University of Mosul

College of Petroleum & Mining Engineering

Newton-Raphson method

Lecture No.5

Dr. Almutasim Abdulmuhsin H. Albaker

Petroleum and Refining Engineering Department

Email:almutasim@uomosul.edu.iq

(4) Newton-Raphson method:

Suppose that the function f is twice continuously differentiable on the interval [a, b], that is $f \in C^2[a, b]$. Assume that the initial approximation x_0 is near the root P. Then the graph of $y = f(x)$ intersects the x-axis at the point (P,0), and the point $(x_0, f(x_0))$ lies on the curve near the point (P,0). Define x_1 to be the point of intersection of the x-axis and the line tangent to the curve at the point $(x_0, f(x_0))$.

$$m = \frac{0-f(x_0)}{x_1-x_0} \qquad (1)$$

which is the slope of the line through $(x_1,0)$ and $(x_0, f(x_0))$, and since the slope at the point $(x_0, f(x_0))$ is

$$m = f'(x_0) \qquad (2)$$

then equating the values of the slope m in equations (1) and (2) and solving for x_1 we have:

$$f'(x_0) = \frac{-f(x_0)}{x_1-x_0}$$

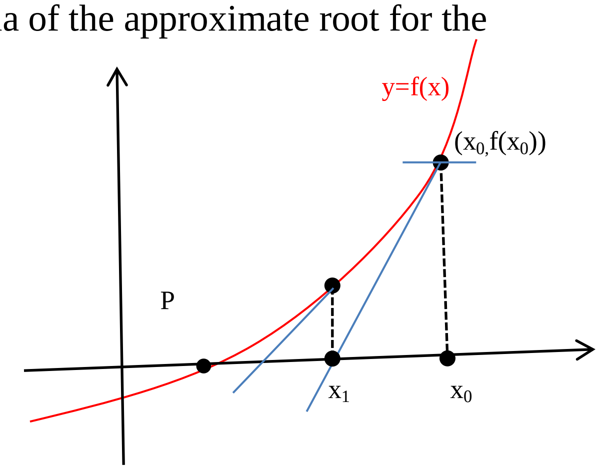
$$(x_1 - x_0)f'(x_0) = -f(x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This process can be repeated to obtain a sequence$\{x_i\}$ defined by the iteration

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} \ , \qquad i = 1,2,3, \ldots \qquad (3)$$

that converges to P. Equation(3) is the formula of the approximate root for the Newton-Raphson method



Example (9):

Find the root of the equation $x^3 - 3x + 2 = 0$ with x_0=-2.4 ,and ε=5*10^{-5} by using Newton-Raphson method.

Solution:

Since $f(x) = x^3 - 3x + 2$ then $f'(x) = 3x^2 - 3$ and x_0=-2.4 ,use the formula

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} , \qquad i = 1,2,3, \dots$$

i=1 $\Longrightarrow$ $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2.4 - \frac{f(-2.4)}{f'(-2.4)}$

$$= -2.4 - \frac{(-2.4)^3 - 3(-2.4) + 2}{3(-2.4)^2 - 3} = -2.0763$$

$$|x_1 - x_0| = |-2.0763 - (-2.4)| = |0.3237| > \varepsilon = 5*10^{-5}$$

$\longrightarrow$ $x_0 = x_1 = -2.0763$

i=2 $\Longrightarrow$ $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2.0763 - \frac{f(-2.0763)}{f'(-2.0763)}$

$$= -2.003699$$

$$|x_2 - x_1| = |-2.003699 - (-2.0763)| > \varepsilon = 5*10^{-5}$$

$\longrightarrow$ $x_1 = x_2 = -2.003699$

i=3 $\Longrightarrow$ $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -2.003699 - \frac{f(-2.003699)}{f'(-2.003699)}$

$$= -2.000009$$

$$|x_3 - x_2| = |-2.000009 - (-2.003699)| = 0.003690 > \varepsilon = 5*10^{-5}$$

$\longrightarrow$ $x_2 = x_3 = -2.000009$

i=4 $\Longrightarrow$ $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -2$

$$|x_4 - x_3| = |-2 - (-2.000009)| = 0.000009 < \varepsilon = 5*10^{-5}$$

The iteration is stop with the root $x_4 = -2$

Algorithm (Newton-Raphson method)

Input: x_0, ε, f(x)

Step(1): Set i=1

Step(2): Compute $x = x_0 - \frac{f(x_0)}{f'(x_0)}$

Step(3): If $|x - x_0| < \varepsilon$ then print x is a root and stop.

Step(4): Else if set x_0= x and i=i+1 then go to step(2).

Special cases of Newton-Raphson method

1- **Find the square roots (i.e. $\sqrt{a}$):**

Let $f(x)=x^2-a$

If f(x)=0 then $x = \sqrt{a}$

By using Newton-Raphson formula in equation(3) we have

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$

$$= x_{i-1} - \frac{x_{i-1}^2 - a}{2x_{i-1}}$$

$$= \frac{2x_{i-1}^2 - x_{i-1}^2 + a}{2x_{i-1}}$$

$$= \frac{1}{2}(x_{i-1} + \frac{a}{x_{i-1}}), \qquad i = 1,2,3,...$$

<u>**Example (10):**</u> Find the square root of the number 5 (i.e. $\sqrt{5}$), take $\varepsilon = 10^{-6}$ by using Newton-Raphson method.

Solution: Set $f(x) = x^2 - 5$ and $f'(x) = 2x$, take $x_0=2$

Now by using Newton-Raphson formula we have:

$$x_i = x_{i-1} - \frac{x_{i-1}^2 - 5}{2x_{i-1}}$$

$$= \frac{1}{2}\left(x_{i-1} + \frac{5}{x_{i-1}}\right)$$

i=1 $\longrightarrow$ $x_1 = \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25$

$|x_1 - x_0| > \varepsilon = 10^{-6}$, $x_0=x_1$

i=2 $\longrightarrow$ $x_2 = \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = 2.236111$

$|x_2 - x_1| > \varepsilon = 10^{-6}$, $x_1=x_2$

i=3 $\longrightarrow$ $x_3 = \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = 2.236068$

$|x_3 - x_2| > \varepsilon = 10^{-6}$, $x_2=x_3$

i=4 $\longrightarrow$ $x_4 = \frac{1}{2}\left(x_3 + \frac{5}{x_3}\right) = 2.236068$

$|x_3 - x_2| < \varepsilon = 10^{-6}$

stop with $x_4=2.236068$ is the root.

2- **Find the k^{th} root for a>0 (i.e. $\sqrt[k]{a}$) :**

Let $f(x)= x^k - a \Rightarrow f'(x) = kx^{k-1}$

If f(x)=0 then $x = \sqrt[k]{a}$, By using Newton-Raphson formula in equation(3) we have

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = x_{i-1} - \frac{x_{i-1}^k - a}{kx_{i-1}^{k-1}} = \left(1 - \frac{1}{k}\right)x_{i-1} + \frac{a}{k}x_{i-1}^{1-k} \qquad ; i=1, 2, \dots .$$

Example (11): Find $\sqrt[3]{2}$, take $\varepsilon = 10^{-4}$ by using Newton-Raphson method.

Solution: Set $f(x) = x^3 - 2$ and $f'(x) = 3x^2$, take x_0=1.2

Now by using Newton-Raphson formula we have:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = x_{i-1} - \frac{x_{i-1}^3 - 2}{3x_{i-1}^2} = \left(1 - \frac{1}{3}\right) x_{i-1} + \frac{2}{3} x_{i-1}^{-2}$$

$x_0 = 1.2$, $x_1 = 1.26296$, $x_2 =$, $x_3 =$, $x_4 =$

3- Find $\frac{1}{a}$, $a \neq 0$ without using division :

Let $f(x) = \frac{1}{x} - a$, $f'(x) = -\frac{1}{x^2}$. If f(x)=0 then $x = \frac{1}{a}$

Now by using Newton-Raphson method we have:

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})} = x_{i-1} - \frac{\frac{1}{x_{i-1}} - a}{-\frac{1}{x_{i-1}^2}} = x_{i-1} + x_{i-1} - a x_{i-1}^2 = (2 - a x_{i-1}) x_{i-1} \quad , \ i=1, 2, \dots .$$

Example (12): Use Newton-Raphson method to find $\frac{1}{4}$, take x_0=0.15, $\varepsilon = 10^{-4}$.

Solution: Let $f(x) = \frac{1}{x} - 4$, $f'(x) = -\frac{1}{x^2}$.

$$x_i = x_{i-1} - \frac{f(x_i)}{f'(x_i)} = (2 - a x_{i-1}) x_{i-1} \qquad , \ i=1, 2, \dots$$

$x_1 = (2 - 4x_0)x_0 = 0.21$

$|x_1 - x_0| > \varepsilon = 10^{-4}$, $x_0=x_1$

$x_2 = (2 - 2x_1)x_1 = 0.2436$

$x_3 = 0.24984$, $x_4 = 0.24999$, $x_5 = 0.249996$

stop since $|x_5 - x_4| \leq \varepsilon = 10^{-4}$.

<u>Home work:</u>

Find an approximate root of equation f(x)=x sin(x)-1=0 in the interval [0,2] by using Newton-Raphson method with $x_0=1$, and $\varepsilon = 10^{-5}$.

Properties of the Newton-Raphson method **خواص الطريقة**

1− سرعة الاقتراب في طريقة Newton-Raphson تربيعية (Quadratic convergent) .

2− تتطلب حساب مشتقة الدالة عند كل تكرار ولايمكن ضمان التقارب عندما $f'(x_i) \approx 0$.

3− نوع الاقتراب محلي (Locally convergent) لان نقطة البداية يجب ان تكون قريبة من الجذر الحقيقي.

4− صيغة الخطأ في طريقة نيوتن رافسون هي $e_{i+1} \le \frac{M}{2m} e_i^2$ حيث ان $m \le |f'(x)|$, $|f''(x)| \le M$

لكل قيم x في الفترة التي تحتوي x_i, P .