



— University of Mosul —
College of Petroleum & Mining Engineering



Fixed-point iteration method

Lecture No.6

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(5) Fixed-point iteration method:

We consider method for determining the solution to an equation $f(x)=0$ that is expressed for some function g in the form $g(x)=x$.

A solution to such an equation is said to be a fixed point of the function g . If a fixed point could be found for any given g , then every root-finding problem could also be solved. If we take x_0 as the initial point then the iterative form is

$$x_{i+1}=g(x_i)$$

that is: $x_1 = g(x_0)$, $x_2 = g(x_1)$, $x_3 = g(x_2)$ and so on.

Example (13): Find the approximate root of the equation $x^2-2x-3=0$ by using fixed-point method with take $x_0=4$.

Solution: We find the exact roots of this equation $f(x)=x^2-2x-3=0 \longrightarrow (x-3)(x+1)=0 \longrightarrow x=3$ and $x=-1$

Case(1): $f(x)=x^2-2x-3=0 \longrightarrow x = g_1(x) = \sqrt{3+2x}$

$$\longrightarrow x_{i+1} = g_1(x_i) = \sqrt{3+2x_i}$$

$$i=0 \longrightarrow x_1 = g_1(x_0) = \sqrt{3+2x_0} = \sqrt{3+2(4)} = 3.3166$$

$$i=1 \longrightarrow x_2 = g_1(x_1) = \sqrt{3+2x_1} = \sqrt{3+2(3.3166)} = 3.1037$$

$$i=2 \longrightarrow x_3 = g_1(x_2) = \sqrt{3+2x_2} = \sqrt{3+2(3.1037)} = 3.0344$$

$$i=3 \longrightarrow x_4 = g_1(x_3) = \sqrt{3+2x_3} = \sqrt{3+2(3.0344)} = 3.01144$$

$$i=4 \longrightarrow x_5 = g_1(x_4) = \sqrt{3+2x_4} = \sqrt{3+2(3.01144)} = 3.0038$$

⋮

So on, we converge to the root ($x=3$).

Case(2): $x^2-2x-3=0 \longrightarrow x^2-2x=3 \longrightarrow x(x-2)=3 \longrightarrow x=3/(x-2)$

$$x = g_2(x) = \frac{3}{x-2}$$

$$\longrightarrow x_{i+1} = g_2(x_i) = \frac{3}{x_i-2}$$

$$i=0 \longrightarrow x_1 = g_2(x_0) = \frac{3}{x_0-2} = \frac{3}{4-2} = 1.5$$

$$i=1 \longrightarrow x_2 = g_2(x_1) = \frac{3}{x_1-2} = \frac{3}{1.5-2} = -6$$

$$i=2 \longrightarrow x_3 = g_2(x_2) = \frac{3}{x_2-2} = -0.375$$

$$i=3 \longrightarrow x_4 = g_2(x_3) = \frac{3}{x_3-2} = -1.2632$$

$$i=4 \longrightarrow x_5 = g_2(x_4) = \frac{3}{x_4-2} = -0.9193$$

⋮

So on , we converge oscillating to the root (x=-1).

$$\text{Case(3): } x^2-2x-3=0 \longrightarrow x^2-3=2x \longrightarrow x=(x^2-3)/2$$

$$\longrightarrow x = g_3(x) = \frac{x^2-3}{2}$$

$$\longrightarrow x_{i+1} = g_3(x_i) = \frac{x_i^2-3}{2}$$

$$i=0 \longrightarrow x_1 = g_3(x_0) = \frac{x_0^2-3}{2} = \frac{(4)^2-3}{2} = 6.5$$

$$i=1 \longrightarrow x_2 = g_3(x_1) = \frac{x_1^2-3}{2} = 19.625$$

$$i=2 \longrightarrow x_3 = g_3(x_2) = \frac{x_2^2-3}{2} = 191.0703$$

$$i=3 \longrightarrow x_4 = g_3(x_3) = \frac{x_3^2-3}{2} = 18252.42977$$

⋮

So on , we obtain divergent.

Note: The following theorem gives sufficient conditions for the existence and uniqueness of a fixed-point.

Theorem:

If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in $[a,b]$. Further, suppose $g'(x)$ exists on $[a,b]$ and $|g'(x)| \leq k < 1$ for all $x \in (a,b)$. Then g has a unique fixed point r in $[a,b]$.

Example (14): Find the approximate root of the equation $x^3 + 4x^2 - 10 = 0$ in the interval $[1,2]$ by using fixed-point method with take $x_0 = 1.5$.

Solution: The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1,2]$. There are many ways to change the equation to the form $x = g(x)$ as follows:

$$\begin{array}{ll} \text{(a)} \ x = g_1(x) = x - x^3 - 4x^2 + 10 & \text{(b)} \ x = g_2(x) = \sqrt{\frac{10}{x} - 4x} \\ \text{(c)} \ x = g_3(x) = \frac{1}{2}\sqrt{10 - x^3} & \text{(d)} \ x = g_4(x) = \sqrt{\frac{10}{4+x}} \\ \text{(e)} \ x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} & \dots\dots\dots \end{array}$$

To approximate the fixed point of a function g , with the initial approximation $x_0 = 1.5$ and generate the sequence $\{x_i\}_{i=0}^{\infty}$ by letting $x_{i+1} = g(x_i)$ for each $i \geq 0$.

$$\begin{array}{ll} \text{(a)} \ |g'_1(1.5)| = |-8.75| > 1 \text{ (diverge)} & \text{(b)} \ |g'_2(1.5)| = |5.17| > 1 \text{ (diverge)} \\ \text{(c)} \ |g'_3(1.5)| = |0.6556| < 1 \text{ (converge)} & \dots\dots\dots \end{array}$$

If we use $g_3(x)$ to find a fixed point:

$$x_1 = g_3(x_0) = 1.2870$$

$$x_2 = g_3(x_1) = 1.4025$$

$$x_3 = g_3(x_2) = 1.3455$$

$\vdots$

$$x_{20} = g_3(x_{19}) = 1.3652$$

Algorithm (Fixed-point method)

Input: $x_0, \varepsilon, g(x)$

Step(1): Set $i=1$

Step(2): Compute $x = g(x_0)$

Step(3): If $|x - x_0| < \varepsilon$ then print x is a fixed point and stop.

Step(4): Else if set $x_0 = x$ and $i=i+1$ then go to step(2).

Home work:

Find the solution of the equation $f(x) = x^2 - x - 2 = 0$ by using Fixed-point method with $x_0 = 2.5$, and $\varepsilon = 5 \cdot 10^{-5}$.