+\cos x)^2}dx$ is equal to

(A) $\dfrac{-1}{\sin x+\cos x}+C$ (B) $\log|\sin x+\cos x|+C$

(C) $\log|\sin x-\cos x|+C$ (D) $\dfrac{1}{(\sin x+\cos x)^2}$

43. If $f(a + b - x) = f(x)$, then $\int_a^b x\, f(x)\, dx$ is equal to

(A) $\frac{a+b}{2}\int_a^b f(b-x)\, dx$ (B) $\frac{a+b}{2}\int_a^b f(b+x)\, dx$

(C) $\frac{b-a}{2}\int_a^b f(x)\, dx$ (D) $\frac{a+b}{2}\int_a^b f(x)\, dx$

44. The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ is

(A) 1 (B) 0 (C) –1 (D) $\frac{\pi}{4}$

Summary

- Integration is the inverse process of differentiation. In the differential calculus, we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation.

  Let $\frac{d}{dx}\text{F}(x) = f(x)$. Then we write $\int f(x)\, dx = \text{F}(x) + \text{C}$. These integrals are called indefinite integrals or general integrals, C is called constant of integration. All these integrals differ by a constant.
- From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the *y*-axis.
- Some properties of indefinite integrals are as follows:

  1. $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

  2. For any real number k, $\int k\, f(x)\, dx = k\int f(x)\, dx$

  More generally, if $f_1, f_2, f_3, ..., f_n$ are functions and $k_1, k_2, ..., k_n$ are real numbers. Then

  $$\int [k_1 f_1(x) + k_2 f_2(x) + ... + k_n f_n(x)]\, dx$$
  $$= k_1 \int f_1(x)\, dx + k_2 \int f_2(x)\, dx + ... + k_n \int f_n(x)\, dx$$

◆ Some standard integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$. Particularly, $\int dx = x + C$

(ii) $\int \cos x \, dx = \sin x + C$

(iii) $\int \sin x \, dx = -\cos x + C$

(iv) $\int \sec^2 x \, dx = \tan x + C$

(v) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

(vi) $\int \sec x \tan x \, dx = \sec x + C$

(vii) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\int e^x dx = e^x + C$

(xiii) $\int a^x dx = \frac{a^x}{\log a} + C$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xvi) $\int \frac{1}{x} dx = \log |x| + C$

◆ Integration by partial fractions

Recall that a rational function is ratio of two polynomials of the form $\frac{P(x)}{Q(x)}$, where P(x) and Q (x) are polynomials in x and Q (x) $\neq$ 0. If degree of the polynomial P (x) is greater than the degree of the polynomial Q (x), then we may divide P (x) by Q (x) so that $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T($x$) is a polynomial in x and degree of $P_1(x)$ is less than the degree of Q(x). T (x) being polynomial can be easily integrated. $\frac{P_1(x)}{Q(x)}$ can be integrated by expressing $\frac{P_1(x)}{Q(x)}$ as the sum of partial fractions of the following type:

1. $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, \ a \neq b$

2. $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

4. $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

5. $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ can not be factorised further.

◆ **Integration by substitution**

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the method of substitution. When the integrand involves some trigonometric functions, we use some well known identities to find the integrals. Using substitution technique, we obtain the following standard integrals.

(i) $\int \tan x \, dx = \log|\sec x| + C$ (ii) $\int \cot x \, dx = \log|\sin x| + C$

(iii) $\int \sec x \, dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$

◆ **Integrals of some special functions**

(i) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(ii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$ (iii) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(iv) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log \left| x + \sqrt{x^2 - a^2} \right| + C$ (v) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

(vi) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log | x + \sqrt{x^2 + a^2} | + C$

- **Integration by parts**

  For given functions f_1 and f_2, we have

  $\int f_1(x) \cdot f_2(x)\, dx = f_1(x) \int f_2(x)\, dx - \int \left[\frac{d}{dx} f_1(x) \cdot \int f_2(x)\, dx \right] dx$, i.e., the integral of the product of two functions = first function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

- $\int e^x [f(x) + f'(x)]\, dx = \int e^x f(x)\, dx + C$

- **Some special types of integrals**

(i) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log \left| x + \sqrt{x^2 - a^2} \right| + C$

(ii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log \left| x + \sqrt{x^2 + a^2} \right| + C$

(iii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

(iv) Integrals of the types $\int \frac{dx}{ax^2 + bx + c}$ or $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a} \right] = a\left[\left(x + \frac{b}{2a} \right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2} \right) \right]$$

(v) Integrals of the types $\int \frac{px + q\, dx}{ax^2 + bx + c}$ or $\int \frac{px + q\, dx}{\sqrt{ax^2 + bx + c}}$ can be transformed into standard form by expressing

$px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A\ (2ax + b) + B$, where A and B are determined by comparing coefficients on both sides.

- We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, $a \le x \le b$, the x-axis and the ordinates $x = a$ and $x = b$. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the **Area function** A (x). This concept of area function leads to the Fundamental Theorems of Integral Calculus.
- **First fundamental theorem of integral calculus**

  Let the area function be defined by $A(x) = \int_a^x f(x)\,dx$ for all $x \ge a$, where the function f is assumed to be continuous on $[a, b]$. Then $A'(x) = f(x)$ for all $x \in [a, b]$.
- **Second fundamental theorem of integral calculus**

  Let f be a continuous function of x defined on the closed interval $[a, b]$ and let F be another function such that $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)\,dx = \left[F(x) + C\right]_a^b = F(b) - F(a)$.

  This is called the definite integral of f over the range $[a, b]$, where a and b are called the limits of integration, a being the lower limit and b the upper limit.

PARTIAL DERIVATIVES

Functions of Several Variables

Many functions depend on more than one independent variable. The function $V = \pi r^2 h$ calculates the volume of a right circular cylinder from its radius and height. The function $f(x, y) = x^2 + y^2$ calculates the height of the paraboloid $z = x^2 + y^2$ above the point $P(x, y)$ from the two coordinates of P. The temperature T of a point on Earth's surface depends on its latitude x and longitude y, expressed by writing $T = f(x, y)$. In this section, we define functions of more than one independent variable and discuss ways to graph them.

Real-valued functions of several independent real variables are defined much the way you would imagine from the single-variable case. The domains are sets of ordered pairs (triples, quadruples, n-tuples) of real numbers, and the ranges are sets of real numbers of the kind we have worked with all along.

DEFINITION Partial Derivative with Respect to x

The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

An equivalent expression for the partial derivative is

$$\left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}.$$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f/\partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0. This is the rate of change of f in the direction of **i** at (x_0, y_0).

The notation for a partial derivative depends on what we want to emphasize:

$\frac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$	"Partial derivative of f with respect to x at (x_0, y_0)" or "f sub x at (x_0, y_0)." Convenient for stressing the point (x_0, y_0).
$\left.\frac{\partial z}{\partial x}\right\|_{(x_0, y_0)}$	"Partial derivative of z with respect to x at (x_0, y_0)." Common in science and engineering when you are dealing with variables and do not mention the function explicitly.
$f_x, \frac{\partial f}{\partial x}, z_x$, or $\frac{\partial z}{\partial x}$	"Partial derivative of f (or z) with respect to x." Convenient when you regard the partial derivative as a function in its own right.

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0.

DEFINITION **Partial Derivative with Respect to y**

The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Figure 14.14) is the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0. This is the rate of change of f in the direction of $\mathbf{j}$ at (x_0, y_0).

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x:

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (Figure 14.15). Is the plane they determine tangent to the surface at P? We will see that it is, but we have to learn more about partial derivatives before we can find out why.

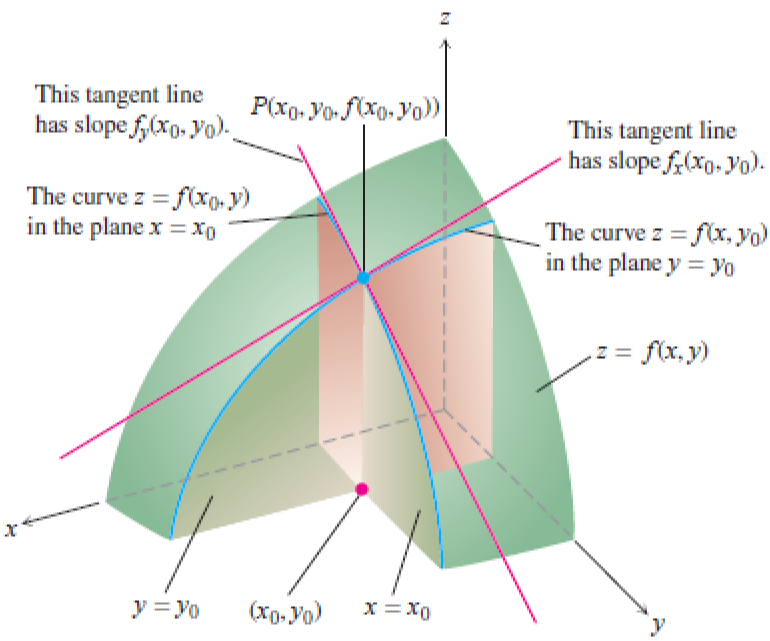


FIGURE 14.15 Figures 14.13 and 14.14 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

EXAMPLE 1 Finding Partial Derivatives at a Point

Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$.

EXAMPLE 2 Finding a Partial Derivative as a Function

Find $\partial f/\partial y$ if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$\dfrac{\partial^2 f}{\partial x^2}$	"*d* squared *f dx* squared"	or	f_{xx}	"*f* sub *xx*"
$\dfrac{\partial^2 f}{\partial y^2}$	"*d* squared *f dy* squared"	or	f_{yy}	"*f* sub *yy*"
$\dfrac{\partial^2 f}{\partial x \partial y}$	"*d* squared *f dx dy*"	or	f_{yx}	"*f* sub *yx*"
$\dfrac{\partial^2 f}{\partial y \partial x}$	"*d* squared *f dy dx*"	or	f_{xy}	"*f* sub *xy*"

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the derivatives are taken:

$$\frac{\partial^2 f}{\partial x \partial y} \qquad \text{Differentiate first with respect to } y, \text{ then with respect to } x.$$

$$f_{yx} = (f_y)_x \qquad \text{Means the same thing.}$$

EXAMPLE 9 Finding Second-Order Partial Derivatives

If $f(x, y) = x \cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x)$$

$$= \cos y + ye^x$$

So

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x)$$

$$= -x \sin y + e^x$$

So

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Calculating a Partial Derivative of Fourth-Order

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$f_y = -4xyz + x^2$$

$$f_{yx} = -4yz + 2x$$

$$f_{yxy} = -4z$$

$$f_{yxyz} = -4$$

■

The Chain Rule

The Chain Rule for functions of a single variable studied in Section 3.5 said that when $w = f(x)$ was a differentiable function of x and $x = g(t)$ was a differentiable function of t, w became a differentiable function of t and dw/dt could be calculated with the formula

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

Functions of Two Variables

The Chain Rule formula for a function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

THEOREM 5 Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

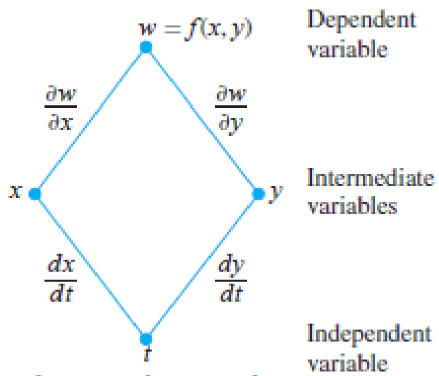
$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

To remember the Chain Rule picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

EXAMPLE 1 Applying the Chain Rule

Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t, y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y} \cdot \frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2} \cdot 2\cos 2t = \cos 2t.$$

THEOREM 6 Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

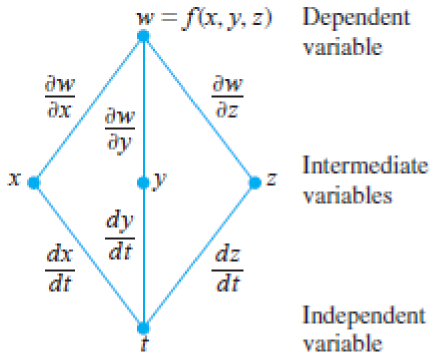
EXAMPLE 2 Changes in a Function's Values Along a Helix

Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Solution

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t.
\end{aligned}$$

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

THEOREM 7 Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The first of these equations can be derived from the Chain Rule in Theorem 6 by holding s fixed and treating r as t. The second can be derived in the same way, holding r fixed and treating s as t. The tree diagrams for both equations are shown in Figure 14.19.

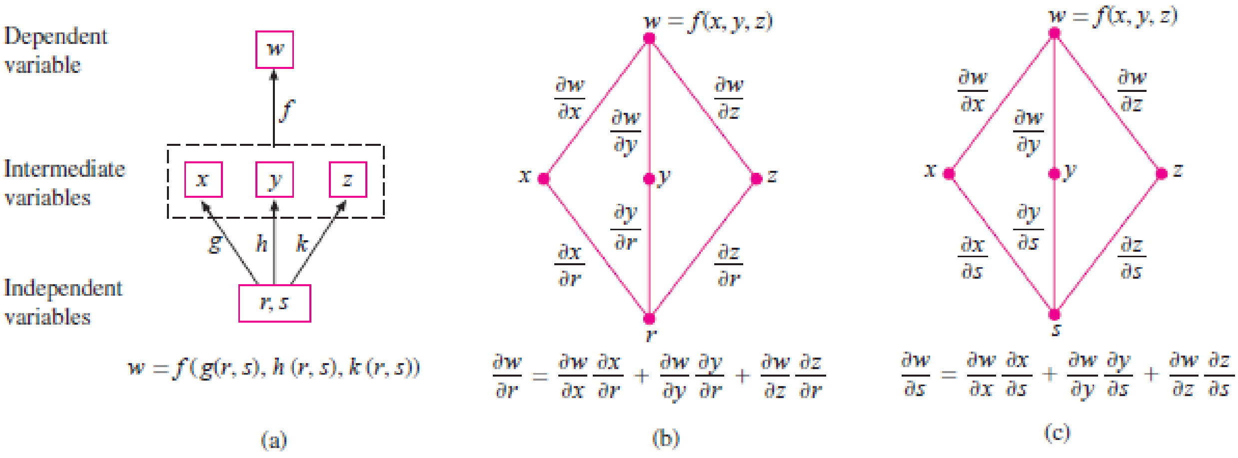


FIGURE 14.19 Composite function and tree diagrams for Theorem 7.

EXAMPLE 3 Partial Derivatives Using Theorem 7

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\ &= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\ &= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r\end{aligned}$$

Substitute for intermediate variable z.

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\ &= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}\end{aligned}$$

Directional Derivatives and Gradient Vectors

Here's a physical interpretation of the directional derivative. Suppose that $T = f(x, y)$ is the temperature at each point (x, y) over a region in the plane. Then $f(x_0, y_0)$ is the temperature at the point $P_0(x_0, y_0)$ and $(D_{\mathbf{u}} f)_{P_0}$ is the instantaneous rate of change of the temperature at P_0 stepping off in the direction $\mathbf{u}$.

DEFINITION Gradient Vector

The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f, read the way it is written.

Equation (3) says that the derivative of a differentiable function f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with the gradient of f at P_0.

EXAMPLE 2 Finding the Directional Derivative Using the Gradient

Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point (2, 0) in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at (2, 0) are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

The gradient of f at (2, 0) is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.26). The derivative of f at (2, 0) in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)|_{(2,0)} = \nabla f|_{(2,0)}\cdot\mathbf{u} \qquad \text{Equation (4)}$$

$$= (\mathbf{i} + 2\mathbf{j})\cdot\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

■

Evaluating the dot product in the formula

$$D_{\mathbf{u}}f = \nabla f\cdot\mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f, reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

EXAMPLE 3 Finding Directions of Maximal, Minimal, and Zero Change

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) Increases most rapidly at the point (1, 1)

(b) Decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient th is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

DEFINITION Total Differential

If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

in the linearization of f is called the **total differential of f**.

EXAMPLE 7 Estimating Change in Volume

Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., b that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estima the resulting absolute change in the volume of the can.

Solution To estimate the absolute change in $V = \pi r^2 h$, we use

$$\Delta V \approx dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh.$$

With $V_r = 2\pi rh$ and $V_h = \pi r^2$, we get

$$dV = 2\pi r_0 h_0\,dr + \pi r_0^2\,dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1)$$
$$= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63 \text{ in.}^3$$

Extreme Values and Saddle Points

THEOREM 11 Second Derivative Test for Local Extreme Values

Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

- **i.** f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- **ii.** f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
- **iii.** f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
- **iv.** **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

EXAMPLE 3 Finding Local Extreme Values

Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. *List the interior points of R* where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. *List the boundary points of R* where f has local maxima and minima and evaluate f at these points. We show how to do this shortly.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.

First-Order Differential Equations

First-Order Linear Equations

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**. Since the exponential growth/decay equation $dy/dx = ky$ (Section 6.5) can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, are not multiplied together, nor do they appear as the argument of a function $\left(\text{such as } \sin y,\ e^y, \text{ or } \sqrt{dy/dx}\right)$.

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int(-3/x)\,dx} \\ &= e^{-3\ln|x|} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

$$\frac{1}{x^3}y = \int \frac{1}{x^2}\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{x^3}y = -\frac{1}{x} + C.$$

Exact ODEs.

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact differential equation** if the **differential form** $M(x, y)\,dx + N(x, y)\,dy$ is **exact,** that is, this form is the differential

(a) $\dfrac{\partial u}{\partial x} = M,$ (b) $\dfrac{\partial u}{\partial y} = N.$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$$u = \int M\,dx + k(y);$$

$$u = \int N\,dy + l(x).$$

Solve

(7) $$\cos(x + y)\,dx + (3y^2 + 2y + \cos(x + y))\,dy = 0.$$

Solution. ***Step 1. Test for exactness.*** Our equation is of the form (1) with

$$M = \cos(x + y),$$

$$N = 3y^2 + 2y + \cos(x + y).$$

Thus

$$\frac{\partial M}{\partial y} = -\sin(x + y),$$

$$\frac{\partial N}{\partial x} = -\sin(x + y).$$

From this and (5) we see that (7) is exact.

Step 2. Implicit general solution. From (6) we obtain by integration

(8) $$u = \int M\,dx + k(y) = \int \cos(x + y)\,dx + k(y) = \sin(x + y) + k(y).$$

To find $k(y)$, we differentiate this formula with respect to y and use formula (4b), obtaining

$$\frac{\partial u}{\partial y} = \cos(x + y) + \frac{dk}{dy} = N = 3y^2 + 2y + \cos(x + y).$$

Hence $dk/dy = 3y^2 + 2y$. By integration, $k = y^3 + y^2 + c^*$. Inserting this result into (8) and observing (3), we obtain the *answer*

$$u(x, y) = \sin(x + y) + y^3 + y^2 = c.$$

Second-Order Differential Equations

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0,$$

THEOREM 3 If r_1 and r_2 are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-2x}.$$

■

THEOREM 4 If r is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$ ■

THEOREM 5 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$ or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$ ■

Nonhomogeneous Linear Equations

In this section we study two methods for solving second-order linear nonhomogeneous differential equations with constant coefficients. These are the methods of *undetermined coefficients* and *variation of parameters*. We begin by considering the form of the general solution.

Form of the General Solution

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x), \tag{1}$$

THEOREM 7 The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** y_c is the general solution to the associated homogeneous equation (2) and y_p is any **particular solution** to the nonhomogeneous equation (1).

The Method of Undetermined Coefficients

EXAMPLE 1 Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

Solution The auxiliary equation for the complementary equation $y'' - 2y' - 3y = 0$ is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots $r = -1$ and $r = 3$ giving the complementary solution

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Now $G(x) = 1 - x^2$ is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if y is a polynomial of degree 2, then $y'' - 2y' - 3y$ is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients A, B, and C. When we substitute the polynomial y_p and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

or, collecting terms with like powers of x,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of x if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of x to get

$$-3A = -1, \qquad -4A - 3B = 0, \qquad \text{and} \qquad 2A - 2B - 3C = 1.$$

These equations imply in turn that $A = 1/3$, $B = -4/9$, and $C = 5/27$. Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

By Theorem 7, the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

■

The Method of Variation of Parameters

Variation of Parameters Procedure

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with the Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation

$$ay'' + by' + cy = 0$$

to find the functions y_1 and y_2.

3. Integrate $v_1{}'$ and $v_2{}'$ to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.

4. Write down the particular solution to nonhomogeneous equation (1) as

$$y_p = v_1 y_1 + v_2 y_2.$$

EXAMPLE 6 Find the general solution to the equation

$$y'' + y = \tan x.$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - (\cos x) \ln |\sec x + \tan x|.$$

CHAPTER 17

THOMAS' CALCULUS

Twelfth Edition

Based on the original work by
George B. Thomas, Jr.
Massachusetts Institute of Technology

as revised by

Maurice D. Weir
Naval Postgraduate School

Joel Hass
University of California, Davis

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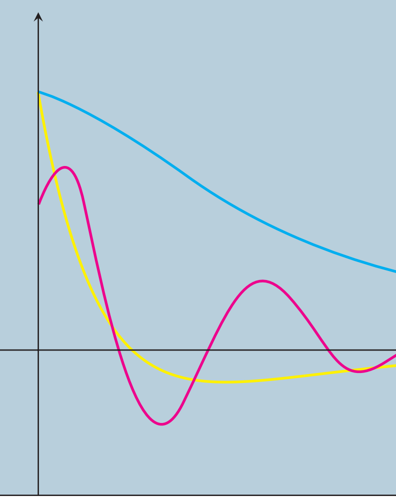

Chapter

17 SECOND-ORDER DIFFERENTIAL EQUATIONS

OVERVIEW In this chapter we extend our study of differential equations to those of *second order*. Second-order differential equations arise in many applications in the sciences and engineering. For instance, they can be applied to the study of vibrating springs and electric circuits. You will learn how to solve such differential equations by several methods in this chapter.

17.1 Second-Order Linear Equations

An equation of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \tag{1}$$

which is linear in y and its derivatives, is called a **second-order linear differential equation**. We assume that the functions P, Q, R, and G are continuous throughout some open interval I. If $G(x)$ is identically zero on I, the equation is said to be **homogeneous**; otherwise it is called **nonhomogeneous**. Therefore, the form of a second-order linear homogeneous differential equation is

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2}$$

We also assume that $P(x)$ is never zero for any $x \in I$.

Two fundamental results are important to solving Equation (2). The first of these says that if we know two solutions y_1 and y_2 of the linear homogeneous equation, then any **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

THEOREM 1—The Superposition Principle If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation (2), then for any constants c_1 and c_2, the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation (2).

Proof Substituting y into Equation (2), we have

$$\begin{aligned}P(x)y'' &+ Q(x)y' + R(x)y\\ &= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2)\\ &= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2)\\ &= c_1\underbrace{(P(x)y_1'' + Q(x)y_1' + R(x)y_1)}_{=\,0,\ y_1 \text{ is a solution}} + c_2\underbrace{(P(x)y_2'' + Q(x)y_2' + R(x)y_2)}_{=\,0,\ y_2 \text{ is a solution}}\\ &= c_1(0) + c_2(0) = 0.\end{aligned}$$

Therefore, $y = c_1y_1 + c_2y_2$ is a solution of Equation (2). ■

Theorem 1 immediately establishes the following facts concerning solutions to the linear homogeneous equation.

1. A sum of two solutions $y_1 + y_2$ to Equation (2) is also a solution. (Choose $c_1 = c_2 = 1$.)
2. A constant multiple ky_1 of any solution y_1 to Equation (2) is also a solution. (Choose $c_1 = k$ and $c_2 = 0$.)
3. The **trivial solution** $y(x) \equiv 0$ is always a solution to the linear homogeneous equation. (Choose $c_1 = c_2 = 0$.)

The second fundamental result about solutions to the linear homogeneous equation concerns its **general solution** or solution containing all solutions. This result says that there are two solutions y_1 and y_2 such that any solution is some linear combination of them for suitable values of the constants c_1 and c_2. However, not just any pair of solutions will do. The solutions must be **linearly independent**, which means that neither y_1 nor y_2 is a constant multiple of the other. For example, the functions $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent, whereas $f(x) = x^2$ and $g(x) = 7x^2$ are not (so they are linearly dependent). These results on linear independence and the following theorem are proved in more advanced courses.

THEOREM 2 If P, Q, and R are continuous over the open interval I and $P(x)$ is never zero on I, then the linear homogeneous equation (2) has two linearly independent solutions y_1 and y_2 on I. Moreover, if y_1 and y_2 are *any* two linearly independent solutions of Equation (2), then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants.

We now turn our attention to finding two linearly independent solutions to the special case of Equation (2), where P, Q, and R are constant functions.

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0, \tag{3}$$

where a, b, and c are constants. To solve Equation (3), we seek a function which when multiplied by a constant and added to a constant times its first derivative plus a constant times its second derivative sums identically to zero. One function that behaves this way is the exponential function $y = e^{rx}$, when r is a constant. Two differentiations of this exponential function give $y' = re^{rx}$ and $y'' = r^2e^{rx}$, which are just constant multiples of the original exponential. If we substitute $y = e^{rx}$ into Equation (3), we obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Since the exponential function is never zero, we can divide this last equation through by e^{rx}. Thus, $y = e^{rx}$ is a solution to Equation (3) if and only if r is a solution to the algebraic equation

$$ar^2 + br + c = 0. \qquad (4)$$

Equation (4) is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. The auxiliary equation is a quadratic equation with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider which depend on the value of the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$. In this case the auxiliary equation has two real and unequal roots r_1 and r_2. Then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions to Equation (3) because e^{r_2x} is not a constant multiple of e^{r_1x} (see Exercise 61). From Theorem 2 we conclude the following result.

THEOREM 3 If r_1 and r_2 are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1e^{3x} + c_2e^{-2x}.$$ ■

Case 2: $b^2 - 4ac = 0$. In this case $r_1 = r_2 = -b/2a$. To simplify the notation, let $r = -b/2a$. Then we have one solution $y_1 = e^{rx}$ with $2ar + b = 0$. Since multiplication of e^{rx} by a constant fails to produce a second linearly independent solution, suppose we try multiplying by a *function* instead. The simplest such function would be $u(x) = x$, so let's see if $y_2 = xe^{rx}$ is also a solution. Substituting y_2 into the differential equation gives

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + (0)xe^{rx} = 0. \end{aligned}$$

The first term is zero because $r = -b/2a$; the second term is zero because r solves the auxiliary equation. The functions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent (see Exercise 62). From Theorem 2 we conclude the following result.

THEOREM 4 If r is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{rx} + c_2xe^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}.$$ ■

Case 3: $b^2 - 4ac < 0$. In this case the auxiliary equation has two complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real numbers and $i^2 = -1$. (These real numbers are $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.) These two complex roots then give rise to two linearly independent solutions

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

(The expressions involving the sine and cosine terms follow from Euler's identity in Section 9.9.) However, the solutions y_1 and y_2 are *complex valued* rather than real valued. Nevertheless, because of the superposition principle (Theorem 1), we can obtain from them the two real-valued solutions

$$y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = e^{\alpha x}\cos\beta x \qquad \text{and} \qquad y_4 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = e^{\alpha x}\sin\beta x.$$

The functions y_3 and y_4 are linearly independent (see Exercise 63). From Theorem 2 we conclude the following result.

THEOREM 5 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$ or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$ ■

Initial Value and Boundary Value Problems

To determine a unique solution to a first-order linear differential equation, it was sufficient to specify the value of the solution at a single point. Since the general solution to a second-order equation contains two arbitrary constants, it is necessary to specify two conditions. One way of doing this is to specify the value of the solution function and the value of its derivative at a single point: $y(x_0) = y_0$ and $y'(x_0) = y_1$. These conditions are called **initial conditions**. The following result is proved in more advanced texts and guarantees the existence of a unique solution for both homogeneous and nonhomogeneous second-order linear initial value problems.

THEOREM 6 If P, Q, R, and G are continuous throughout an open interval I, then there exists one and only one function $y(x)$ satisfying both the differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x)$$

on the interval I, and the initial conditions

$$y(x_0) = y_0 \qquad \text{and} \qquad y'(x_0) = y_1$$

at the specified point $x_0 \in I$.

It is important to realize that any real values can be assigned to y_0 and y_1 and Theorem 6 applies. Here is an example of an initial value problem for a homogeneous equation.

EXAMPLE 4 Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$

Solution The auxiliary equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

The repeated real root is $r = 1$, giving the general solution

$$y = c_1 e^x + c_2 x e^x.$$

Then,

$$y' = c_1 e^x + c_2 (x + 1) e^x.$$

From the initial conditions we have

$$1 = c_1 + c_2 \cdot 0 \qquad \text{and} \qquad -1 = c_1 + c_2 \cdot 1.$$

Thus, $c_1 = 1$ and $c_2 = -2$. The unique solution satisfying the initial conditions is

$$y = e^x - 2xe^x.$$

The solution curve is shown in Figure 17.1. ■

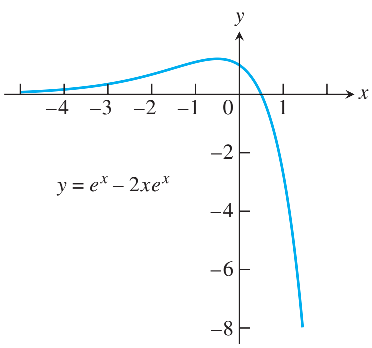

FIGURE 17.1 Particular solution curve for Example 4.

Another approach to determine the values of the two arbitrary constants in the general solution to a second-order differential equation is to specify the values of the solution function at *two different points* in the interval I. That is, we solve the differential equation subject to the **boundary values**

$$y(x_1) = y_1 \qquad \text{and} \qquad y(x_2) = y_2,$$

where x_1 and x_2 both belong to I. Here again the values for y_1 and y_2 can be any real numbers. The differential equation together with specified boundary values is called a **boundary value problem**. Unlike the result stated in Theorem 6, boundary value problems do not always possess a solution or more than one solution may exist (see Exercise 65). These problems are studied in more advanced texts, but here is an example for which there is a unique solution.

EXAMPLE 5 Solve the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 0, \quad y\left(\frac{\pi}{12}\right) = 1.$$

Solution The auxiliary equation is $r^2 + 4 = 0$, which has the complex roots $r = \pm 2i$. The general solution to the differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

The boundary conditions are satisfied if

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y\left(\frac{\pi}{12}\right) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = 1.$$

It follows that $c_1 = 0$ and $c_2 = 2$. The solution to the boundary value problem is

$$y = 2 \sin 2x.$$ ■

EXERCISES 17.1

In Exercises 1–30, find the general solution of the given equation.

1. $y'' - y' - 12y = 0$
2. $3y'' - y' = 0$
3. $y'' + 3y' - 4y = 0$
4. $y'' - 9y = 0$
5. $y'' - 4y = 0$
6. $y'' - 64y = 0$
7. $2y'' - y' - 3y = 0$
8. $9y'' - y = 0$
9. $8y'' - 10y' - 3y = 0$
10. $3y'' - 20y' + 12y = 0$
11. $y'' + 9y = 0$
12. $y'' + 4y' + 5y = 0$
13. $y'' + 25y = 0$
14. $y'' + y = 0$
15. $y'' - 2y' + 5y = 0$
16. $y'' + 16y = 0$
17. $y'' + 2y' + 4y = 0$
18. $y'' - 2y' + 3y = 0$
19. $y'' + 4y' + 9y = 0$
20. $4y'' - 4y' + 13y = 0$
21. $y'' = 0$
22. $y'' + 8y' + 16y = 0$
23. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$
24. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$
25. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$
26. $4\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 9y = 0$
27. $4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$
28. $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$
29. $9\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + y = 0$
30. $9\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0$

In Exercises 31–40, find the unique solution of the second-order initial value problem.

31. $y'' + 6y' + 5y = 0, \quad y(0) = 0,\ y'(0) = 3$
32. $y'' + 16y = 0, \quad y(0) = 2,\ y'(0) = -2$
33. $y'' + 12y = 0, \quad y(0) = 0,\ y'(0) = 1$
34. $12y'' + 5y' - 2y = 0, \quad y(0) = 1,\ y'(0) = -1$
35. $y'' + 8y = 0, \quad y(0) = -1,\ y'(0) = 2$
36. $y'' + 4y' + 4y = 0, \quad y(0) = 0,\ y'(0) = 1$
37. $y'' - 4y' + 4y = 0, \quad y(0) = 1,\ y'(0) = 0$
38. $4y'' - 4y' + y = 0, \quad y(0) = 4,\ y'(0) = 4$
39. $4\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 9y = 0, \quad y(0) = 2,\ \dfrac{dy}{dx}(0) = 1$
40. $9\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0, \quad y(0) = -1,\ \dfrac{dy}{dx}(0) = 1$

In Exercises 41–55, find the general solution.

41. $y'' - 2y' - 3y = 0$
42. $6y'' - y' - y = 0$
43. $4y'' + 4y' + y = 0$
44. $9y'' + 12y' + 4y = 0$
45. $4y'' + 20y = 0$
46. $y'' + 2y' + 2y = 0$
47. $25y'' + 10y' + y = 0$
48. $6y'' + 13y' - 5y = 0$
49. $4y'' + 4y' + 5y = 0$
50. $y'' + 4y' + 6y = 0$
51. $16y'' - 24y' + 9y = 0$
52. $6y'' - 5y' - 6y = 0$
53. $9y'' + 24y' + 16y = 0$
54. $4y'' + 16y' + 52y = 0$
55. $6y'' - 5y' - 4y = 0$

In Exercises 56–60, solve the initial value problem.

56. $y'' - 2y' + 2y = 0, \quad y(0) = 0,\ y'(0) = 2$
57. $y'' + 2y' + y = 0, \quad y(0) = 1,\ y'(0) = 1$
58. $4y'' - 4y' + y = 0, \quad y(0) = -1,\ y'(0) = 2$
59. $3y'' + y' - 14y = 0, \quad y(0) = 2,\ y'(0) = -1$
60. $4y'' + 4y' + 5y = 0, \quad y(\pi) = 1,\ y'(\pi) = 0$
61. Prove that the two solution functions in Theorem 3 are linearly independent.
62. Prove that the two solution functions in Theorem 4 are linearly independent.
63. Prove that the two solution functions in Theorem 5 are linearly independent.
64. Prove that if y_1 and y_2 are linearly independent solutions to the homogeneous equation (2), then the functions $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ are also linearly independent solutions.
65. **a.** Show that there is no solution to the boundary value problem
$$y'' + 4y = 0, \quad y(0) = 0,\ y(\pi) = 1.$$
b. Show that there are infinitely many solutions to the boundary value problem
$$y'' + 4y = 0, \quad y(0) = 0,\ y(\pi) = 0.$$
66. Show that if a, b, and c are positive constants, then all solutions of the homogeneous differential equation
$$ay'' + by' + cy = 0$$
approach zero as $x \to \infty$.

17.2 Nonhomogeneous Linear Equations

In this section we study two methods for solving second-order linear nonhomogeneous differential equations with constant coefficients. These are the methods of *undetermined coefficients* and *variation of parameters*. We begin by considering the form of the general solution.

Form of the General Solution

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x), \tag{1}$$

where a, b, and c are constants and G is continuous over some open interval I. Let $y_c = c_1y_1 + c_2y_2$ be the general solution to the associated **complementary equation**

$$ay'' + by' + cy = 0. \tag{2}$$

(We learned how to find y_c in Section 17.1.) Now suppose we could somehow come up with a particular function y_p that solves the nonhomogeneous equation (1). Then the sum

$$y = y_c + y_p \tag{3}$$

also solves the nonhomogeneous equation (1) because

$$\begin{aligned} a(y_c + y_p)'' &+ b(y_c + y_p)' + c(y_c + y_p) \\ &= (ay_c'' + by_c' + cy_c) + (ay_p'' + by_p' + cy_p) \\ &= 0 + G(x) \qquad y_c \text{ solves Eq. (2) and } y_p \text{ solves Eq. (1)} \\ &= G(x). \end{aligned}$$

Moreover, if $y = y(x)$ is the general solution to the nonhomogeneous equation (1), it must have the form of Equation (3). The reason for this last statement follows from the observation that for any function y_p satisfying Equation (1), we have

$$\begin{aligned} a(y - y_p)'' &+ b(y - y_p)' + c(y - y_p) \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0. \end{aligned}$$

Thus, $y_c = y - y_p$ is the general solution to the homogeneous equation (2). We have established the following result.

THEOREM 7 The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** y_c is the general solution to the associated homogeneous equation (2) and y_p is any **particular solution** to the nonhomogeneous equation (1).

The Method of Undetermined Coefficients

This method for finding a particular solution y_p to the nonhomogeneous equation (1) applies to special cases for which $G(x)$ is a sum of terms of various polynomials $p(x)$ multiplying an exponential with possibly sine or cosine factors. That is, $G(x)$ is a sum of terms of the following forms:

$$p_1(x)e^{rx}, \qquad p_2(x)e^{\alpha x}\cos\beta x, \qquad p_3(x)e^{\alpha x}\sin\beta x.$$

For instance, $1 - x$, e^{2x}, xe^x, $\cos x$, and $5e^x - \sin 2x$ represent functions in this category. (Essentially these are functions solving homogeneous linear differential equations with constant coefficients, but the equations may be of order higher than two.) We now present several examples illustrating the method.

EXAMPLE 1 Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

Solution The auxiliary equation for the complementary equation $y'' - 2y' - 3y = 0$ is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots $r = -1$ and $r = 3$ giving the complementary solution

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Now $G(x) = 1 - x^2$ is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if y is a polynomial of degree 2, then $y'' - 2y' - 3y$ is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients A, B, and C. When we substitute the polynomial y_p and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

or, collecting terms with like powers of x,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of x if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of x to get

$$-3A = -1, \qquad -4A - 3B = 0, \qquad \text{and} \qquad 2A - 2B - 3C = 1.$$

These equations imply in turn that $A = 1/3$, $B = -4/9$, and $C = 5/27$. Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

By Theorem 7, the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$ ■

EXAMPLE 2 Find a particular solution of $y'' - y' = 2 \sin x$.

Solution If we try to find a particular solution of the form

$$y_p = A \sin x$$

and substitute the derivatives of y_p in the given equation, we find that A must satisfy the equation

$$-A \sin x + A \cos x = 2 \sin x$$

for all values of x. Since this requires A to equal both -2 and 0 at the same time, we conclude that the nonhomogeneous differential equation has no solution of the form $A \sin x$.

It turns out that the required form is the sum

$$y_p = A \sin x + B \cos x.$$

The result of substituting the derivatives of this new trial solution into the differential equation is

$$-A \sin x - B \cos x - (A \cos x - B \sin x) = 2 \sin x$$

or

$$(B - A) \sin x - (A + B) \cos x = 2 \sin x.$$

This last equation must be an identity. Equating the coefficients for like terms on each side then gives

$$B - A = 2 \qquad \text{and} \qquad A + B = 0.$$

Simultaneous solution of these two equations gives $A = -1$ and $B = 1$. Our particular solution is

$$y_p = \cos x - \sin x.$$

■

EXAMPLE 3 Find a particular solution of $y'' - 3y' + 2y = 5e^x$.

Solution If we substitute

$$y_p = Ae^x$$

and its derivatives in the differential equation, we find that

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x$$

or

$$0 = 5e^x.$$

However, the exponential function is never zero. The trouble can be traced to the fact that $y = e^x$ is already a solution of the related homogeneous equation

$$y'' - 3y' + 2y = 0.$$

The auxiliary equation is

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0,$$

which has $r = 1$ as a root. So we would expect Ae^x to become zero when substituted into the left-hand side of the differential equation.

The appropriate way to modify the trial solution in this case is to multiply Ae^x by x. Thus, our new trial solution is

$$y_p = Axe^x.$$

The result of substituting the derivatives of this new candidate into the differential equation is

$$(Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2Axe^x = 5e^x$$

or

$$-Ae^x = 5e^x.$$

Thus, $A = -5$ gives our sought-after particular solution

$$y_p = -5xe^x. \quad ■$$

EXAMPLE 4 Find a particular solution of $y'' - 6y' + 9y = e^{3x}$.

Solution The auxiliary equation for the complementary equation

$$r^2 - 6r + 9 = (r - 3)^2 = 0$$

has $r = 3$ as a repeated root. The appropriate choice for y_p in this case is neither Ae^{3x} nor Axe^{3x} because the complementary solution contains both of those terms already. Thus, we choose a term containing the next higher power of x as a factor. When we substitute

$$y_p = Ax^2e^{3x}$$

and its derivatives in the given differential equation, we get

$$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x}$$

or

$$2Ae^{3x} = e^{3x}.$$

Thus, $A = 1/2$, and the particular solution is

$$y_p = \frac{1}{2}x^2e^{3x}. \quad ■$$

When we wish to find a particular solution of Equation (1) and the function $G(x)$ is the sum of two or more terms, we choose a trial function for each term in $G(x)$ and add them.

EXAMPLE 5 Find the general solution to $y'' - y' = 5e^x - \sin 2x$.

Solution We first check the auxiliary equation

$$r^2 - r = 0.$$

Its roots are $r = 1$ and $r = 0$. Therefore, the complementary solution to the associated homogeneous equation is

$$y_c = c_1e^x + c_2.$$

We now seek a particular solution y_p. That is, we seek a function that will produce $5e^x - \sin 2x$ when substituted into the left-hand side of the given differential equation. One part of y_p is to produce $5e^x$, the other $-\sin 2x$.

Since any function of the form c_1e^x is a solution of the associated homogeneous equation, we choose our trial solution y_p to be the sum

$$y_p = Axe^x + B\cos 2x + C\sin 2x,$$

including xe^x where we might otherwise have included only e^x. When the derivatives of y_p are substituted into the differential equation, the resulting equation is

$$(Axe^x + 2Ae^x - 4B\cos 2x - 4C\sin 2x)$$
$$- (Axe^x + Ae^x - 2B\sin 2x + 2C\cos 2x) = 5e^x - \sin 2x$$

or

$$Ae^x - (4B + 2C)\cos 2x + (2B - 4C)\sin 2x = 5e^x - \sin 2x.$$

This equation will hold if

$$A = 5, \qquad 4B + 2C = 0, \qquad 2B - 4C = -1,$$

or $A = 5$, $B = -1/10$, and $C = 1/5$. Our particular solution is

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$

The general solution to the differential equation is

$$y = y_c + y_p = c_1 e^x + c_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$ ■

You may find the following table helpful in solving the problems at the end of this section.

TABLE 17.1 The method of undetermined coefficients for selected equations of the form

$$ay'' + by' + cy = G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	And if	Then include this expression in the trial function for y_p.
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}
$\sin kx$, $\cos kx$	ki is not a root of the auxiliary equation	$B\cos kx + C\sin kx$
$px^2 + qx + m$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

The Method of Variation of Parameters

This is a general method for finding a particular solution of the nonhomogeneous equation (1) once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring (in a way to be explained) that the resulting expression satisfy the nonhomogeneous equation (1). There are two functions to be determined, and requiring that Equation (1) be satisfied is only one condition. As a second condition, we also require that

$$v_1'y_1 + v_2'y_2 = 0. \tag{4}$$

Then we have

$$\begin{aligned} y &= v_1y_1 + v_2y_2, \\ y' &= v_1y_1' + v_2y_2', \\ y'' &= v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'. \end{aligned}$$

If we substitute these expressions into the left-hand side of Equation (1), we obtain

$$v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) + a(v_1'y_1' + v_2'y_2') = G(x).$$

The first two parenthetical terms are zero since y_1 and y_2 are solutions of the associated homogeneous equation (2). So the nonhomogeneous equation (1) is satisfied if, in addition to Equation (4), we require that

$$a(v_1'y_1' + v_2'y_2') = G(x). \tag{5}$$

Equations (4) and (5) can be solved together as a pair

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= \frac{G(x)}{a} \end{aligned}$$

for the unknown functions v_1' and v_2'. The usual procedure for solving this simple system is to use the *method of determinants* (also known as *Cramer's Rule*), which will be demonstrated in the examples to follow. Once the derivative functions v_1' and v_2' are known, the two functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ can be found by integration. Here is a summary of the method.

Variation of Parameters Procedure

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation

$$ay'' + by' + cy = 0$$

to find the functions y_1 and y_2.

2. Solve the equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= \frac{G(x)}{a} \end{aligned}$$

simultaneously for the derivative functions v_1' and v_2'.

3. Integrate v_1' and v_2' to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.

4. Write down the particular solution to nonhomogeneous equation (1) as

$$y_p = v_1y_1 + v_2y_2.$$

EXAMPLE 6 Find the general solution to the equation

$$y'' + y = \tan x.$$

Solution The solution of the homogeneous equation

$$y'' + y = 0$$

is given by

$$y_c = c_1 \cos x + c_2 \sin x.$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$, the conditions to be satisfied in Equations (4) and (5) are

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0, \\ -v_1' \sin x + v_2' \cos x &= \tan x. \qquad a = 1 \end{aligned}$$

Solution of this system gives

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x.$$

After integrating v_1' and v_2', we have

$$\begin{aligned} v_1(x) &= \int \frac{-\sin^2 x}{\cos x}\, dx \\ &= -\int (\sec x - \cos x)\, dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2(x) = \int \sin x\, dx = -\cos x.$$

Note that we have omitted the constants of integration in determining v_1 and v_2. They would merely be absorbed into the arbitrary constants in the complementary solution.

Substituting v_1 and v_2 into the expression for y_p in Step 4 gives

$$\begin{aligned} y_p &= [-\ln|\sec x + \tan x| + \sin x] \cos x + (-\cos x) \sin x \\ &= (-\cos x) \ln|\sec x + \tan x|. \end{aligned}$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - (\cos x) \ln|\sec x + \tan x|.$$ ■

EXAMPLE 7 Solve the nonhomogeneous equation

$$y'' + y' - 2y = xe^x.$$

Solution The auxiliary equation is

$$r^2 + r - 2 = (r + 2)(r - 1) = 0$$

giving the complementary solution

$$y_c = c_1e^{-2x} + c_2e^x.$$

The conditions to be satisfied in Equations (4) and (5) are

$$\begin{aligned} v_1'e^{-2x} + v_2'e^x &= 0, \\ -2v_1'e^{-2x} + v_2'e^x &= xe^x. \qquad a = 1 \end{aligned}$$

Solving the above system for v_1' and v_2' gives

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix}} = \frac{-xe^{2x}}{3e^{-x}} = -\frac{1}{3}xe^{3x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix}}{3e^{-x}} = \frac{xe^{-x}}{3e^{-x}} = \frac{x}{3}.$$

Integrating to obtain the parameter functions, we have

$$\begin{aligned} v_1(x) &= \int -\frac{1}{3}xe^{3x}\,dx \\ &= -\frac{1}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx\right) \\ &= \frac{1}{27}(1 - 3x)e^{3x}, \end{aligned}$$

and

$$v_2(x) = \int \frac{x}{3}\,dx = \frac{x^2}{6}.$$

Therefore,

$$\begin{aligned} y_p &= \left[\frac{(1 - 3x)e^{3x}}{27}\right]e^{-2x} + \left(\frac{x^2}{6}\right)e^x \\ &= \frac{1}{27}e^x - \frac{1}{9}xe^x + \frac{1}{6}x^2e^x. \end{aligned}$$

The general solution to the differential equation is

$$y = c_1e^{-2x} + c_2e^x - \frac{1}{9}xe^x + \frac{1}{6}x^2e^x,$$

where the term $(1/27)e^x$ in y_p has been absorbed into the term c_2e^x in the complementary solution. ■

EXERCISES 17.2

Solve the equations in Exercises 1–16 by the method of undetermined coefficients.

1. $y'' - 3y' - 10y = -3$
2. $y'' - 3y' - 10y = 2x - 3$
3. $y'' - y' = \sin x$
4. $y'' + 2y' + y = x^2$
5. $y'' + y = \cos 3x$
6. $y'' + y = e^{2x}$
7. $y'' - y' - 2y = 20 \cos x$
8. $y'' + y = 2x + 3e^x$
9. $y'' - y = e^x + x^2$
10. $y'' + 2y' + y = 6 \sin 2x$
11. $y'' - y' - 6y = e^{-x} - 7 \cos x$
12. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$
13. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 15x^2$
14. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = -8x + 3$
15. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = e^{3x} - 12x$
16. $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} = 42x^2 + 5x + 1$

Solve the equations in Exercises 17–28 by variation of parameters.

17. $y'' + y' = x$
18. $y'' + y = \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
19. $y'' + y = \sin x$
20. $y'' + 2y' + y = e^x$
21. $y'' + 2y' + y = e^{-x}$
22. $y'' - y = x$
23. $y'' - y = e^x$
24. $y'' - y = \sin x$
25. $y'' + 4y' + 5y = 10$
26. $y'' - y' = 2^x$
27. $\dfrac{d^2y}{dx^2} + y = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
28. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x \cos x, \quad x > 0$

In each of Exercises 29–32, the given differential equation has a particular solution y_p of the form given. Determine the coefficients in y_p. Then solve the differential equation.

29. $y'' - 5y' = xe^{5x}, \quad y_p = Ax^2e^{5x} + Bxe^{5x}$
30. $y'' - y' = \cos x + \sin x, \quad y_p = A \cos x + B \sin x$
31. $y'' + y = 2 \cos x + \sin x, \quad y_p = Ax \cos x + Bx \sin x$
32. $y'' + y' - 2y = xe^x, \quad y_p = Ax^2e^x + Bxe^x$

In Exercises 33–36, solve the given differential equations **(a)** by variation of parameters and **(b)** by the method of undetermined coefficients.

33. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x + e^{-x}$
34. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 2e^{2x}$
35. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = e^x + 4$
36. $\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} = 9e^{9x}$

Solve the differential equations in Exercises 37–46. Some of the equations can be solved by the method of undetermined coefficients, but others cannot.

37. $y'' + y = \cot x, \quad 0 < x < \pi$
38. $y'' + y = \csc x, \quad 0 < x < \pi$
39. $y'' - 8y' = e^{8x}$
40. $y'' + 4y = \sin x$
41. $y'' - y' = x^3$
42. $y'' + 4y' + 5y = x + 2$
43. $y'' + 2y' = x^2 - e^x$
44. $y'' + 9y = 9x - \cos x$
45. $y'' + y = \sec x \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
46. $y'' - 3y' + 2y = e^x - e^{2x}$

The method of undetermined coefficients can sometimes be used to solve first-order ordinary differential equations. Use the method to solve the equations in Exercises 47–50.

47. $y' - 3y = e^x$
48. $y' + 4y = x$
49. $y' - 3y = 5e^{3x}$
50. $y' + y = \sin x$

Solve the differential equations in Exercises 51 and 52 subject to the given initial conditions.

51. $\dfrac{d^2y}{dx^2} + y = \sec^2 x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}; \quad y(0) = y'(0) = 1$
52. $\dfrac{d^2y}{dx^2} + y = e^{2x}; \quad y(0) = 0,\ y'(0) = \dfrac{2}{5}$

In Exercises 53–58, verify that the given function is a particular solution to the specified nonhomogeneous equation. Find the general solution and evaluate its arbitrary constants to find the unique solution satisfying the equation and the given initial conditions.

53. $y'' + y' = x, \quad y_p = \dfrac{x^2}{2} - x, \quad y(0) = 0,\ y'(0) = 0$
54. $y'' + y = x, \quad y_p = 2 \sin x + x, \quad y(0) = 0,\ y'(0) = 0$
55. $\dfrac{1}{2}y'' + y' + y = 4e^x(\cos x - \sin x),$
    $y_p = 2e^x \cos x, \quad y(0) = 0,\ y'(0) = 1$
56. $y'' - y' - 2y = 1 - 2x, \quad y_p = x - 1, \quad y(0) = 0,\ y'(0) = 1$
57. $y'' - 2y' + y = 2e^x, \quad y_p = x^2e^x, \quad y(0) = 1,\ y'(0) = 0$
58. $y'' - 2y' + y = x^{-1}e^x,\ x > 0,$
    $y_p = xe^x \ln x, \quad y(1) = e,\ y'(1) = 0$

In Exercises 59 and 60, two linearly independent solutions y_1 and y_2 are given to the associated homogeneous equation of the variable-coefficient nonhomogeneous equation. Use the method of variation of parameters to find a particular solution to the nonhomogeneous equation. Assume $x > 0$ in each exercise.

59. $x^2y'' + 2xy' - 2y = x^2, \quad y_1 = x^{-2},\ y_2 = x$
60. $x^2y'' + xy' - y = x, \quad y_1 = x^{-1},\ y_2 = x$

17.3 Applications

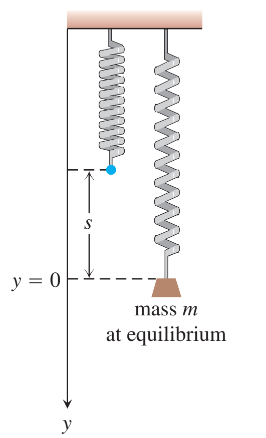

FIGURE 17.2 Mass m stretches a spring by length s to the equilibrium position at $y = 0$.

In this section we apply second-order differential equations to the study of vibrating springs and electric circuits.

Vibrations

A spring has its upper end fastened to a rigid support, as shown in Figure 17.2. An object of mass m is suspended from the spring and stretches it a length s when the spring comes to rest in an equilibrium position. According to Hooke's Law (Section 6.5), the tension force in the spring is ks, where k is the spring constant. The force due to gravity pulling down on the spring is mg, and equilibrium requires that

$$ks = mg. \tag{1}$$

Suppose that the object is pulled down an additional amount y_0 beyond the equilibrium position and then released. We want to study the object's motion, that is, the vertical position of its center of mass at any future time.

Let y, with positive direction downward, denote the displacement position of the object away from the equilibrium position $y = 0$ at any time t after the motion has started. Then the forces acting on the object are (see Figure 17.3)

$F_p = mg$, the propulsion force due to gravity,

$F_s = k(s + y)$, the restoring force of the spring's tension,

$F_r = \delta \dfrac{dy}{dt}$, a frictional force assumed proportional to velocity.

The frictional force tends to retard the motion of the object. The resultant of these forces is $F = F_p - F_s - F_r$, and by Newton's second law $F = ma$, we must then have

$$m\frac{d^2y}{dt^2} = mg - ks - ky - \delta\frac{dy}{dt}.$$

By Equation (1), $mg - ks = 0$, so this last equation becomes

$$m\frac{d^2y}{dt^2} + \delta\frac{dy}{dt} + ky = 0, \tag{2}$$

subject to the initial conditions $y(0) = y_0$ and $y'(0) = 0$. (Here we use the prime notation to denote differentiation with respect to time t.)

You might expect that the motion predicted by Equation (2) will be oscillatory about the equilibrium position $y = 0$ and eventually damp to zero because of the retarding frictional force. This is indeed the case, and we will show how the constants m, δ, and k determine the nature of the damping. You will also see that if there is no friction (so $\delta = 0$), then the object will simply oscillate indefinitely.

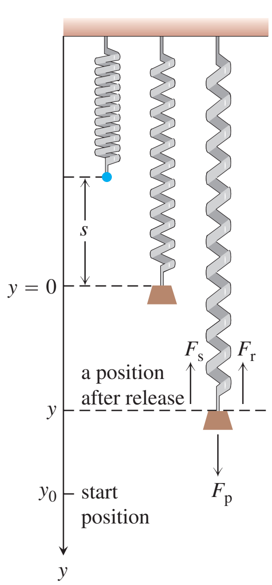

FIGURE 17.3 The propulsion force (weight) F_p pulls the mass downward, but the spring restoring force F_s and frictional force F_r pull the mass upward. The motion starts at $y = y_0$ with the mass vibrating up and down.

Simple Harmonic Motion

Suppose first that there is no retarding frictional force. Then $\delta = 0$ and there is no damping. If we substitute $\omega = \sqrt{k/m}$ to simplify our calculations, then the second-order equation (2) becomes

$$y'' + \omega^2 y = 0, \qquad \text{with} \qquad y(0) = y_0 \qquad \text{and} \qquad y'(0) = 0.$$

The auxiliary equation is

$$r^2 + \omega^2 = 0,$$

having the imaginary roots $r = \pm\omega i$. The general solution to the differential equation in (2) is

$$y = c_1 \cos \omega t + c_2 \sin \omega t. \tag{3}$$

To fit the initial conditions, we compute

$$y' = -c_1\omega \sin \omega t + c_2\omega \cos \omega t$$

and then substitute the conditions. This yields $c_1 = y_0$ and $c_2 = 0$. The particular solution

$$y = y_0 \cos \omega t \tag{4}$$

describes the motion of the object. Equation (4) represents **simple harmonic motion** of amplitude y_0 and period $T = 2\pi/\omega$.

The general solution given by Equation (3) can be combined into a single term by using the trigonometric identity

$$\sin(\omega t + \phi) = \cos \omega t \sin \phi + \sin \omega t \cos \phi.$$

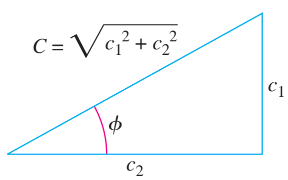

FIGURE 17.4 $c_1 = C \sin \phi$ and $c_2 = C \cos \phi$.

To apply the identity, we take (see Figure 17.4)

$$c_1 = C \sin \phi \qquad \text{and} \qquad c_2 = C \cos \phi,$$

where

$$C = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \phi = \tan^{-1} \frac{c_1}{c_2}.$$

Then the general solution in Equation (3) can be written in the alternative form

$$y = C \sin(\omega t + \phi). \tag{5}$$

Here C and ϕ may be taken as two new arbitrary constants, replacing the two constants c_1 and c_2. Equation (5) represents simple harmonic motion of amplitude C and period $T = 2\pi/\omega$. The angle $\omega t + \phi$ is called the **phase angle**, and ϕ may be interpreted as its initial value. A graph of the simple harmonic motion represented by Equation (5) is given in Figure 17.5.

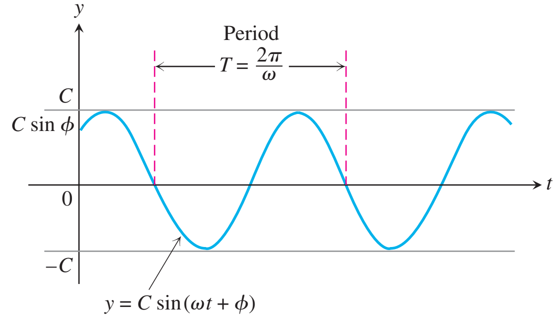

FIGURE 17.5 Simple harmonic motion of amplitude C and period T with initial phase angle ϕ (Equation 5).

Damped Motion

Assume now that there is friction in the spring system, so $\delta \neq 0$. If we substitute $\omega = \sqrt{k/m}$ and $2b = \delta/m$, then the differential equation (2) is

$$y'' + 2by' + \omega^2 y = 0. \tag{6}$$

The auxiliary equation is

$$r^2 + 2br + \omega^2 = 0,$$

with roots $r = -b \pm \sqrt{b^2 - \omega^2}$. Three cases now present themselves, depending upon the relative sizes of b and ω.

Case 1: $b = \omega$. The double root of the auxiliary equation is real and equals $r = \omega$. The general solution to Equation (6) is

$$y = (c_1 + c_2 t)e^{-\omega t}.$$

This situation of motion is called **critical damping** and is not oscillatory. Figure 17.6a shows an example of this kind of damped motion.

Case 2: $b > \omega$. The roots of the auxiliary equation are real and unequal, given by $r_1 = -b + \sqrt{b^2 - \omega^2}$ and $r_2 = -b - \sqrt{b^2 - \omega^2}$. The general solution to Equation (6) is given by

$$y = c_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}.$$

Here again the motion is not oscillatory and both r_1 and r_2 are negative. Thus y approaches zero as time goes on. This motion is referred to as **overdamping** (see Figure 17.6b).

Case 3: $b < \omega$. The roots to the auxiliary equation are complex and given by $r = -b \pm i\sqrt{\omega^2 - b^2}$. The general solution to Equation (6) is given by

$$y = e^{-bt}\left(c_1 \cos\sqrt{\omega^2 - b^2}\, t + c_2 \sin\sqrt{\omega^2 - b^2}\, t\right).$$

This situation, called **underdamping**, represents damped oscillatory motion. It is analogous to simple harmonic motion of period $T = 2\pi/\sqrt{\omega^2 - b^2}$ except that the amplitude is not constant but damped by the factor e^{-bt}. Therefore, the motion tends to zero as t increases, so the vibrations tend to die out as time goes on. Notice that the period $T = 2\pi/\sqrt{\omega^2 - b^2}$ is larger than the period $T_0 = 2\pi/\omega$ in the friction-free system. Moreover, the larger the value of $b = \delta/2m$ in the exponential damping factor, the more quickly the vibrations tend to become unnoticeable. A curve illustrating underdamped motion is shown in Figure 17.6c.

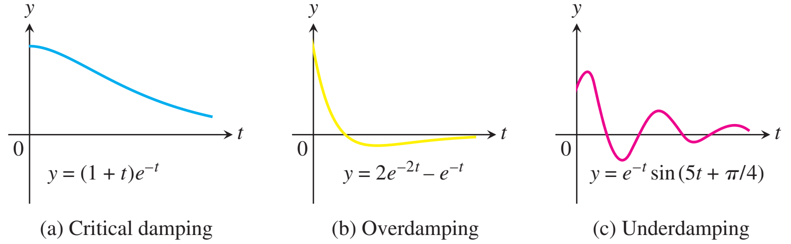

FIGURE 17.6 Three examples of damped vibratory motion for a spring system with friction, so $\delta \neq 0$.

An external force $F(t)$ can also be added to the spring system modeled by Equation (2). The forcing function may represent an external disturbance on the system. For instance, if the equation models an automobile suspension system, the forcing function might represent periodic bumps or potholes in the road affecting the performance of the suspension system; or it might represent the effects of winds when modeling the vertical motion of a suspension bridge. Inclusion of a forcing function results in the second-order nonhomogeneous equation

$$m\frac{d^2y}{dt^2} + \delta\frac{dy}{dt} + ky = F(t). \tag{7}$$

We leave the study of such spring systems to a more advanced course.

Electric Circuits

The basic quantity in electricity is the **charge** q (analogous to the idea of mass). In an electric field we use the flow of charge, or **current** $I = dq/dt$, as we might use velocity in a gravitational field. There are many similarities between motion in a gravitational field and the flow of electrons (the carriers of charge) in an electric field.

Consider the electric circuit shown in Figure 17.7. It consists of four components: voltage source, resistor, inductor, and capacitor. Think of electrical flow as being like a fluid flow, where the voltage source is the pump and the resistor, inductor, and capacitor tend to block the flow. A battery or generator is an example of a source, producing a voltage that causes the current to flow through the circuit when the switch is closed. An electric light bulb or appliance would provide resistance. The inductance is due to a magnetic field that opposes any change in the current as it flows through a coil. The capacitance is normally created by two metal plates that alternate charges and thus reverse the current flow. The following symbols specify the quantities relevant to the circuit:

- q: charge at a cross section of a conductor measured in **coulombs** (abbreviated c);
- I: current or rate of change of charge dq/dt (flow of electrons) at a cross section of a conductor measured in **amperes** (abbreviated A);
- E: electric (potential) source measured in **volts** (abbreviated V);
- V: difference in potential between two points along the conductor measured in **volts** (V).

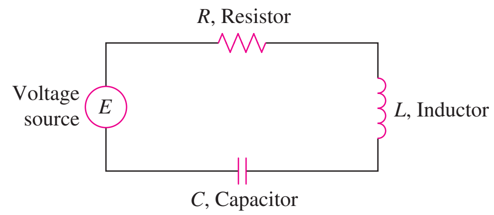

FIGURE 17.7 An electric circuit.

Ohm observed that the current I flowing through a resistor, caused by a potential difference across it, is (approximately) proportional to the potential difference (voltage drop). He named his constant of proportionality $1/R$ and called R the **resistance**. So *Ohm's law* is

$$I = \frac{1}{R}V.$$

Similarly, it is known from physics that the voltage drops across an inductor and a capacitor are

$$L\,\frac{dI}{dt} \qquad \text{and} \qquad \frac{q}{C},$$

where L is the **inductance** and C is the **capacitance** (with q the charge on the capacitor).

The German physicist Gustav R. Kirchhoff (1824–1887) formulated the law that the sum of the voltage drops in a closed circuit is equal to the supplied voltage $E(t)$. Symbolically, this says that

$$RI + L\,\frac{dI}{dt} + \frac{q}{C} = E(t).$$

Since $I = dq/dt$, Kirchhoff's law becomes

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = E(t). \tag{8}$$

The second-order differential equation (8), which models an electric circuit, has exactly the same form as Equation (7) modeling vibratory motion. Both models can be solved using the methods developed in Section 17.2.

Summary

The following chart summarizes our analogies for the physics of motion of an object in a spring system versus the flow of charged particles in an electrical circuit.

Linear Second-Order Constant-Coefficient Models

Mechanical System		**Electrical System**	
$my'' + \delta y' + ky = F(t)$		$Lq'' + Rq' + \frac{1}{C}q = E(t)$	
y:	displacement	q:	charge
y':	velocity	q':	current
y'':	acceleration	q'':	change in current
m:	mass	L:	inductance
δ:	damping constant	R:	resistance
k:	spring constant	$1/C$:	where C is the capacitance
$F(t)$:	forcing function	$E(t)$:	voltage source

EXERCISES 17.3

1. A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 1 lb/ft. The resistance in the spring–mass system is numerically equal to the instantaneous velocity. At $t = 0$ the weight is set in motion from a position 2 ft below its equilibrium position by giving it a downward velocity of 2 ft/sec. Write an initial value problem that models the given situation.

2. An 8-lb weight stretches a spring 4 ft. The spring–mass system resides in a medium offering a resistance to the motion that is numerically equal to 1.5 times the instantaneous velocity. If the weight is released at a position 2 ft above its equilibrium position with a downward velocity of 3 ft/sec, write an initial value problem modeling the given situation.

3. A 20-lb weight is hung on an 18-in. spring and stretches it 6 in. The weight is pulled down 5 in. and 5 lb are added to the weight. If the weight is now released with a downward velocity of v_0 in./sec, write an initial value problem modeling the vertical displacement.

4. A 10-lb weight is suspended by a spring that is stretched 2 in. by the weight. Assume a resistance whose magnitude is $20/\sqrt{g}$ lb times the instantaneous velocity v in feet per second. If the weight is pulled down 3 in. below its equilibrium position and released, formulate an initial value problem modeling the behavior of the spring–mass system.

5. An (open) electrical circuit consists of an inductor, a resistor, and a capacitor. There is an initial charge of 2 coulombs on the capacitor. At the instant the circuit is closed, a current of 3 amperes is present and a voltage of $E(t) = 20 \cos t$ is applied. In this circuit the voltage drop across the resistor is 4 times the instantaneous change in the charge, the voltage drop across the capacitor is 10 times the charge, and the voltage drop across the inductor is 2 times the instantaneous change in the current. Write an initial value problem to model the circuit.

6. An inductor of 2 henrys is connected in series with a resistor of 12 ohms, a capacitor of 1/16 farad, and a 300 volt battery. Initially, the charge on the capacitor is zero and the current is zero. Formulate an initial value problem modeling this electrical circuit.

Mechanical units in the British and metric systems may be helpful in doing the following problems.

Unit	British System	MKS System
Distance	Feet (ft)	Meters (m)
Mass	Slugs	Kilograms (kg)
Time	Seconds (sec)	Seconds (sec)
Force	Pounds (lb)	Newtons (N)
g(earth)	32 ft/sec^2	9.81 m/sec^2

7. A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 1 lb/ft. The resistance in the spring–mass system is numerically equal to the instantaneous velocity. At $t = 0$ the weight is set in motion from a position 2 ft below its equilibrium position by giving it a downward velocity of 2 ft/sec. At the end of π sec, determine whether the mass is above or below the equilibrium position and by what distance.

8. An 8-lb weight stretches a spring 4 ft. The spring–mass system resides in a medium offering a resistance to the motion equal to 1.5 times the instantaneous velocity. If the weight is released at a position 2 ft above its equilibrium position with a downward velocity of 3 ft/sec, find its position relative to the equilibrium position 2 sec later.

9. A 20-lb weight is hung on an 18-in. spring stretching it 6 in. The weight is pulled down 5 in. and 5 lb are added to the weight. If the weight is now released with a downward velocity of v_0 in./sec, find the position of mass relative to the equilibrium in terms of v_0 and valid for any time $t \geq 0$.

10. A mass of 1 slug is attached to a spring whose constant is 25/4 lb/ft. Initially the mass is released 1 ft above the equilibrium position with a downward velocity of 3 ft/sec, and the subsequent motion takes place in a medium that offers a damping force numerically equal to 3 times the instantaneous velocity. An external force $f(t)$ is driving the system, but assume that initially $f(t) \equiv 0$. Formulate and solve an initial value problem that models the given system. Interpret your results.

11. A 10-lb weight is suspended by a spring that is stretched 2 in. by the weight. Assume a resistance whose magnitude is $40/\sqrt{g}$ lb times the instantaneous velocity in feet per second. If the weight is pulled down 3 in. below its equilibrium position and released, find the time required to reach the equilibrium position for the first time.

12. A weight stretches a spring 6 in. It is set in motion at a point 2 in. below its equilibrium position with a downward velocity of 2 in./sec.

a. When does the weight return to its starting position?

b. When does it reach its highest point?

c. Show that the maximum velocity is $2\sqrt{2g + 1}$ in./sec.

13. A weight of 10 lb stretches a spring 10 in. The weight is drawn down 2 in. below its equilibrium position and given an initial velocity of 4 in./sec. An identical spring has a different weight attached to it. This second weight is drawn down from its equilibrium position a distance equal to the amplitude of the first motion and then given an initial velocity of 2 ft/sec. If the amplitude of the second motion is twice that of the first, what weight is attached to the second spring?

14. A weight stretches one spring 3 in. and a second weight stretches another spring 9 in. If both weights are simultaneously pulled down 1 in. below their respective equilibrium positions and then released, find the first time after $t = 0$ when their velocities are equal.

15. A weight of 16 lb stretches a spring 4 ft. The weight is pulled down 5 ft below the equilibrium position and then released. What initial velocity v_0 given to the weight would have the effect of doubling the amplitude of the vibration?

16. A mass weighing 8 lb stretches a spring 3 in. The spring–mass system resides in a medium with a damping constant of 2 lb-sec/ft. If the mass is released from its equilibrium position with a velocity of 4 in./sec in the downward direction, find the time required for the mass to return to its equilibrium position for the first time.

17. A weight suspended from a spring executes damped vibrations with a period of 2 sec. If the damping factor decreases by 90% in 10 sec, find the acceleration of the weight when it is 3 in. below its equilibrium position and is moving upward with a speed of 2 ft/sec.

18. A 10-lb weight stretches a spring 2 ft. If the weight is pulled down 6 in. below its equilibrium position and released, find the highest point reached by the weight. Assume the spring–mass system resides in a medium offering a resistance of $10/\sqrt{g}$ lb times the instantaneous velocity in feet per second.

19. An *LRC* circuit is set up with an inductance of $1/5$ henry, a resistance of 1 ohm, and a capacitance of $5/6$ farad. Assuming the initial charge is 2 coulombs and the initial current is 4 amperes, find the solution function describing the charge on the capacitor at any time. What is the charge on the capacitor after a long period of time?

20. An (open) electrical circuit consists of an inductor, a resistor, and a capacitor. There is an initial charge of 2 coulombs on the capacitor. At the instant the circuit is closed, a current of 3 amperes is present but no external voltage is being applied. In this circuit the voltage drops at three points are numerically related as follows: across the capacitor, 10 times the charge; across the resistor, 4 times the instantaneous change in the charge; and across the inductor, 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time.

21. A 16-lb weight stretches a spring 4 ft. This spring–mass system is in a medium with a damping constant of 4.5 lb-sec/ft, and an external force given by $f(t) = 4 + e^{-2t}$ (in pounds) is being applied. What is the solution function describing the position of the mass at any time if the mass is released from 2 ft below the equilibrium position with an initial velocity of 4 ft/sec downward?

22. A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force given by $f(t) = 5 \sin t$ (in newtons). The mass is in a viscous medium with a coefficient of resistance equal to 90 N-sec/m. Formulate an initial value problem that models the given system; solve the model and interpret the results.

23. A 2-kg mass is attached to the lower end of a coil spring suspended from the ceiling. The mass comes to rest in its equilibrium position thereby stretching the spring 1.96 m. The mass is in a viscous medium that offers a resistance in newtons numerically equal to 4 times the instantaneous velocity measured in meters per second. The mass is then pulled down 2 m below its equilibrium position and released with a downward velocity of 3 m/sec. At this same instant an external force given by $f(t) = 20 \cos t$ (in newtons) is applied to the system. At the end of π sec determine if the mass is above or below its equilibrium position and by how much.

24. An 8-lb weight stretches a spring 4 ft. The spring–mass system resides in a medium offering a resistance to the motion equal to 1.5 times the instantaneous velocity, and an external force given by $f(t) = 6 + e^{-t}$ (in pounds) is being applied. If the weight is released at a position 2 ft above its equilibrium position with downward velocity of 3 ft/sec, find its position relative to the equilibrium after 2 sec have elapsed.

25. Suppose $L = 10$ henrys, $R = 10$ ohms, $C = 1/500$ farads, $E = 100$ volts, $q(0) = 10$ coulombs, and $q'(0) = i(0) = 0$. Formulate and solve an initial value problem that models the given *LRC* circuit. Interpret your results.

26. A series circuit consisting of an inductor, a resistor, and a capacitor is open. There is an initial charge of 2 coulombs on the capacitor, and 3 amperes of current is present in the circuit at the instant the circuit is closed. A voltage given by $E(t) = 20 \cos t$ is applied. In this circuit the voltage drops are numerically equal to the following: across the resistor to 4 times the instantaneous change in the charge, across the capacitor to 10 times the charge, and across the inductor to 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time. Determine the charge on the capacitor and the current at time $t = 10$.

17.4 Euler Equations

In Section 17.1 we introduced the second-order linear homogeneous differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

and showed how to solve this equation when the coefficients P, Q, and R are constants. If the coefficients are not constant, we cannot generally solve this differential equation in terms of elementary functions we have studied in calculus. In this section you will learn how to solve the equation when the coefficients have the special forms

$$P(x) = ax^2, \qquad Q(x) = bx, \qquad \text{and} \qquad R(x) = c,$$

where a, b, and c are constants. These special types of equations are called **Euler equations**, in honor of Leonhard Euler who studied them and showed how to solve them. Such equations arise in the study of mechanical vibrations.

The General Solution of Euler Equations

Consider the Euler equation

$$ax^2y'' + bxy' + cy = 0, \quad x > 0. \tag{1}$$

To solve Equation (1), we first make the change of variables

$$z = \ln x \qquad \text{and} \qquad y(x) = Y(z).$$

We next use the chain rule to find the derivatives $y'(x)$ and $y''(x)$:

$$y'(x) = \frac{d}{dx}Y(z) = \frac{d}{dz}Y(z)\frac{dz}{dx} = Y'(z)\frac{1}{x}$$

and

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}Y'(z)\frac{1}{x} = -\frac{1}{x^2}Y'(z) + \frac{1}{x}Y''(z)\frac{dz}{dx} = -\frac{1}{x^2}Y'(z) + \frac{1}{x^2}Y''(z).$$

Substituting these two derivatives into the left-hand side of Equation (1), we find

$$\begin{aligned} ax^2y'' + bxy' + cy &= ax^2\left(-\frac{1}{x^2}Y'(z) + \frac{1}{x^2}Y''(z)\right) + bx\left(\frac{1}{x}Y'(z)\right) + cY(z) \\ &= aY''(z) + (b - a)Y'(z) + cY(z). \end{aligned}$$

Therefore, the substitutions give us the second-order linear differential equation with constant coefficients

$$aY''(z) + (b - a)Y'(z) + cY(z) = 0. \tag{2}$$

We can solve Equation (2) using the method of Section 17.1. That is, we find the roots to the associated auxiliary equation

$$ar^2 + (b - a)r + c = 0 \tag{3}$$

to find the general solution for $Y(z)$. After finding $Y(z)$, we can determine $y(x)$ from the substitution $z = \ln x$.

EXAMPLE 1 Find the general solution of the equation $x^2y'' + 2xy' - 2y = 0$.

Solution This is an Euler equation with $a = 1$, $b = 2$, and $c = -2$. The auxiliary equation (3) for $Y(z)$ is

$$r^2 + (2 - 1)r - 2 = (r - 1)(r + 2) = 0,$$

with roots $r = -2$ and $r = 1$. The solution for $Y(z)$ is given by

$$Y(z) = c_1e^{-2z} + c_2e^z.$$

Substituting $z = \ln x$ gives the general solution for $y(x)$:

$$y(x) = c_1e^{-2\ln x} + c_2e^{\ln x} = c_1x^{-2} + c_2x$$ ■

EXAMPLE 2 Solve the Euler equation $x^2y'' - 5xy' + 9y = 0$.

Solution Since $a = 1$, $b = -5$, and $c = 9$, the auxiliary equation (3) for $Y(z)$ is

$$r^2 + (-5 - 1)r + 9 = (r - 3)^2 = 0.$$

The auxiliary equation has the double root $r = 3$ giving

$$Y(z) = c_1e^{3z} + c_2ze^{3z}.$$

Substituting $z = \ln x$ into this expression gives the general solution

$$y(x) = c_1e^{3\ln x} + c_2\ln x\, e^{3\ln x} = c_1x^3 + c_2x^3\ln x$$ ■

EXAMPLE 3 Find the particular solution to $x^2y'' - 3xy' + 68y = 0$ that satisfies the initial conditions $y(1) = 0$ and $y'(1) = 1$.

Solution Here $a = 1$, $b = -3$, and $c = 68$ substituted into the auxiliary equation (3) gives

$$r^2 - 4r + 68 = 0.$$

The roots are $r = 2 + 8i$ and $r = 2 - 8i$ giving the solution

$$Y(z) = e^{2z}(c_1 \cos 8z + c_2 \sin 8z).$$

Substituting $z = \ln x$ into this expression gives

$$y(x) = e^{2 \ln x}\big(c_1 \cos(8 \ln x) + c_2 \sin(8 \ln x)\big).$$

From the initial condition $y(1) = 0$, we see that $c_1 = 0$ and

$$y(x) = c_2 x^2 \sin(8 \ln x).$$

To fit the second initial condition, we need the derivative

$$y'(x) = c_2\big(8x \cos(8 \ln x) + 2x \sin(8 \ln x)\big).$$

Since $y'(1) = 1$, we immediately obtain $c_2 = 1/8$. Therefore, the particular solution satisfying both initial conditions is

$$y(x) = \frac{1}{8}x^2 \sin(8 \ln x).$$

Since $-1 \le \sin(8 \ln x) \le 1$, the solution satisfies

$$-\frac{x^2}{8} \le y(x) \le \frac{x^2}{8}.$$

A graph of the solution is shown in Figure 17.8. ■

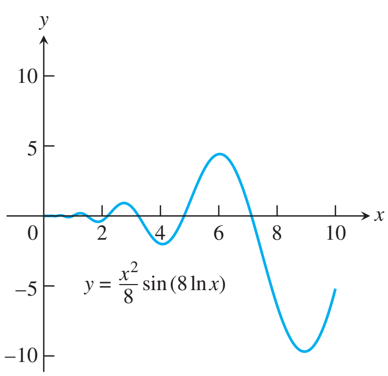

FIGURE 17.8 Graph of the solution to Example 3.

EXERCISES 17.4

In Exercises 1–24, find the general solution to the given Euler equation. Assume $x > 0$ throughout.

1. $x^2y'' + 2xy' - 2y = 0$
2. $x^2y'' + xy' - 4y = 0$
3. $x^2y'' - 6y = 0$
4. $x^2y'' + xy' - y = 0$
5. $x^2y'' - 5xy' + 8y = 0$
6. $2x^2y'' + 7xy' + 2y = 0$
7. $3x^2y'' + 4xy' = 0$
8. $x^2y'' + 6xy' + 4y = 0$
9. $x^2y'' - xy' + y = 0$
10. $x^2y'' - xy' + 2y = 0$
11. $x^2y'' - xy' + 5y = 0$
12. $x^2y'' + 7xy' + 13y = 0$
13. $x^2y'' + 3xy' + 10y = 0$
14. $x^2y'' - 5xy' + 10y = 0$
15. $4x^2y'' + 8xy' + 5y = 0$
16. $4x^2y'' - 4xy' + 5y = 0$
17. $x^2y'' + 3xy' + y = 0$
18. $x^2y'' - 3xy' + 9y = 0$
19. $x^2y'' + xy' = 0$
20. $4x^2y'' + y = 0$
21. $9x^2y'' + 15xy' + y = 0$
22. $16x^2y'' - 8xy' + 9y = 0$
23. $16x^2y'' + 56xy' + 25y = 0$
24. $4x^2y'' - 16xy' + 25y = 0$

In Exercises 25–30, solve the given initial value problem.

25. $x^2y'' + 3xy' - 3y = 0, \quad y(1) = 1,\ y'(1) = -1$
26. $6x^2y'' + 7xy' - 2y = 0, \quad y(1) = 0,\ y'(1) = 1$
27. $x^2y'' - xy' + y = 0, \quad y(1) = 1,\ y'(1) = 1$
28. $x^2y'' + 7xy' + 9y = 0, \quad y(1) = 1,\ y'(1) = 0$
29. $x^2y'' - xy' + 2y = 0, \quad y(1) = -1,\ y'(1) = 1$
30. $x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1,\ y'(1) = 0$

17.5 Power-Series Solutions

In this section we extend our study of second-order linear homogeneous equations with variable coefficients. With the Euler equations in Section 17.4, the power of the variable x in the nonconstant coefficient had to match the order of the derivative with which it was paired: x^2 with y'', x^1 with y', and $x^0\,(=1)$ with y. Here we drop that requirement so we can solve more general equations.

Method of Solution

The **power-series method** for solving a second-order homogeneous differential equation consists of finding the coefficients of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \tag{1}$$

which solves the equation. To apply the method we substitute the series and its derivatives into the differential equation to determine the coefficients $c_0, c_1, c_2, \ldots$. The technique for finding the coefficients is similar to that used in the method of undetermined coefficients presented in Section 17.2.

In our first example we demonstrate the method in the setting of a simple equation whose general solution we already know. This is to help you become more comfortable with solutions expressed in series form.

EXAMPLE 1 Solve the equation $y'' + y = 0$ by the power-series method.

Solution We assume the series solution takes the form of

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substitution of these forms into the second-order equation gives us

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient Equation		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2} c_0$
x^1	$3(2)c_3 + c_1 = 0$	or	$c_3 = -\frac{1}{3\cdot 2} c_1$
x^2	$4(3)c_4 + c_2 = 0$	or	$c_4 = -\frac{1}{4\cdot 3} c_2$
x^3	$5(4)c_5 + c_3 = 0$	or	$c_5 = -\frac{1}{5\cdot 4} c_3$
x^4	$6(5)c_6 + c_4 = 0$	or	$c_6 = -\frac{1}{6\cdot 5} c_4$
$\vdots$	$\vdots$		$\vdots$
x^{n-2}	$n(n-1)c_n + c_{n-2} = 0$	or	$c_n = -\frac{1}{n(n-1)} c_{n-2}$

From the table we notice that the coefficients with even indices ($n = 2k, k = 1, 2, 3, \ldots$) are related to each other and the coefficients with odd indices ($n = 2k + 1$) are also interrelated. We treat each group in turn.

Even indices: Here $n = 2k$, so the power is x^{2k-2}. From the last line of the table, we have

$$2k(2k - 1)c_{2k} + c_{2k-2} = 0$$

or

$$c_{2k} = -\frac{1}{2k(2k - 1)} c_{2k-2}.$$

From this recursive relation we find

$$\begin{aligned} c_{2k} &= \left[-\frac{1}{2k(2k - 1)}\right]\left[-\frac{1}{(2k - 2)(2k - 3)}\right]\cdots\left[-\frac{1}{4(3)}\right]\left[-\frac{1}{2}\right]c_0 \\ &= \frac{(-1)^k}{(2k)!} c_0. \end{aligned}$$

Odd indices: Here $n = 2k + 1$, so the power is x^{2k-1}. Substituting this into the last line of the table yields

$$(2k + 1)(2k)c_{2k+1} + c_{2k-1} = 0$$

or

$$c_{2k+1} = -\frac{1}{(2k + 1)(2k)} c_{2k-1}.$$

Thus,

$$\begin{aligned} c_{2k+1} &= \left[-\frac{1}{(2k + 1)(2k)}\right]\left[-\frac{1}{(2k - 1)(2k - 2)}\right]\cdots\left[-\frac{1}{5(4)}\right]\left[-\frac{1}{3(2)}\right]c_1 \\ &= \frac{(-1)^k}{(2k + 1)!} c_1. \end{aligned}$$

Writing the power series by grouping its even and odd powers together and substituting for the coefficients yields

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}. \end{aligned}$$

From Table 9.1 in Section 9.10, we see that the first series on the right-hand side of the last equation represents the cosine function and the second series represents the sine. Thus, the general solution to $y'' + y = 0$ is

$$y = c_0 \cos x + c_1 \sin x.$$

EXAMPLE 2 Find the general solution to $y'' + xy' + y = 0$.

Solution We assume the series solution form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Substitution of these forms into the second-order equation yields

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient Equation		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 + c_1 + c_1 = 0$	or	$c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 + 2c_2 + c_2 = 0$	or	$c_4 = -\frac{1}{4}c_2$
x^3	$5(4)c_5 + 3c_3 + c_3 = 0$	or	$c_5 = -\frac{1}{5}c_3$
x^4	$6(5)c_6 + 4c_4 + c_4 = 0$	or	$c_6 = -\frac{1}{6}c_4$
$\vdots$	$\vdots$		$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$	or	$c_{n+2} = -\frac{1}{n+2}c_n$

From the table notice that the coefficients with even indices are interrelated and the coefficients with odd indices are also interrelated.

Even indices: Here $n = 2k - 2$, so the power is x^{2k-2}. From the last line in the table, we have

$$c_{2k} = -\frac{1}{2k}c_{2k-2}.$$

From this recurrence relation we obtain

$$c_{2k} = \left(-\frac{1}{2k}\right)\left(-\frac{1}{2k-2}\right)\cdots\left(-\frac{1}{6}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)c_0$$

$$= \frac{(-1)^k}{(2)(4)(6)\cdots(2k)}c_0.$$

Odd indices: Here $n = 2k - 1$, so the power is x^{2k-1}. From the last line in the table, we have

$$c_{2k+1} = -\frac{1}{2k+1}c_{2k-1}.$$

From this recurrence relation we obtain

$$c_{2k+1} = \left(-\frac{1}{2k+1}\right)\left(-\frac{1}{2k-1}\right)\cdots\left(-\frac{1}{5}\right)\left(-\frac{1}{3}\right)c_1$$

$$= \frac{(-1)^k}{(3)(5)\cdots(2k+1)}c_1.$$

Writing the power series by grouping its even and odd powers and substituting for the coefficients yields

$$y = \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1}$$
$$= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)(4)\cdots(2k)} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(3)(5)\cdots(2k+1)} x^{2k+1}.$$ ■

EXAMPLE 3 Find the general solution to

$$(1 - x^2)y'' - 6xy' - 4y = 0, \qquad |x| < 1.$$

Solution Notice that the leading coefficient is zero when $x = \pm 1$. Thus, we assume the solution interval I: $-1 < x < 1$. Substitution of the series form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and its derivatives gives us

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6\sum_{n=1}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0,$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - 6\sum_{n=1}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient Equation		
x^0	$2(1)c_2 - 4c_0 = 0$	or	$c_2 = \frac{4}{2}c_0$
x^1	$3(2)c_3 - 6(1)c_1 - 4c_1 = 0$	or	$c_3 = \frac{5}{3}c_1$
x^2	$4(3)c_4 - 2(1)c_2 - 6(2)c_2 - 4c_2 = 0$	or	$c_4 = \frac{6}{4}c_2$
x^3	$5(4)c_5 - 3(2)c_3 - 6(3)c_3 - 4c_3 = 0$	or	$c_5 = \frac{7}{5}c_3$
⋮	⋮		⋮
x^n	$(n+2)(n+1)c_{n+2} - [n(n-1) + 6n + 4]c_n = 0$		
	$(n+2)(n+1)c_{n+2} - (n+4)(n+1)c_n = 0$	or	$c_{n+2} = \dfrac{n+4}{n+2}c_n$

Again we notice that the coefficients with even indices are interrelated and those with odd indices are interrelated.

Even indices: Here $n = 2k - 2$, so the power is x^{2k}. From the right-hand column and last line of the table, we get

$$c_{2k} = \frac{2k+2}{2k}c_{2k-2}$$
$$= \left(\frac{2k+2}{2k}\right)\left(\frac{2k}{2k-2}\right)\left(\frac{2k-2}{2k-4}\right)\cdots\frac{6}{4}\left(\frac{4}{2}\right)c_0$$
$$= (k+1)c_0.$$

Odd indices: Here $n = 2k - 1$, so the power is x^{2k+1}. The right-hand column and last line of the table gives us

$$\begin{aligned} c_{2k+1} &= \frac{2k+3}{2k+1} c_{2k-1} \\ &= \left(\frac{2k+3}{2k+1}\right)\left(\frac{2k+1}{2k-1}\right)\left(\frac{2k-1}{2k-3}\right)\cdots\frac{7}{5}\left(\frac{5}{3}\right)c_1 \\ &= \frac{2k+3}{3} c_1. \end{aligned}$$

The general solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} (k+1) x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{2k+3}{3} x^{2k+1}. \end{aligned}$$

■

EXAMPLE 4 Find the general solution to $y'' - 2xy' + y = 0$.

Solution Assuming that

$$y = \sum_{n=0}^{\infty} c_n x^n,$$

substitution into the differential equation gives us

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We next determine the coefficients, listing them in the following table.

Power of x	**Coefficient Equation**		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 - 2c_1 + c_1 = 0$	or	$c_3 = \frac{1}{3 \cdot 2}c_1$
x^2	$4(3)c_4 - 4c_2 + c_2 = 0$	or	$c_4 = \frac{3}{4 \cdot 3}c_2$
x^3	$5(4)c_5 - 6c_3 + c_3 = 0$	or	$c_5 = \frac{5}{5 \cdot 4}c_3$
x^4	$6(5)c_6 - 8c_4 + c_4 = 0$	or	$c_6 = \frac{7}{6 \cdot 5}c_4$
⋮	⋮		⋮
x^n	$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$	or	$c_{n+2} = \frac{2n-1}{(n+2)(n+1)}c_n$

From the recursive relation

$$c_{n+2} = \frac{2n - 1}{(n + 2)(n + 1)} c_n,$$

we write out the first few terms of each series for the general solution:

$$y = c_0\left(1 - \frac{1}{2}x^2 - \frac{3}{4!}x^4 - \frac{21}{6!}x^6 - \cdots\right)$$

$$+ c_1\left(x + \frac{1}{3!}x^3 + \frac{5}{5!}x^5 + \frac{45}{7!}x^7 + \cdots\right).$$

EXERCISES 17.5

In Exercises 1–18, use power series to find the general solution of the differential equation.

1. $y'' + 2y' = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 4y = 0$
4. $y'' - 3y' + 2y = 0$
5. $x^2y'' - 2xy' + 2y = 0$
6. $y'' - xy' + y = 0$
7. $(1 + x)y'' - y = 0$
8. $(1 - x^2)y'' - 4xy' + 6y = 0$
9. $(x^2 - 1)y'' + 2xy' - 2y = 0$
10. $y'' + y' - x^2y = 0$
11. $(x^2 - 1)y'' - 6y = 0$
12. $xy'' - (x + 2)y' + 2y = 0$
13. $(x^2 - 1)y'' + 4xy' + 2y = 0$
14. $y'' - 2xy' + 4y = 0$
15. $y'' - 2xy' + 3y = 0$
16. $(1 - x^2)y'' - xy' + 4y = 0$
17. $y'' - xy' + 3y = 0$
18. $x^2y'' - 4xy' + 6y = 0$