— University of Mosul —
College of Petroleum & Mining Engineering

Numerical Solution of Linear Systems
Lecture No.8

Dr. Almutasim Abdulmuhsin H. Albaker

Petroleum and Refining Engineering Department

Email:almutasim@uomosul.edu.iq

Chapter Three
Numerical Solution of Linear Systems

There are two methods for solving linear system of equations:

(i) **Direct method**
- Gauss eliminations
- Gauss elimination with partial pivoting
- Gauss-Jordan elimination
- Cramer Rule
- Decomposition method

(ii) **Indirect method (iterative methods)**
- Jacobi method
- Gauss-Seidel method

(i) Direct method:

(1) Gauss elimination with partial pivoting

Algorithm for Gauss elimination: To solve n×n system

R1: $a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=a_{1,n+1}$
R2: $a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=a_{2,n+1}$
$\vdots$
Rn: $a_{n1}x_1+a_{n2}x_2+\ldots a_{nn}x_n=a_{n,n+1}$
Augmented matrix $A=(a_{ij})$, where $1\leq i\leq n$, $1\leq j\leq n+1$.

Step 1: For i=1, 2, …,n-1 do step 2-4 (Elimination Process)
Step 2: Let p be the smallest integer with $1\leq p\leq n$ and $a_{pi}\neq 0$.
If no integer p can be found
Then output ('no unique solution exists');
Stop
Step 3: If $p\neq I$ then perform $(R_p)\leftrightarrow(R_i)$.
Step 4: For j=i+1, i+2, …,n do step 5 and 6.
Step 5: Set $m_{ji}=\dfrac{a_{ji}}{a_{ii}}$.
Step 6: perform $(R_j-m_{ji}R_i)\rightarrow R_j$.
Step 7: If $a_{nn}=0$ then output ('no unique solution exists');
Stop

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Step 8: set $x_n = \frac{a_{n,n+1}}{a_{nn}}$ (Start backward substitution)

Step 9: For i=n-1, n-2, …,1 set $x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}$.

Step 10: Output (x_1, x_2, …,x_n); (procedure complete successfully)
Stop.

Example 1:

Use Gauss elimination to solve

$$2x_1 - x_2 + x_3 = -1$$
$$3x_1 + 3x_2 + 9x_3 = 0$$
$$3x_1 + 3x_2 + 5x_3 = 4.$$

Solution:

$$\begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 3 & 3 & 9 & 0 \\ 3 & 3 & 5 & 4 \end{array}\right) \quad a_{11}=2,\ m_{ji} = \frac{a_{ji}}{a_{ii}} \Rightarrow m_{21} = \frac{a_{21}}{a_{11}} = \frac{3}{2};\quad m_{31} = \frac{a_{31}}{a_{11}} = \frac{3}{2}$$

$(R_2 - m_{21}R_1) \rightarrow R_2$
$(R_3 - m_{31}R_1) \rightarrow R_3$

$$\begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & 9/2 & 15/2 & 3/1 \\ 0 & 9/2 & 7/2 & 11/2 \end{array}\right),$$

$$a_{22} = \frac{9}{2},\quad m_{ji} = \frac{a_{ji}}{a_{ii}} \Rightarrow m_{32} = \frac{a_{32}}{a_{33}} = \frac{9/2}{9/2} = 1$$

$(R_3 - m_{32}R_2) \rightarrow R_3$

$$\begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 0 & 9/2 & 15/2 & 3/1 \\ 0 & 0 & -4 & 4 \end{array}\right)$$

$$x_3 = \frac{a_{33}}{a_{34}} = \frac{-4}{4} = -1,\quad x_2 = \frac{\frac{3}{2} - \frac{15}{2}x_3}{\frac{9}{2}} = 2,\quad x_1 = \frac{-1 + x_2 - x_3}{2} = 1.$$

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Example 2: Solve

$$4x_1+\ x_2+x_3=6$$
$$2x_1+5x_2+2x_3=3$$
$$x_1+2x_2+4x_3=11$$

The exact solution to each system is $x_1=1$, $x_2=-1$, $x_3=3$.

Gauss elimination with partial pivoting:

Example 3:

The linear system

R1: $0.003x_1+59.14x_2=59.17$
R2: $5.291x_1-6.130x_2=46.78$

has the exact solution $x_1=10$ and $x_2=1$.

Solution: (by Gauss elimination) ; $m_{21}=\dfrac{a_{21}}{a_{11}}=\dfrac{5.291}{0.003}=1763.66$

$(R_2-m_{21}R_1)\rightarrow R_2$ R1: $0.003x_1+59.14x_2=59.17$
R2: $-104300x_2=-104400$

Backward substitution implies $x_2=1.001$ and $x_1=-10$.

We note that the large error in the numerical solution for x_1 resulted from the small error of 0.001 in solving for x_2. Above example illustrates that difficulties can arise in some case when the pivot element $a_{k,k}^{(k)}$ is small relative to entries $a_{i,j}^{(k)}$, for $k\le i\le n$ and $k\le j\le n$. Partial pivoting in general are accomplished by selecting a new element for the pivot $a_{pq}^{(k)}$ and interchanging the k^{th} and p^{th} rows, followed by the interchange of the k^{th} and q^{th} columns, if necessary. The simplest strategy is to select the element in the same column that is below the diagonal and has the largest absolute value that is determine p such that $\left|a_{pk}^{(k)}\right|=\max\limits_{k\le i\le n}\left|a_{ik}^{(k)}\right|$ and perform $(Rk)\leftrightarrow(Rp)$.

Example 4:

Reconsider the linear system

R1: $0.003x_1+59.14x_2=59.17$
R2: $5.291x_1-6.130x_2=46.78$

Solution: Use the pivoting procedure:

$$\max\left\{\left|a_{11}^{(1)}\right|,\left|a_{21}^{(1)}\right|\right\}=\max\{0.003,5.291\}=5.291.$$

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The operation (R2)$\leftrightarrow$(R1) is performed to give the system

$5.291x_1-6.130x_2=46.78$

$0.003x_1+59.14x_2=59.17$; $\quad m_{21}=\dfrac{a_{21}^{(1)}}{a_{11}^{(1)}}=\dfrac{0.003}{5.291}=0.0005670$

The operation (R2-m21R1)$\rightarrow$R1 reduce the system to

$5.291x_1-6.130x_2=46.78$

$\qquad 59.14x_2=59.14$

$\Rightarrow x_2=1$ and $x_1=1$

(2) Gauss-Jordan elimination method

Example 5:

Use Gauss-Jordan elimination method to solve

$$\begin{aligned} 4x_1-9x_2+2x_3&=5\\ 2x_1-4x_2+6x_3&=3\\ x_1-x_2+3x_3&=4. \end{aligned}$$

Solution:

$$\begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 4 & -9 & 2 & 5 \\ 2 & -4 & 6 & 3 \\ 1 & -1 & 3 & 4 \end{array}\right)$$

$$\begin{array}{l} (R_2-m_{21}R_1)\rightarrow R_2 \\ (R_3-m_{31}R_1)\rightarrow R_3 \\ (R_3-m_{32}R_2)\rightarrow R_3 \end{array} \quad \begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 4 & -9 & 2 & 5 \\ 0 & 0.5 & 5 & 0.5 \\ 0 & 0 & -10 & 1.5 \end{array}\right), \quad \begin{array}{l} R1/10 \\ \\ \\ \end{array}$$

$$\begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 4 & -9 & 0 & 5.3 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & -0.15 \end{array}\right), \quad R2/0.5$$

$$\begin{array}{l} R1: \\ R2: \\ R3: \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6.95 \\ 0 & 1 & 0 & 2.5 \\ 0 & 0 & 1 & -0.15 \end{array}\right), \quad \begin{array}{l} \\ \\ R1/4 \end{array}$$

$$x_1=6.95,\ x_2=2.5\ ,\ x_3=-0.15$$

Dr.F.S.Al-Rawi