



— University of Mosul —
College of Petroleum & Mining Engineering



Simpson's One-Third Method

Lecture No.12

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2.3 Simpson's One-Third Method

The derivative of simple Simpson's One-Third rule formula to approximate value of the integral in eq.(1) we approximate the function $f(x)$ as Newton forward interpolation formula and then integration over the interval $[x_0, x_2]$, where

$$x = x_m = x_0 + mh, \quad m = 0, 1, 2, \dots, n$$

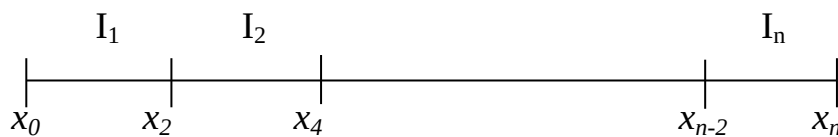
Now, let $n=2$, we have

$$\begin{aligned} \int_{a=x_0}^{b=x_2} f(x) dx &= \int_0^2 f(x_m) dx \frac{dm}{dh} = \int_0^2 f(x_m) dm \cdot h = h \int_0^2 f(x_m) dm \\ &= h \int_0^2 \left[f_0 + m\Delta f_0 + \frac{m(m-1)}{2!} \Delta^2 f_0 + \frac{m(m-1)(m-2)}{3!} \Delta^3 f_0 + \dots \right] dm \\ &= h \left[mf_0 + \frac{m^2}{2} \Delta f_0 + \left(\frac{m^3}{6} - \frac{m^2}{4} \right) \Delta^2 f_0 + \left(\frac{m^4}{24} - \frac{m^3}{6} + \frac{m^2}{6} \right) \Delta^3 f_0 + \dots \right]_0^2 \\ &= h \left[2f_0 + 2\Delta f_0 + \frac{1}{3} \Delta^2 f_0 + \dots - 0 \right] \\ \int_{a=x_0}^{b=x_2} f(x) dx &\cong \frac{h}{3} [f_0 + 4f_1 + f_2] \end{aligned} \quad (6)$$

Equation (6) is called **Simple Simpson's One-Third Rule formula**, the local truncation error in equation (6) is

$$\text{L. T. E.} = \frac{-h^5}{90} f^{(4)}(\theta), \quad \theta \in (x_0, x_2)$$

In general, we have subintervals $[x_{i-2}, x_i]$ and we apply eq.(6) at each subinterval we get:



$$\int_{x_i}^{x_{i+2}} f(x)dx \cong \frac{h}{3} [f_i + 4f_{i+1} + f_{i+2}]$$

then

$$\begin{aligned} \int_{a=x_0}^{b=x_n} f(x)dx &= \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)dx \\ &= \frac{h}{3} [f_0 + 4f_1 + f_2] + \frac{h}{3} [f_2 + 4f_3 + f_4] + \cdots + \frac{h}{3} [f_{n-2} + 4f_{n-1} + f_n] \\ &= \frac{h}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 4f_{n-1} + f_n] \\ &= \frac{h}{3} [f_0 + 4 \sum_{i=1}^n f(x_{2i-1}) + 2 \sum_{i=1}^{n-1} f(x_{2i}) + f_n] \end{aligned} \quad (7)$$

Equation (7) is called **Composite Simpson's One-Third Rule formula**, and the total truncation error in equation (7) is

$$\begin{aligned} E_s &= \left[\frac{-h^5}{90} f^4(\theta_1) \right] + \left[\frac{-h^5}{90} f^4(\theta_2) \right] + \cdots + \left[\frac{-h^5}{90} f^4(\theta_n) \right] \\ E_s &= -n \left[\frac{-h^5}{180} f^4(\theta) \right] = -\frac{(b-a)^5}{180n^4} f^4(\theta) \quad , \quad \theta \in (a, b) \end{aligned}$$

This is **Composite error term** in Composite Simpson's One-Third Rule.

Example(4): Evaluate the following integral using Simpson's One-Third Rule taking $h=0.25$.

$$\int_0^1 \frac{1}{1+x^2} dx$$

Solution:

$$a=0, b=1, \quad f(x) = \frac{1}{1+x^2} \quad \text{and} \quad h=0.25, \quad n = \frac{b-a}{h} = 4$$

$$x_i = x_{i-1} + h \implies$$

$$x_0 = 0, \quad f(x_0) = f(0) = 1$$

$$x_1 = 0 + 0.25 = 0.25, \quad f(x_1) = f(0.25) = 0.9412$$

$$x_2 = 0.25 + 0.25 = 0.5, \quad f(x_2) = f(0.5) = 0.8$$

$$x_3 = 0.5 + 0.25 = 0.75, \quad f(x_3) = f(0.75) = 0.64$$

$$x_4 = 0.75 + 0.25 = 1, \quad f(x_4) = f(1) = 0.5$$

$$I[f] = \int_0^1 \frac{1}{1+x^2} dx = \frac{h}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 4f_{n-1} + f_n]$$

$$= \frac{0.25}{3} [1 + 4(0.9412) + 2(0.8) + 4(0.64) + 0.5]$$

$$= 0.7854$$

Example(5): Use Simpson's One-Third Rule to evaluate the integral

$$\int_0^1 (x^3 + 1) dx$$

Considering $n=2$.

Solution:

$$a=0, b=1, \quad f(x) = (x^3 + 1) \quad \text{and}$$

$$n=2 \implies h = \frac{b-a}{n} = \frac{1}{2}$$

$$x_i = x_{i-1} + h \quad \Longrightarrow$$

$$x_0 = 0, \quad f(x_0) = f(0) = 1$$

$$x_1 = 0 + 1/2 = 1/2, \quad f(x_1) = f(1/2) = 9/8$$

$$x_2 = 1/2 + 1/2 = 1, \quad f(x_2) = f(1) = 2$$

$$I[f] = \int_0^1 (x^3 + 1) dx = \frac{h}{3} [f_0 + 4f_1 + f_2]$$

$$= \frac{1/2}{3} \left[1 + 4\left(\frac{9}{8}\right) + 2 \right] = \frac{12}{15} = 1.25$$

Evaluation the number of subintervals in Simpson's One-Third rule :

From the error formula

$$E_S = -\frac{(b-a)^5}{180n^4} f^4(\theta), \quad \theta \in (a, b)$$

Let $|f^4(\theta)| \leq M$ then the error E_S for the composite Simpson's One-Third rule is less than accuracy ϵ

$$|E_S| = \left| -\frac{(b-a)^5}{180n^4} f^4(\theta) \right| \leq \epsilon$$

$$\Longrightarrow \frac{(b-a)^5}{180n^4} M \leq \epsilon$$

$$\Longrightarrow n \geq \sqrt[4]{\frac{(b-a)^5}{180\epsilon} M}$$

Example(6): Find the value of the integral with accuracy $\epsilon = 0.001$ by using Simpson's One-Third rule

$$I(f) = \int_{0.5}^1 \cos x \, dx$$

Solution:

$$a=0.5, b=1, \quad f(x) = \cos x \quad \text{and} \quad \epsilon = 0.001$$

$$I[f] = \int_{0.5}^1 \cos x \, dx \cong 0.36206$$

Algorithm of Simpson's One-Third rule:

Input: $a, b, n, f(x)$

Step(1): Evaluate $h=(b-a)/n$

Step(2): for $i=0,1,2,\dots,n$

set $x_i=a+ih$ and $y_i=f(x_i)$

Step(3): set $s1=0$ and $s2=0$

Step(4): for $i=1,2,\dots,n-1$ set

$s1 = s1 + y_i$ (for i is even)

$s2 = s2 + y_i$ (for i is odd)

Step(5): evaluate $I=(h/3).[y_0+2.s1+4.s2+y_n]$

Step(6): print I and stop.