قسم هندسة الاتصالات والانظمة الرقمية الذكية

Electrical Circuit Analysis

تحليل الدوائر الكهربائية

مدرس المادة

م.د. مروان عبدالخالق ذنون

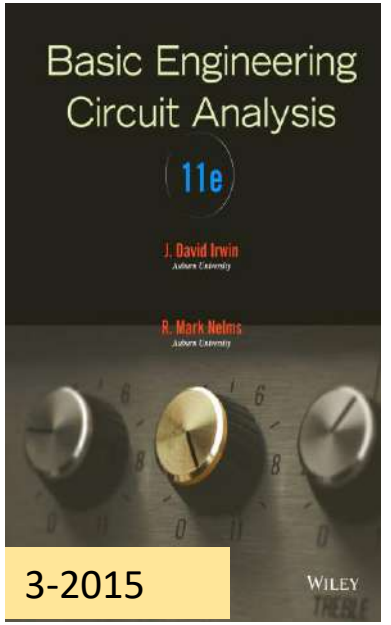
م.م محمد ادريس داؤد

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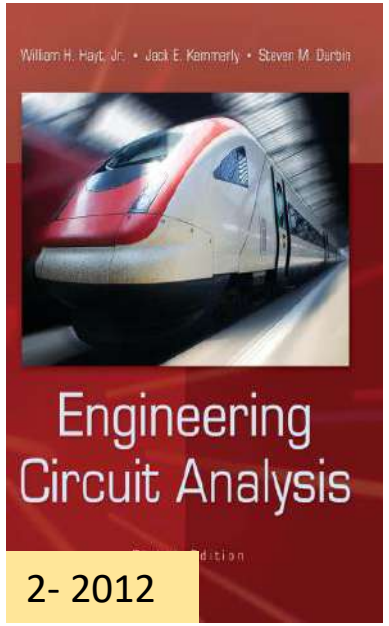
Electrical Circuit Analysis

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➢ Introduction [1]

Standard Form Notation

Standard form is a method of writing large and small numbers in a form that is more convenient than writing a large number of trailing or leading zeroes.
For example the speed of light is approximately 300 000 000m/s. When written in standard form this figure would appears as:

$$3.0 \times 10^8 \text{ m/s, where } 10^8 \text{ represents } 100\,000\,000$$

Similarly, if the wavelength of ‘ red ’ light is approximately 0.000 000 767 m, it is more convenient to write it in standard form as:

$$7.67 \times 10^{-7} \text{ m, where } 10^{-7} = 1/10\,000\,000$$

when using *standard form* is that only the first non-zero digit must appear before the decimal point.

Thus, 46 500 is written as

$$4.65 \times 10^4 \text{ and } \textit{not} \text{ as } 46.5 \times 10^3$$

Similarly, 0.002 69 is written as

$$2.69 \times 10^{-3} \text{ and } \textit{not} \text{ as } 26.9 \times 10^{-4} \text{ or } 269 \times 10^{-5}$$

‘ Scientific ’ Notation

For example if a body has a mass *m* =500 000 g.
In standard form this would be written as

$$m = 5.0 \times 10^5 \text{ g}.$$

Using scientific notation it would appear as
m =500 kg (500 kilogram)

Unit prefixes used in ‘scientific’ notation

Multiplying factor	Prefix name	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p

Worked Example 1

Write the following quantities in a concise form using (a) standard form, and (b) scientific notation
(i) 0.000 018 A (ii) 15 000 V (iii) 250 000 000 W

A

(a) (i) $0.000\,018\,\text{A} = 1.8 \times 10^{-5}\,\text{A}$
(ii) $15\,000\,\text{V} = 1.5 \times 10^{4}\,\text{V}$
(iii) $250\,000\,000\,\text{W} = 2.5 \times 10^{8}\,\text{W}$

(b) (i) $0.000\,018\,\text{A} = 18\,\mu\text{A}$
(ii) $15\,000\,\text{V} = 15\,\text{kV}$
(iii) $250\,000\,000\,\text{W} = 250\,\text{MW}$ **Ans**

Worked Example 2

Write the following quantities in scientific (engineering) notation. (a) 25×10^{-5} A, (b) 3×10^{-4} W, (c) 850 000 J, (d) 0.0016 V.

A

(a) $25 \times 10^{-5} = 25 \times 10 \times 10^{-6}$

$= 250 \times 10^{-6}$

and since 10^{-6} is represented by μ (micro)

then 25×10^{-5} A $= 250\,\mu$A **Ans**

Alternatively, $25 \times 10^{-5} = 25 \times 10^{-2} \times 10^{-3}$

$= 0.25 \times 10^{-3}$

so 25×10^{-5} A $= 0.25$ mA **Ans**

(b) $3 \times 10^{-4} = 3 \times 10^{-1} \times 10^{-3}$ or 300×10^{-6}

so 3×10^{-4} W $= 0.3$ mW or 300 μW **Ans**

(c) $850\,000 = 850 \times 10^{3}$ or 0.85×10^{6}

so 850 000 J = 850 kJ or 0.85 MJ **Ans**

(d) $0.0016 = 1.6 \times 10^{-3}$

so 0.0016 V = 1.6 mV **Ans**

➢ Charge, Current, Voltage, Power, and Energy [2]

Charge and Current:

There are many devices (such as batteries, diodes, and transistors) in which positive charge motion is important to understanding internal operation, but external to the device we typically concentrate on the electrons which flow through the connecting wires**. Charge (Q)** in motion represents a **current (I).**

The fundamental unit of charge (Q) is the coulomb (C). It is defined in terms of the **ampere** by counting the total charge that passes through an arbitrary cross section of a wire during an interval of one second.

The definition of current (I) illustrated using current flowing through a wire; 1 ampere corresponds to 1 coulomb of charge passing through the arbitrarily chosen cross section in 1 second.

$$I = \frac{Q}{t} \text{ amp, or } Q = It \text{ coulomb}$$

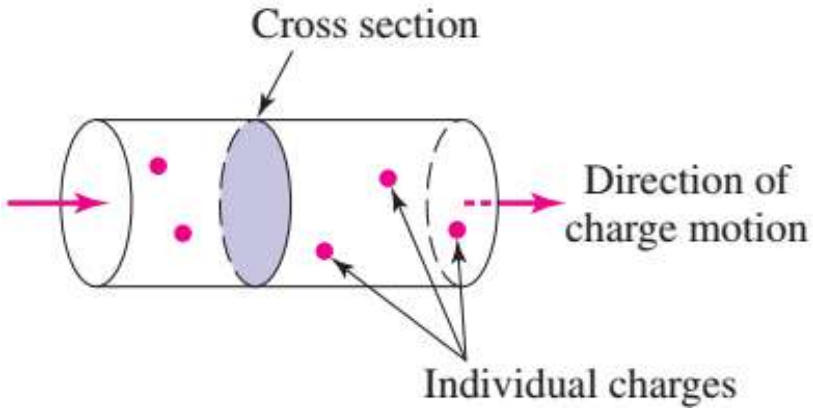
Example: A charge of 35 mC is transferred between two points in a circuit in a time of 20 ms. Calculate the value of current flowing

$$Q = 35 \times 10^{-3}\,\text{C}; t = 20 \times 10^{-3}\,\text{s}$$

$$I = \frac{Q}{t} \text{ amp}$$

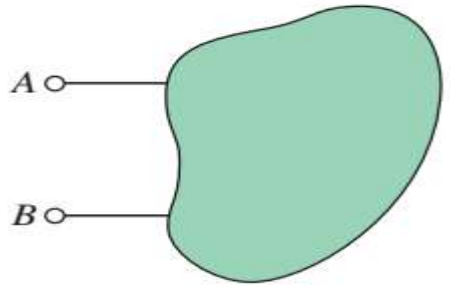
$$= \frac{35 \times 10^{-3}}{20 \times 10^{-3}}$$

$$I = 1.75 \text{ A } \textbf{Ans}$$

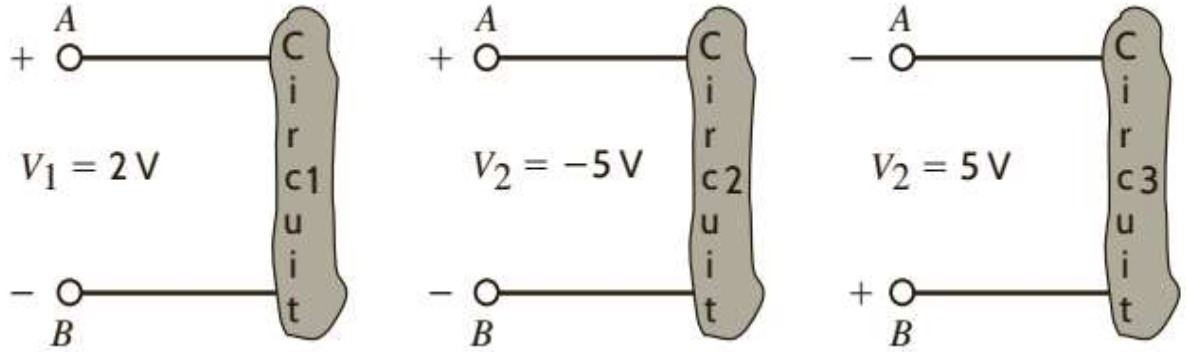


Voltage:

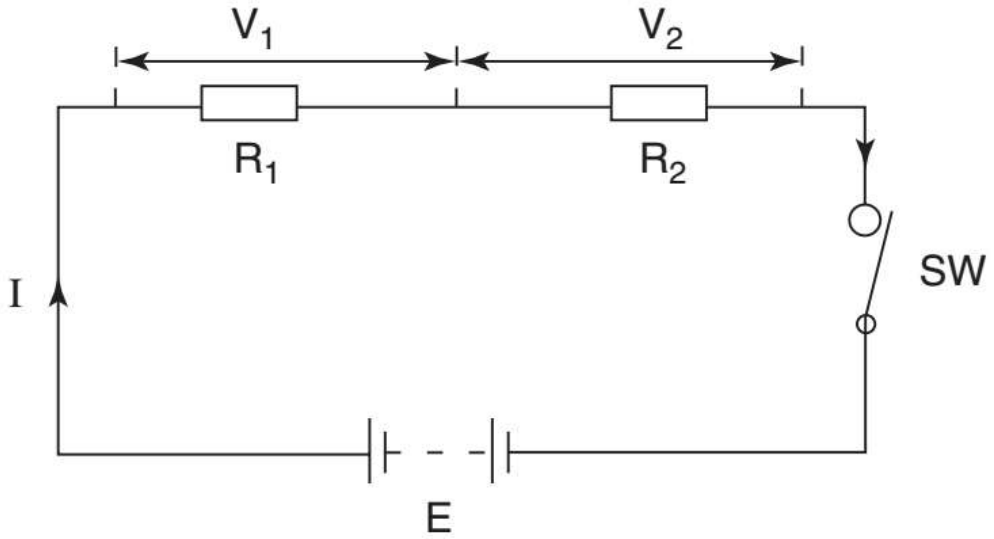
Let us suppose that a current is sent into terminal *A* and back out of terminal *B*. Let us also assume that pushing charge through the element requires a spending of energy. We then say that an electrical voltage (or a *potential difference p.d.*) exists between the two terminals, or that there is a voltage "across" the element.



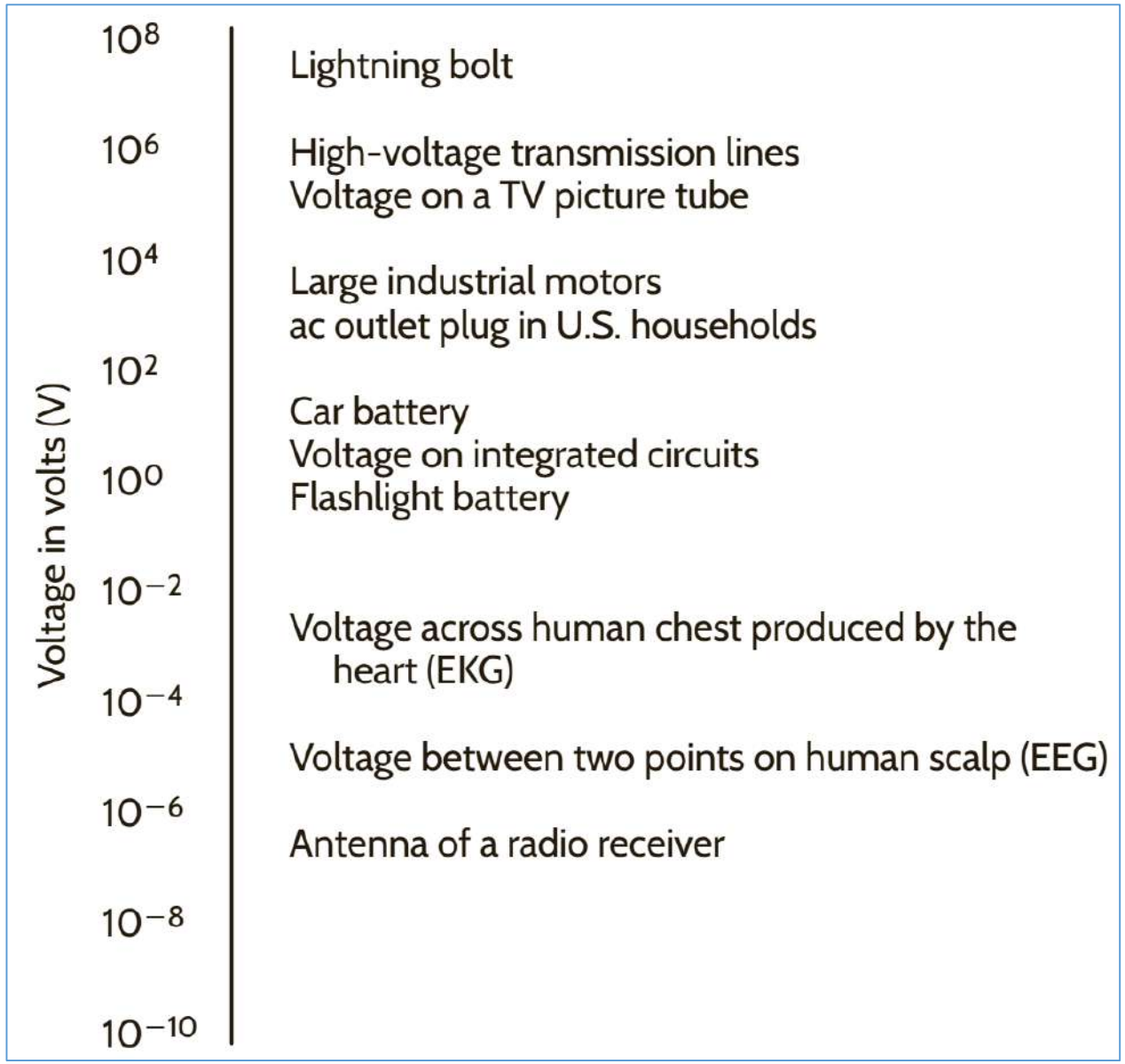
Whenever current (I) flows through an electrical circuit, there will be a ***Potential Difference*** developed across it and it is measured in Volt.



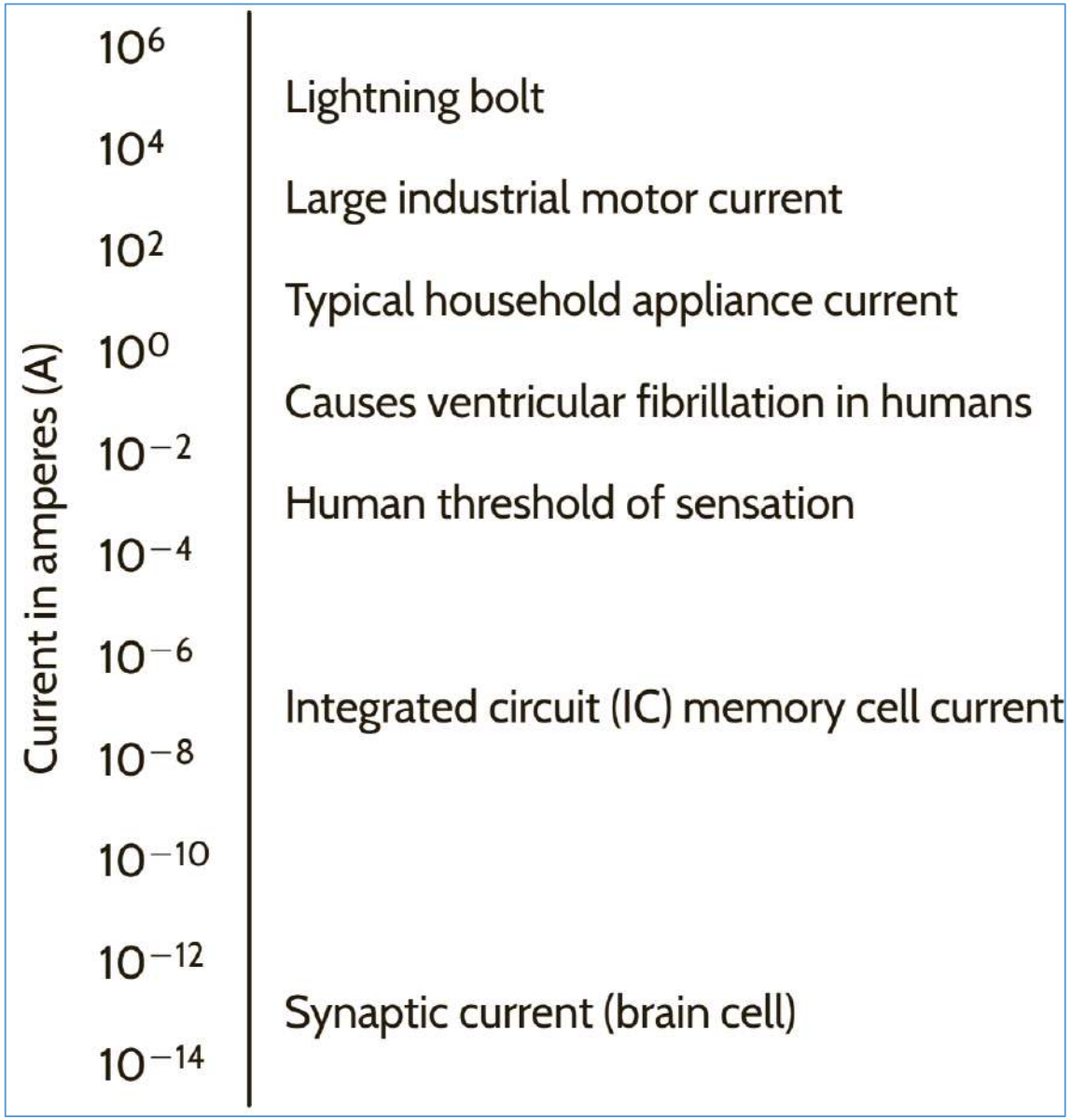
A correct definition includes both a symbol (V_1 , V_2) for the variable and a plus-minus symbol pair.



The equivalent electrical circuit, comprising a battery, the connecting conductors (cables or leads) and two resistors.



Typical voltage magnitudes



Typical current magnitudes

Power and Energy:

Power: the product of V and I gives the power absorbed by the resistor. The absorbed power appears physically as heat and/or light and is always positive; a (positive) resistor is a passive element that cannot deliver power or store energy.

$$p = vi = i^2R = v^2/R$$

Energy (W): Whenever work is done energy is transferred from a system to another one. The most common form into which energy is transformed is heat. **Joule** reached the conclusion that the amount of heat so produced was proportional to the value of the square of the current (I^2) flowing and the time for which it flowed. The heat produced (or energy dissipated) is given by the equation:

$$W = I^2Rt \text{ joule}$$

$$W = \frac{V^2t}{R} \text{ joule}$$

Example:

A current of 200 mA flows through a resistance of 750 Ω for a time of 5 minutes. Calculate (a) the p.d. developed, and (b) the energy dissipated.

A

$$I = 200\,\text{mA} = 0.2\text{A};\ t = 5 \times 60 = 300\,\text{s};\ R = 750\,\Omega$$

(a) $V = IR$ volt

$= 0.2 \times 750$

$V = 150\,\text{V}$ **Ans**

(b) $W = I^2Rt$ joule

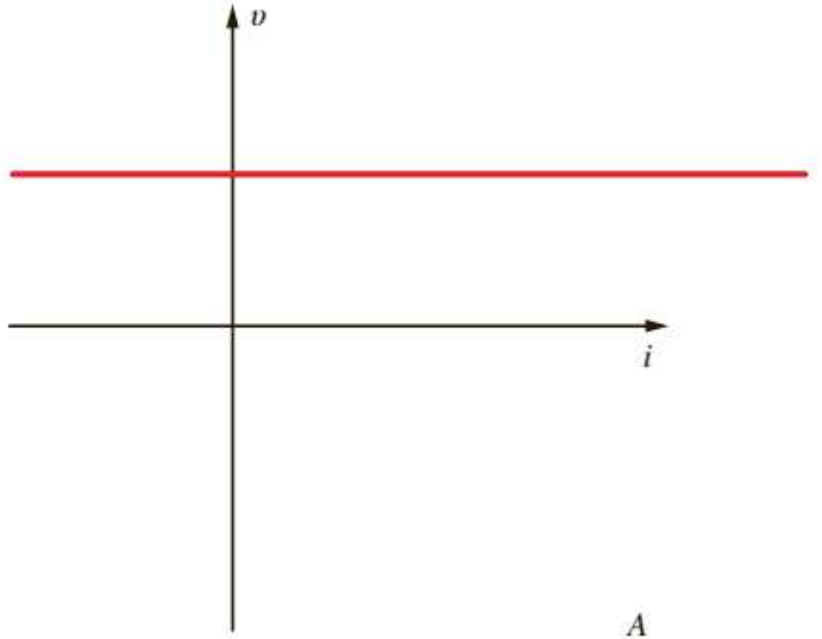
$= 0.2 \times 750 \times 300$

$W = 9000\,\text{J}$ or $9\,\text{kJ}$ **Ans**

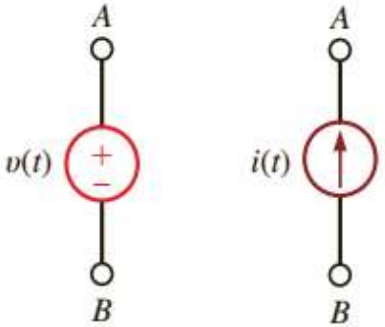
INDEPENDENT SOURCES

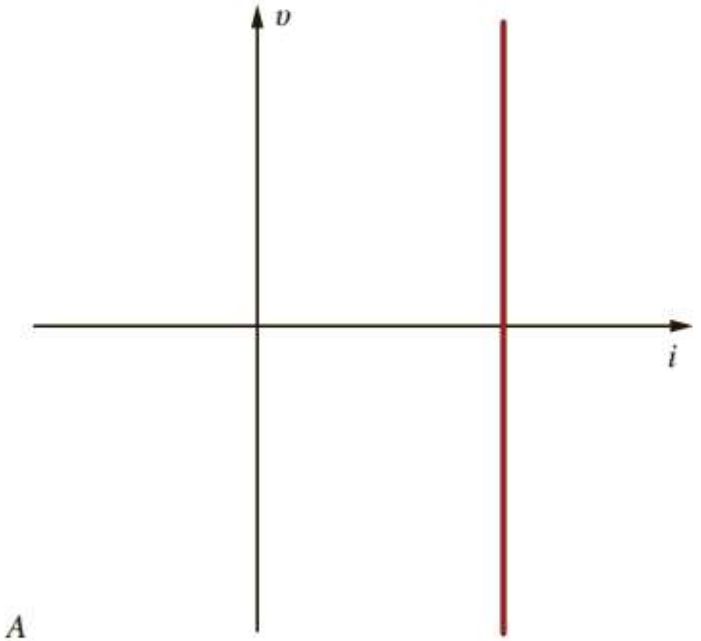
An *independent voltage source* is a two-terminal element that maintains a specified voltage between its terminals *regardless of the current through it* as shown by the υ-*i* plot (a).

The *independent current source* is a two terminal element that maintains a specified current regardless of the voltage across its terminals, as illustrated by the υ-i plot (b).

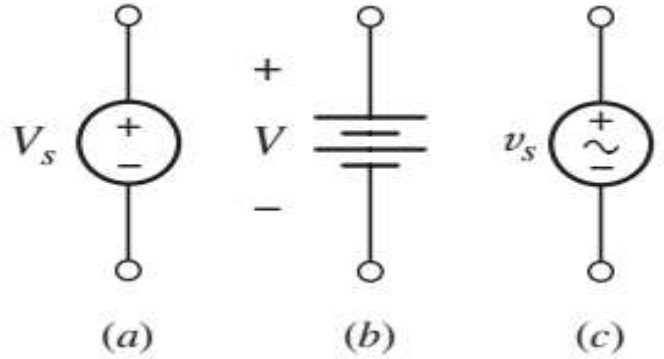


(a)





(b)



(a) DC voltage source symbol;
(*b*) battery symbol;
(*c*) ac voltage source symbol.

OHM'S LAW

Ohm's law states that the voltage across a resistance is directly proportional to the current flowing through it.

The constant of proportionality between the voltage and current (R) is called the ***resistance.*** The unit of resistance is the *ohm*, which is 1 V/A and customarily abbreviated by a capital omega, **Ω**.

Example: The 560 Ω resistor is connected to a circuit which causes a current of 42.4 mA to flow through it. Calculate the voltage across the resistor and the power it is dissipating.

The voltage across the resistor is given by Ohm's law:

$$v = Ri = (560)(0.0424) = 23.7 \text{ V}$$

The dissipated power can be calculated in several different ways. For instance,

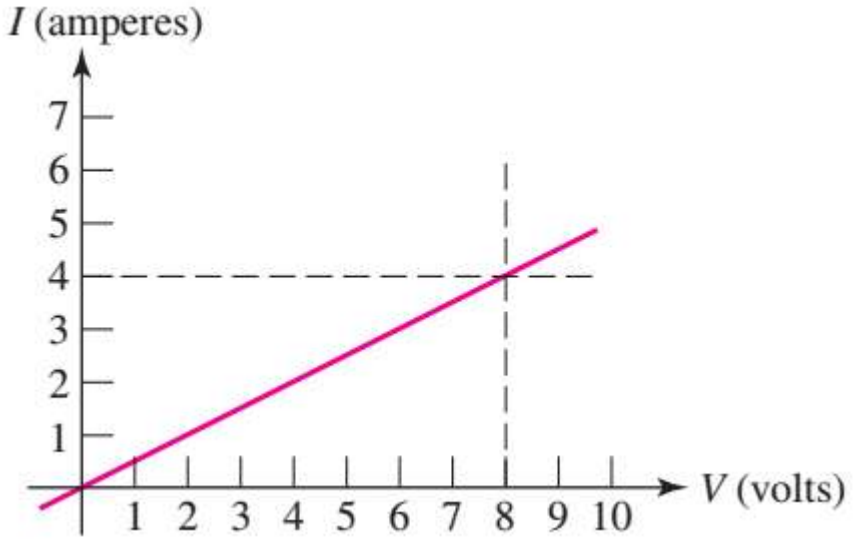
$$p = vi = (23.7)(0.0424) = 1.005 \text{ W}$$

Alternatively,

$$p = v^2/R = (23.7)^2/560 = 1.003 \text{ W}$$

or

$$p = i^2R = (0.0424)^2(560) = 1.007 \text{ W}$$



Current-voltage relationship for an example 2Ω linear resistor. Note the slope of the line is 0.5 A/V, or 500 $m\Omega^{-1}$

10.3 | Kirchhoff's Rules

Learning Objectives

By the end of the section, you will be able to:

- State Kirchhoff's junction rule
- State Kirchhoff's loop rule
- Analyze complex circuits using Kirchhoff's rules

We have just seen that some circuits may be analyzed by reducing a circuit to a single voltage source and an equivalent resistance. Many complex circuits cannot be analyzed with the series-parallel techniques developed in the preceding sections. In this section, we elaborate on the use of Kirchhoff's rules to analyze more complex circuits. For example, the circuit in **Figure 10.19** is known as a multi-loop circuit, which consists of junctions. A junction, also known as a node, is a connection of three or more wires. In this circuit, the previous methods cannot be used, because not all the resistors are in clear series or parallel configurations that can be reduced. Give it a try. The resistors R_1 and R_2 are in series and can be reduced to an equivalent resistance. The same is true of resistors R_4 and R_5. But what do you do then?

Even though this circuit cannot be analyzed using the methods already learned, two circuit analysis rules can be used to analyze any circuit, simple or complex. The rules are known as **Kirchhoff's rules**, after their inventor Gustav Kirchhoff (1824–1887).

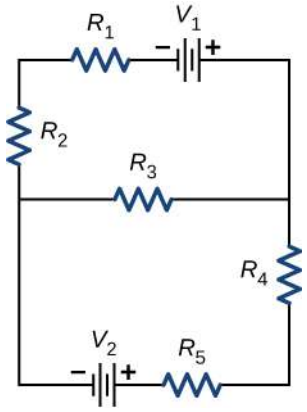


Figure 10.19 This circuit cannot be reduced to a combination of series and parallel connections. However, we can use Kirchhoff's rules to analyze it.

Kirchhoff's Rules

- Kirchhoff's first rule—the junction rule. The sum of all currents entering a junction must equal the sum of all currents leaving the junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}}. \tag{10.4}$$

- Kirchhoff's second rule—the loop rule. The algebraic sum of changes in potential around any closed circuit path (loop) must be zero:

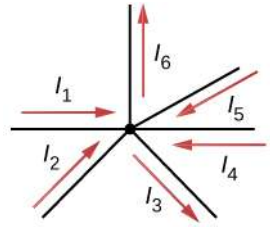
$$\sum V = 0. \tag{10.5}$$

We now provide explanations of these two rules, followed by problem-solving hints for applying them and a worked example that uses them.

Kirchhoff's First Rule

Kirchhoff's first rule (the **junction rule**) applies to the charge entering and leaving a junction (**Figure 10.20**). As stated

earlier, a junction, or node, is a connection of three or more wires. Current is the flow of charge, and charge is conserved; thus, whatever charge flows into the junction must flow out.



$$\sum I_{in} = \sum I_{out}$$
$$I_1 + I_2 + I_4 + I_5 = I_3 + I_6$$

Figure 10.20 Charge must be conserved, so the sum of currents into a junction must be equal to the sum of currents out of the junction.

Although it is an over-simplification, an analogy can be made with water pipes connected in a plumbing junction. If the wires in **Figure 10.20** were replaced by water pipes, and the water was assumed to be incompressible, the volume of water flowing into the junction must equal the volume of water flowing out of the junction.

Kirchhoff's Second Rule

Kirchhoff's second rule (the **loop rule**) applies to potential differences. The loop rule is stated in terms of potential V rather than potential energy, but the two are related since $U = qV$. In a closed loop, whatever energy is supplied by a voltage source, the energy must be transferred into other forms by the devices in the loop, since there are no other ways in which energy can be transferred into or out of the circuit. Kirchhoff's loop rule states that the algebraic sum of potential differences, including voltage supplied by the voltage sources and resistive elements, in any loop must be equal to zero. For example, consider a simple loop with no junctions, as in **Figure 10.21**.

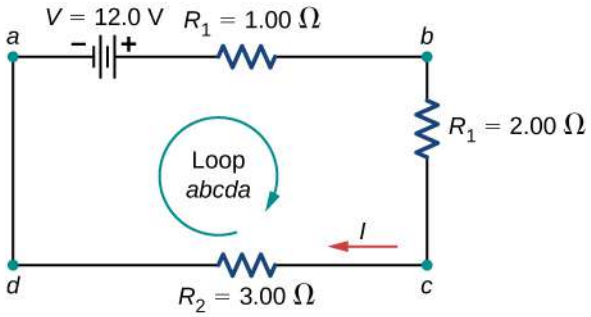


Figure 10.21 A simple loop with no junctions. Kirchhoff's loop rule states that the algebraic sum of the voltage differences is equal to zero.

The circuit consists of a voltage source and three external load resistors. The labels *a*, *b*, *c*, and *d* serve as references, and have no other significance. The usefulness of these labels will become apparent soon. The loop is designated as Loop *abcda*, and the labels help keep track of the voltage differences as we travel around the circuit. Start at point *a* and travel to point *b*. The voltage of the voltage source is added to the equation and the potential drop of the resistor R_1 is subtracted. From point *b* to *c*, the potential drop across R_2 is subtracted. From *c* to *d*, the potential drop across R_3 is subtracted. From points *d* to *a*, nothing is done because there are no components.

Figure 10.22 shows a graph of the voltage as we travel around the loop. Voltage increases as we cross the battery, whereas voltage decreases as we travel across a resistor. The potential drop, or change in the electric potential, is equal to the current through the resistor times the resistance of the resistor. Since the wires have negligible resistance, the voltage remains constant as we cross the wires connecting the components.

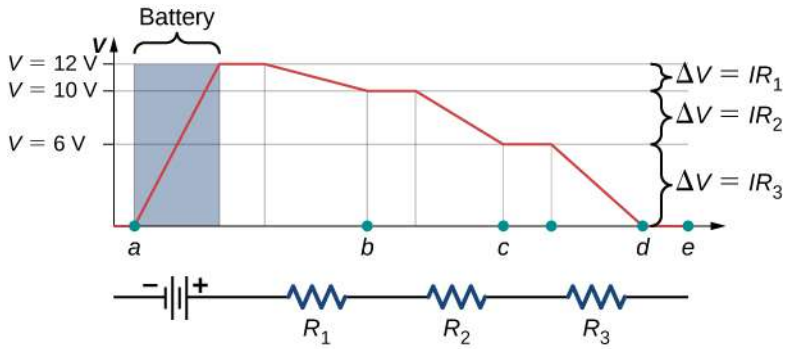


Figure 10.22 A voltage graph as we travel around the circuit. The voltage increases as we cross the battery and decreases as we cross each resistor. Since the resistance of the wire is quite small, we assume that the voltage remains constant as we cross the wires connecting the components.

Then Kirchhoff's loop rule states

$$V - IR_1 - IR_2 - IR_3 = 0.$$

The loop equation can be used to find the current through the loop:

$$I = \frac{V}{R_1 + R_2 + R_2} = \frac{12.00\ \text{V}}{1.00\ \Omega + 2.00\ \Omega + 3.00\ \Omega} = 2.00\ \text{A}.$$

This loop could have been analyzed using the previous methods, but we will demonstrate the power of Kirchhoff's method in the next section.

Applying Kirchhoff's Rules

By applying Kirchhoff's rules, we generate a set of linear equations that allow us to find the unknown values in circuits. These may be currents, voltages, or resistances. Each time a rule is applied, it produces an equation. If there are as many independent equations as unknowns, then the problem can be solved.

Using Kirchhoff's method of analysis requires several steps, as listed in the following procedure.

Problem-Solving Strategy: Kirchhoff's Rules

1. Label points in the circuit diagram using lowercase letters *a*, *b*, *c*, …. These labels simply help with orientation.
2. Locate the junctions in the circuit. The junctions are points where three or more wires connect. Label each junction with the currents and directions into and out of it. Make sure at least one current points into the junction and at least one current points out of the junction.
3. Choose the loops in the circuit. Every component must be contained in at least one loop, but a component may be contained in more than one loop.
4. Apply the junction rule. Again, some junctions should not be included in the analysis. You need only use enough nodes to include every current.
5. Apply the loop rule. Use the map in **Figure 10.23**.

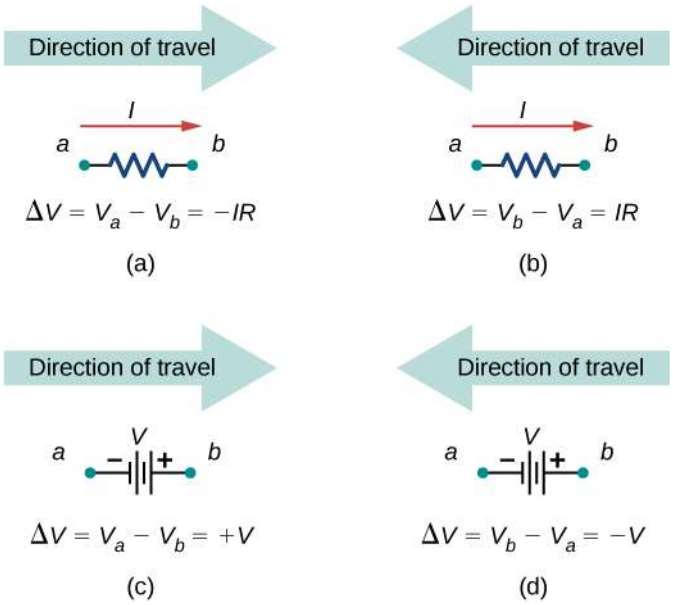


Figure 10.23 Each of these resistors and voltage sources is traversed from *a* to *b*. (a) When moving across a resistor in the same direction as the current flow, subtract the potential drop. (b) When moving across a resistor in the opposite direction as the current flow, add the potential drop. (c) When moving across a voltage source from the negative terminal to the positive terminal, add the potential drop. (d) When moving across a voltage source from the positive terminal to the negative terminal, subtract the potential drop.

Let's examine some steps in this procedure more closely. When locating the junctions in the circuit, do not be concerned about the direction of the currents. If the direction of current flow is not obvious, choosing any direction is sufficient as long as at least one current points into the junction and at least one current points out of the junction. If the arrow is in the opposite direction of the conventional current flow, the result for the current in question will be negative but the answer will still be correct.

The number of nodes depends on the circuit. Each current should be included in a node and thus included in at least one junction equation. Do not include nodes that are not linearly independent, meaning nodes that contain the same information.

Consider **Figure 10.24**. There are two junctions in this circuit: Junction *b* and Junction *e*. Points *a*, *c*, *d*, and *f* are not junctions, because a junction must have three or more connections. The equation for Junction *b* is $I_1 = I_2 + I_3$, and the equation for Junction *e* is $I_2 + I_3 = I_1$. These are equivalent equations, so it is necessary to keep only one of them.

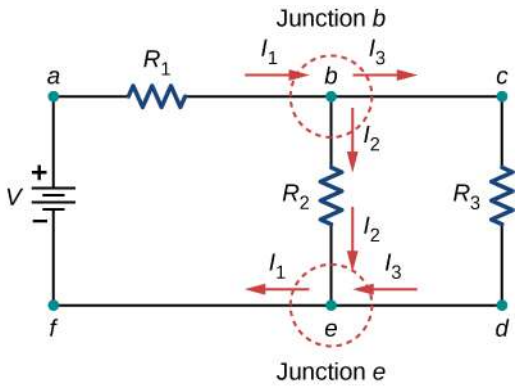


Figure 10.24 At first glance, this circuit contains two junctions, Junction *b* and Junction *e*, but only one should be considered because their junction equations are equivalent.

When choosing the loops in the circuit, you need enough loops so that each component is covered once, without repeating loops. **Figure 10.25** shows four choices for loops to solve a sample circuit; choices (a), (b), and (c) have a sufficient amount of loops to solve the circuit completely. Option (d) reflects more loops than necessary to solve the circuit.

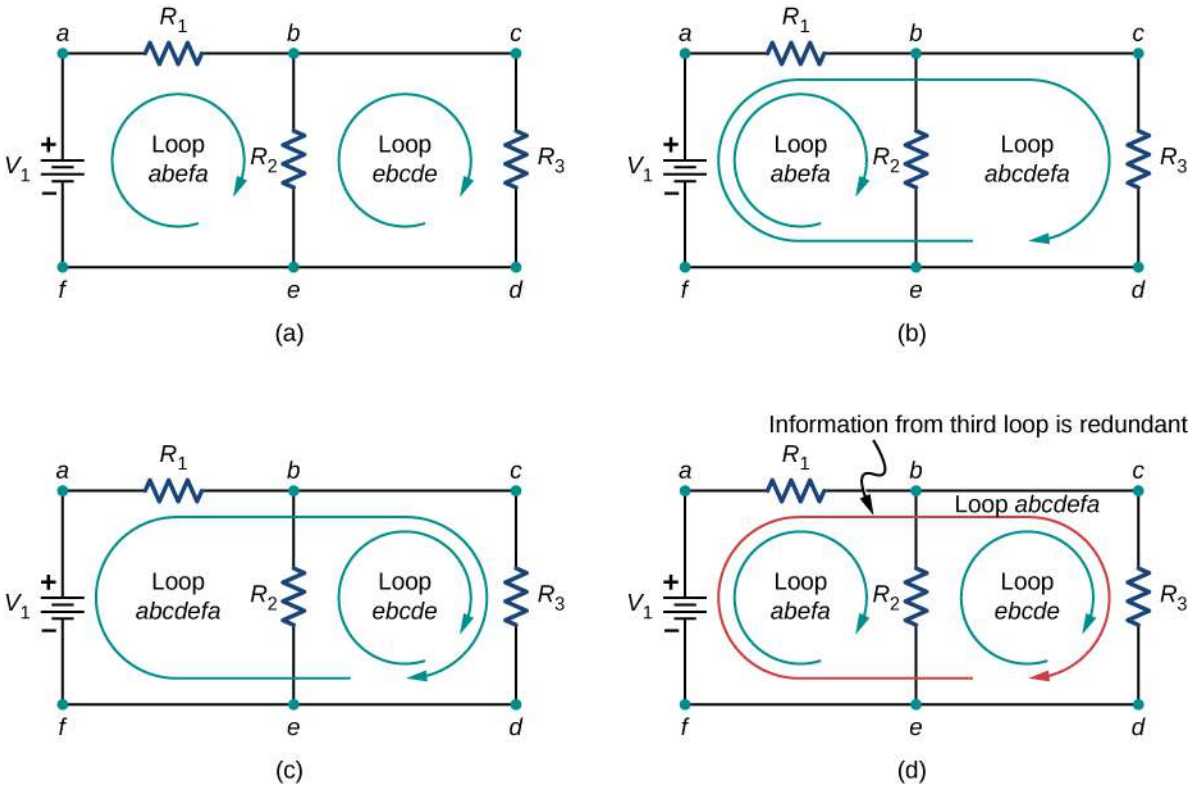


Figure 10.25 Panels (a)–(c) are sufficient for the analysis of the circuit. In each case, the two loops shown contain all the circuit elements necessary to solve the circuit completely. Panel (d) shows three loops used, which is more than necessary. Any two loops in the system will contain all information needed to solve the circuit. Adding the third loop provides redundant information.

Consider the circuit in **Figure 10.26**(a). Let us analyze this circuit to find the current through each resistor. First, label the circuit as shown in part (b).

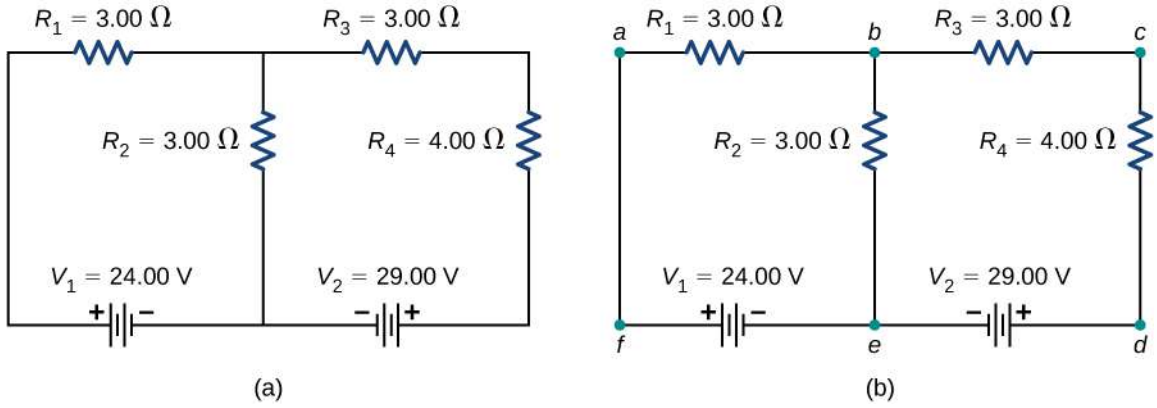


Figure 10.26 (a) A multi-loop circuit. (b) Label the circuit to help with orientation.

Next, determine the junctions. In this circuit, points *b* and *e* each have three wires connected, making them junctions. Start to apply Kirchhoff's junction rule $\left(\sum I_{\text{in}} = \sum I_{\text{out}}\right)$ by drawing arrows representing the currents and labeling each arrow, as shown in **Figure 10.27**(b). Junction *b* shows that $I_1 = I_2 + I_3$ and Junction *e* shows that $I_2 + I_3 = I_1$. Since Junction

e gives the same information of Junction *b*, it can be disregarded. This circuit has three unknowns, so we need three linearly independent equations to analyze it.

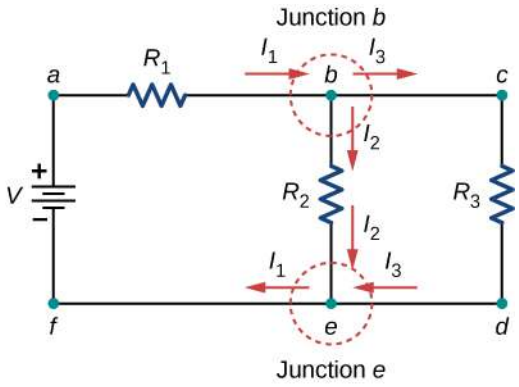


Figure 10.27 (a) This circuit has two junctions, labeled *b* and *e*, but only node *b* is used in the analysis. (b) Labeled arrows represent the currents into and out of the junctions.

Next we need to choose the loops. In **Figure 10.28**, Loop *abefa* includes the voltage source V_1 and resistors R_1 and R_2. The loop starts at point *a*, then travels through points *b*, *e*, and *f*, and then back to point *a*. The second loop, Loop *ebcde*, starts at point *e* and includes resistors R_2 and R_3, and the voltage source V_2.

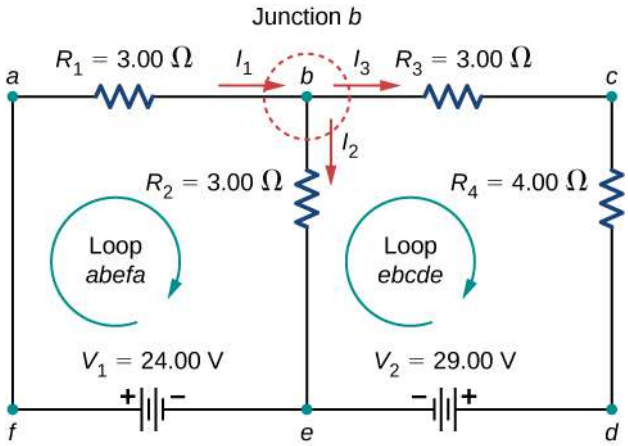


Figure 10.28 Choose the loops in the circuit.

Now we can apply Kirchhoff's loop rule, using the map in **Figure 10.23**. Starting at point *a* and moving to point *b*, the resistor R_1 is crossed in the same direction as the current flow I_1, so the potential drop I_1R_1 is subtracted. Moving from point *b* to point *e*, the resistor R_2 is crossed in the same direction as the current flow I_2 so the potential drop I_2R_2 is subtracted. Moving from point *e* to point *f*, the voltage source V_1 is crossed from the negative terminal to the positive terminal, so V_1 is added. There are no components between points *f* and *a*. The sum of the voltage differences must equal zero:

$$\text{Loop } abefa: \quad -I_1R_1 - I_2R_2 + V_1 = 0 \text{ or } V_1 = I_1R_1 + I_2R_2.$$

Finally, we check loop *ebcde*. We start at point *e* and move to point *b*, crossing R_2 in the opposite direction as the current flow I_2. The potential drop I_2R_2 is added. Next, we cross R_3 and R_4 in the same direction as the current flow I_3 and subtract the potential drops I_3R_3 and I_3R_4. Note that the current is the same through resistors R_3 and R_4, because they are connected in series. Finally, the voltage source is crossed from the positive terminal to the negative terminal, and the

voltage source V_2 is subtracted. The sum of these voltage differences equals zero and yields the loop equation

$$\text{Loop } ebcde: I_2R_2 - I_3(R_3 + R_4) - V_2 = 0.$$

We now have three equations, which we can solve for the three unknowns.

$$\begin{aligned}&(1)\ \text{Junction } b: I_1 - I_2 - I_3 = 0.\\ &(2)\ \text{Loop } abefa: \quad I_1R_1 + I_2R_2 = V_1.\\ &(3)\ \text{Loop } ebcde: I_2R_2 - I_3(R_3 + R_4) = V_2.\end{aligned}$$

To solve the three equations for the three unknown currents, start by eliminating current I_2. First add Eq. (1) times R_2 to Eq. (2). The result is labeled as Eq. (4):

$$\begin{aligned}(R_1 + R_2)I_1 - R_2I_3 &= V_1.\\ (4)\ 6\ \Omega I_1 - 3\ \Omega I_3 &= 24\ \text{V}.\end{aligned}$$

Next, subtract Eq. (3) from Eq. (2). The result is labeled as Eq. (5):

$$\begin{aligned}I_1R_1 + I_3(R_3 + R_4) &= V_1 - V_2.\\ (5)\ 3\ \Omega I_1 + 7\ \Omega I_3 &= -5\ \text{V}.\end{aligned}$$

We can solve Eqs. (4) and (5) for current I_1. Adding seven times Eq. (4) and three times Eq. (5) results in $51\ \Omega I_1 = 153\ \text{V}$, or $I_1 = 3.00\ \text{A}$. Using Eq. (4) results in $I_3 = -2.00\ \text{A}$. Finally, Eq. (1) yields $I_2 = I_1 - I_3 = 5.00\ \text{A}$. One way to check that the solutions are consistent is to check the power supplied by the voltage sources and the power dissipated by the resistors:

$$\begin{aligned}P_{\text{in}} &= I_1V_1 + I_3V_2 = 130\ \text{W},\\ P_{\text{out}} &= I_1^2R_1 + I_2^2R_2 + I_3^2R_3 + I_3^2R_4 = 130\ \text{W}.\end{aligned}$$

Note that the solution for the current I_3 is negative. This is the correct answer, but suggests that the arrow originally drawn in the junction analysis is the direction opposite of conventional current flow. The power supplied by the second voltage source is 58 W and not −58 W.

Example 10.6

Calculating Current by Using Kirchhoff's Rules

Find the currents flowing in the circuit in **Figure 10.29**.

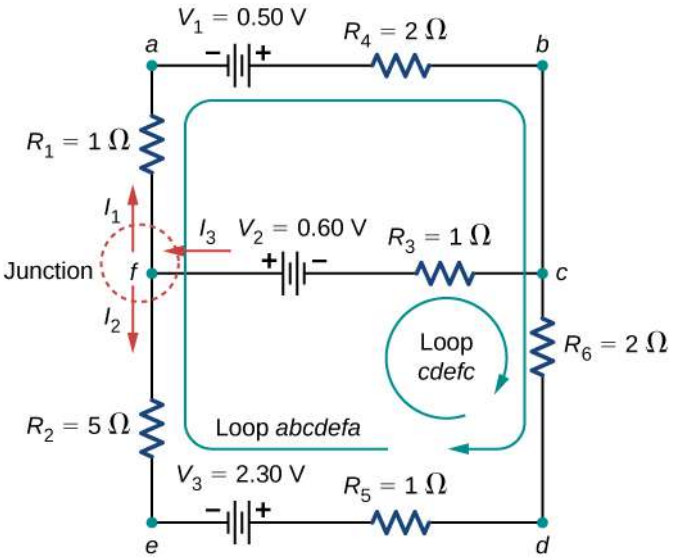


Figure 10.29 This circuit is combination of series and parallel configurations of resistors and voltage sources. This circuit cannot be analyzed using the techniques discussed in **Electromotive Force** but can be analyzed using Kirchhoff's rules.

Strategy

This circuit is sufficiently complex that the currents cannot be found using Ohm's law and the series-parallel techniques—it is necessary to use Kirchhoff's rules. Currents have been labeled I_1, I_2, and I_3 in the figure, and assumptions have been made about their directions. Locations on the diagram have been labeled with letters *a* through *h*. In the solution, we apply the junction and loop rules, seeking three independent equations to allow us to solve for the three unknown currents.

Solution

Applying the junction and loop rules yields the following three equations. We have three unknowns, so three equations are required.

$$\text{Junction } c: \ I_1 + I_2 = I_3.$$
$$\text{Loop } abcdefa: \ I_1(R_1 + R_4) - I_2(R_2 + R_5 + R_6) = V_1 - V_3.$$
$$\text{Loop } cdefc: \ I_2(R_2 + R_5 + R_6) + I_3 R_3 = V_2 + V_3.$$

Simplify the equations by placing the unknowns on one side of the equations.

$$\text{Junction } c: \ I_1 + I_2 - I_3 = 0.$$
$$\text{Loop } abcdefa: \ I_1(3\ \Omega) - I_2(8\ \Omega) = 0.5\ \text{V} - 2.30\ \text{V}.$$
$$\text{Loop } cdefc: \ I_2(8\ \Omega) + I_3(1\ \Omega) = 0.6\ \text{V} + 2.30\ \text{V}.$$

Simplify the equations. The first loop equation can be simplified by dividing both sides by 3.00. The second loop equation can be simplified by dividing both sides by 6.00.

$$\text{Junction } c: \ I_1 + I_2 - I_3 = 0.$$
$$\text{Loop } abcdefa: \ I_1(3\ \Omega) - I_2(8\ \Omega) = -1.8\ \text{V}.$$
$$\text{Loop } cdefc: \ I_2(8\ \Omega) + I_3(1\ \Omega) = 2.9\ \text{V}.$$

The results are

$$I_1 = 0.20\ \text{A}, \quad I_2 = 0.30\ \text{A}, \quad I_3 = 0.50\ \text{A}.$$

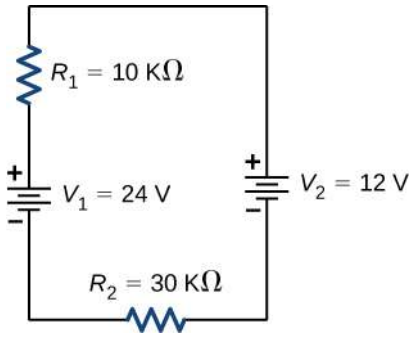
Significance

A method to check the calculations is to compute the power dissipated by the resistors and the power supplied by the voltage sources:

$$P_{R_1} = I_1^2 R_1 = 0.04\,\text{W}.$$
$$P_{R_2} = I_2^2 R_2 = 0.45\,\text{W}.$$
$$P_{R_3} = I_3^2 R_3 = 0.25\,\text{W}.$$
$$P_{R_4} = I_1^2 R_4 = 0.08\,\text{W}.$$
$$P_{R_5} = I_2^2 R_5 = 0.09\,\text{W}.$$
$$P_{R_6} = I_2^2 R_6 = 0.18\,\text{W}.$$
$$P_{\text{dissipated}} = 1.09\,\text{W}.$$
$$P_{\text{source}} = I_1 V_1 + I_2 V_3 + I_3 V_2 = 0.10\,\text{W} + 0.69\,\text{W} + 0.30\,\text{W} = 1.09\,\text{W}.$$

The power supplied equals the power dissipated by the resistors.

10.6 Check Your Understanding In considering the following schematic and the power supplied and consumed by a circuit, will a voltage source always provide power to the circuit, or can a voltage source consume power?



Example 10.7

Calculating Current by Using Kirchhoff's Rules

Find the current flowing in the circuit in **Figure 10.30**.

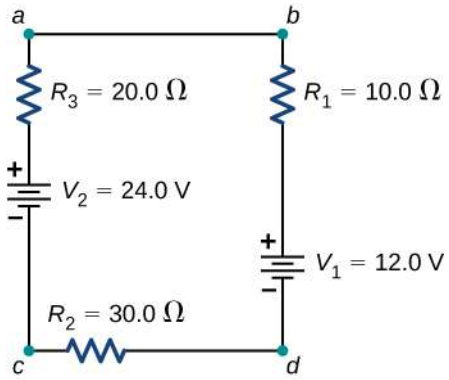


Figure 10.30 This circuit consists of three resistors and two batteries connected in series. Note that the batteries are connected with opposite polarities.

Strategy

This circuit can be analyzed using Kirchhoff's rules. There is only one loop and no nodes. Choose the direction of current flow. For this example, we will use the clockwise direction from point *a* to point *b*. Consider Loop *abcda* and use **Figure 10.23** to write the loop equation. Note that according to **Figure 10.23**, battery V_1 will be added and battery V_2 will be subtracted.

Solution

Applying the junction rule yields the following three equations. We have one unknown, so one equation is required:

$$\text{Loop } abcda: \quad -IR_1 - V_1 - IR_2 + V_2 - IR_3 = 0.$$

Simplify the equations by placing the unknowns on one side of the equations. Use the values given in the figure.

$$I(R_1 + R_2 + R_3) = V_2 - V_1.$$

$$I = \frac{V_2 - V_1}{R_1 + R_2 + R_3} = \frac{24\text{ V} - 12\text{ V}}{10.0\,\Omega + 30.0\,\Omega + 10.0\,\Omega} = 0.20\text{ A}.$$

Significance

The power dissipated or consumed by the circuit equals the power supplied to the circuit, but notice that the current in the battery V_1 is flowing through the battery from the positive terminal to the negative terminal and consumes power.

$$P_{R_1} = I^2 R_1 = 0.40\text{ W}$$

$$P_{R_2} = I^2 R_2 = 1.20\text{ W}$$

$$P_{R_3} = I^2 R_3 = 0.80\text{ W}$$

$$P_{V_1} = IV_1 = 2.40\text{ W}$$

$$P_{\text{dissipated}} = 4.80\text{ W}$$

$$P_{\text{source}} = IV_2 = 4.80\text{ W}$$

The power supplied equals the power dissipated by the resistors and consumed by the battery V_1.

10.7 Check Your Understanding When using Kirchhoff's laws, you need to decide which loops to use and the direction of current flow through each loop. In analyzing the circuit in **Example 10.7**, the direction of current flow was chosen to be clockwise, from point *a* to point *b*. How would the results change if the direction of the current was chosen to be counterclockwise, from point *b* to point *a*?

Multiple Voltage Sources

Many devices require more than one battery. Multiple voltage sources, such as batteries, can be connected in series configurations, parallel configurations, or a combination of the two.

In series, the positive terminal of one battery is connected to the negative terminal of another battery. Any number of voltage sources, including batteries, can be connected in series. Two batteries connected in series are shown in **Figure 10.31**. Using Kirchhoff's loop rule for the circuit in part (b) gives the result

$$\varepsilon_1 - Ir_1 + \varepsilon_2 - Ir_2 - IR = 0,$$
$$[(\varepsilon_1 + \varepsilon_2) - I(r_1 + r_2)] - IR = 0.$$

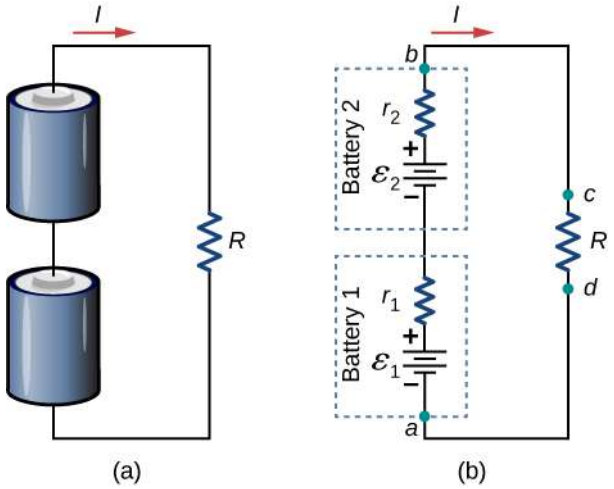


Figure 10.31 (a) Two batteries connected in series with a load resistor. (b) The circuit diagram of the two batteries and the load resistor, with each battery modeled as an idealized emf source and an internal resistance.

When voltage sources are in series, their internal resistances can be added together and their emfs can be added together to get the total values. Series connections of voltage sources are common—for example, in flashlights, toys, and other appliances. Usually, the cells are in series in order to produce a larger total emf. In **Figure 10.31**, the terminal voltage is

$$V_{\text{terminal}} = (\varepsilon_1 - Ir_1) + (\varepsilon_2 - Ir_2) = [(\varepsilon_1 + \varepsilon_2) - I(r_1 + r_2)] = (\varepsilon_1 + \varepsilon_2) + Ir_{\text{eq}}.$$

Note that the same current I is found in each battery because they are connected in series. The disadvantage of series connections of cells is that their internal resistances are additive.

Batteries are connected in series to increase the voltage supplied to the circuit. For instance, an LED flashlight may have two AAA cell batteries, each with a terminal voltage of 1.5 V, to provide 3.0 V to the flashlight.

Any number of batteries can be connected in series. For N batteries in series, the terminal voltage is equal to

$$V_{\text{terminal}} = (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{N-1} + \varepsilon_N) - I(r_1 + r_2 + \cdots + r_{N-1} + r_N) = \sum_{i=1}^{N} \varepsilon_i - Ir_{\text{eq}} \quad (10.6)$$

where the equivalent resistance is $r_{\text{eq}} = \sum_{i=1}^{N} r_i$.

When a load is placed across voltage sources in series, as in **Figure 10.32**, we can find the current:

$$(\varepsilon_1 - Ir_1) + (\varepsilon_2 - Ir_2) = IR,$$
$$Ir_1 + Ir_2 + IR = \varepsilon_1 + \varepsilon_2,$$
$$I = \frac{\varepsilon_1 + \varepsilon_2}{r_1 + r_2 + R}.$$

As expected, the internal resistances increase the equivalent resistance.

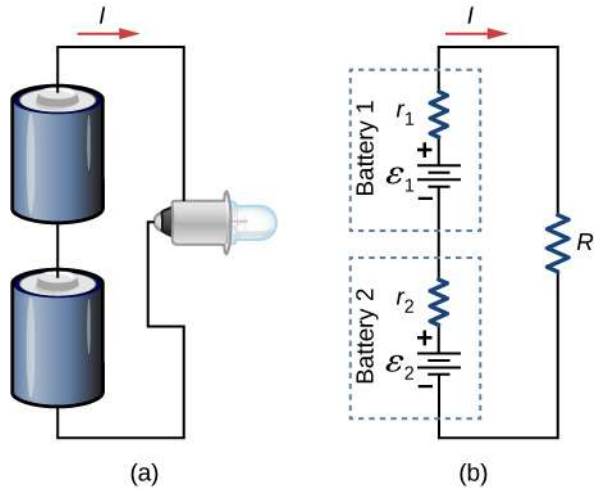


Figure 10.32 Two batteries connect in series to an LED bulb, as found in a flashlight.

Voltage sources, such as batteries, can also be connected in parallel. **Figure 10.33** shows two batteries with identical emfs in parallel and connected to a load resistance. When the batteries are connect in parallel, the positive terminals are connected together and the negative terminals are connected together, and the load resistance is connected to the positive and negative terminals. Normally, voltage sources in parallel have identical emfs. In this simple case, since the voltage sources are in parallel, the total emf is the same as the individual emfs of each battery.

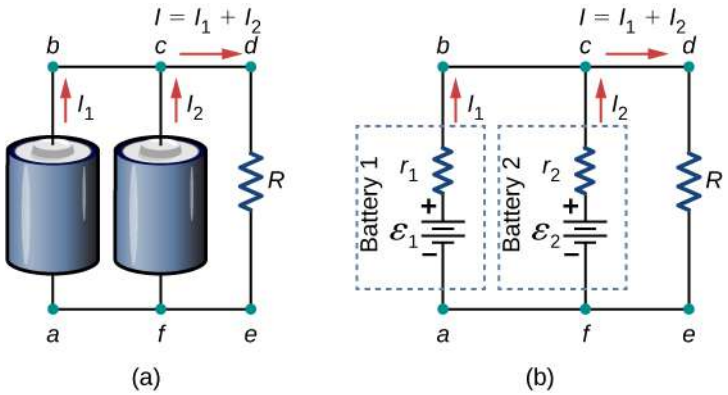


Figure 10.33 (a) Two batteries connect in parallel to a load resistor. (b) The circuit diagram shows the shows battery as an emf source and an internal resistor. The two emf sources have identical emfs (each labeled by ε) connected in parallel that produce the same emf.

Consider the Kirchhoff analysis of the circuit in **Figure 10.33**(b). There are two loops and a node at point b and $\varepsilon = \varepsilon_1 = \varepsilon_2$.

Node *b*: $I_1 + I_2 - I = 0$.

Loop *abcfa*:
$$\begin{aligned} \varepsilon - I_1 r_1 + I_2 r_2 - \varepsilon &= 0, \\ I_1 r_1 &= I_2 r_2. \end{aligned}$$

Loop *fcdef*:
$$\begin{aligned} \varepsilon_2 - I_2 r_2 - IR &= 0, \\ \varepsilon - I_2 r_2 - IR &= 0. \end{aligned}$$

Solving for the current through the load resistor results in $I = \frac{\varepsilon}{r_{\text{eq}} + R}$, where $r_{\text{eq}} = \left(\frac{1}{r_1} + \frac{1}{r_2}\right)^{-1}$. The terminal voltage is equal to the potential drop across the load resistor $IR = \left(\frac{\varepsilon}{r_{\text{eq}} + R}\right)$. The parallel connection reduces the internal resistance and thus can produce a larger current.

Any number of batteries can be connected in parallel. For *N* batteries in parallel, the terminal voltage is equal to

$$V_{\text{terminal}} = \varepsilon - I\left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_{N-1}} + \frac{1}{r_N}\right)^{-1} = \varepsilon - Ir_{\text{eq}} \tag{10.7}$$

where the equivalent resistance is $r_{\text{eq}} = \left(\sum_{i=1}^{N} \frac{1}{r_i}\right)^{-1}$.

As an example, some diesel trucks use two 12-V batteries in parallel; they produce a total emf of 12 V but can deliver the larger current needed to start a diesel engine.

In summary, the terminal voltage of batteries in series is equal to the sum of the individual emfs minus the sum of the internal resistances times the current. When batteries are connected in parallel, they usually have equal emfs and the terminal voltage is equal to the emf minus the equivalent internal resistance times the current, where the equivalent internal resistance is smaller than the individual internal resistances. Batteries are connected in series to increase the terminal voltage to the load. Batteries are connected in parallel to increase the current to the load.

Solar Cell Arrays

Another example dealing with multiple voltage sources is that of combinations of solar cells—wired in both series and parallel combinations to yield a desired voltage and current. Photovoltaic generation, which is the conversion of sunlight directly into electricity, is based upon the photoelectric effect. The photoelectric effect is beyond the scope of this chapter and is covered in **Photons and Matter Waves (http://cnx.org/content/m58757/latest/)** , but in general, photons hitting the surface of a solar cell create an electric current in the cell.

Most solar cells are made from pure silicon. Most single cells have a voltage output of about 0.5 V, while the current output is a function of the amount of sunlight falling on the cell (the incident solar radiation known as the insolation). Under bright noon sunlight, a current per unit area of about $100\ \text{mA/cm}^2$ of cell surface area is produced by typical single-crystal cells.

Individual solar cells are connected electrically in modules to meet electrical energy needs. They can be wired together in series or in parallel—connected like the batteries discussed earlier. A solar-cell array or module usually consists of between 36 and 72 cells, with a power output of 50 W to 140 W.

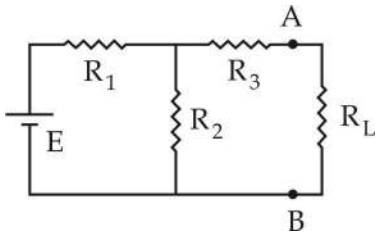
Solar cells, like batteries, provide a direct current (dc) voltage. Current from a dc voltage source is unidirectional. Most household appliances need an alternating current (ac) voltage.

NETWORK THEOREMS

1. Thevenin's theorem

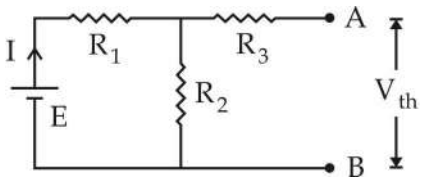
Statement: *A linear network consisting of a number of voltage sources and resistances can be replaced by an equivalent network having a single voltage source called Thevenin's voltage* (V_{Th}) *and a single resistance called Thevenin's resistance* (R_{Th}).

Explanation:



Consider a network or a circuit as shown. Let E be the emf of the cell having its internal resistance $r = 0$. $R_L \rightarrow$ load resistance across AB.

To find V_{Th}:

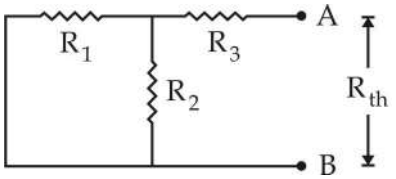


The load resistance R_L is removed. The current I in the circuit is $I = \dfrac{E}{R_1 + R_2}$.

The voltage across $AB =$ Thevenin's voltage V_{Th}.

$$V_{Th} = I R_2 \quad \Rightarrow \quad \boxed{V_{Th} = \frac{E R_2}{R_1 + R_2}}$$

To find R_{Th}:



The load resistance R_L is removed. The cell is disconnected and the wires are short as shown.

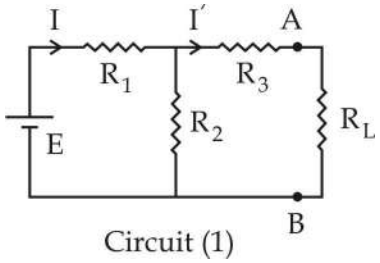
The effective resistance across $AB =$ Thevenin's resistance R_{Th}.

$\boxed{R_{Th} = R_3 + \dfrac{R_1 R_2}{R_1 + R_2}}$ [R_1 is parallel to R_2 and this combination in series with R_3]

If the cell has internal resistance r, then $V_{Th} = \dfrac{E R_2}{R_1 + R_2 + r}$ and $R_{Th} = R_3 + \dfrac{(R_1 + r) R_2}{R_1 + r + R_2}$.

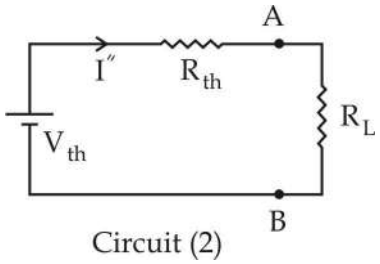
Proof of Thevenin's theorem:

Consider the network as shown below



Circuit (1)

The equivalent circuit is given by



Circuit (2)

The effective resistance of the network in (1) is R_3 and R_L in series and this combination is parallel to R_2 which in turn is in series with R_1.

Thus, $R_{eff} = R_1 + \dfrac{R_2(R_3 + R_L)}{R_2 + R_3 + R_L}$ ---------- (1)

The current I in the circuit is $I = \dfrac{E}{R_{eff}} = \dfrac{E}{R_1 + \dfrac{R_2(R_3 + R_L)}{R_2 + R_3 + R_L}}$

or $I = \dfrac{E(R_2 + R_3 + R_L)}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L}$ ---------- (2)

The current through the load resistance (I') is found using branch current method.

$I' = \dfrac{I\,R_2}{R_2 + R_3 + R_L}$ ---------- (3)

Substituting for I from (2) in (3)

$I' = \dfrac{E(R_2 + R_3 + R_L)R_2}{(R_2 + R_3 + R_L)(R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L)}$

or $I' = \dfrac{ER_2}{R_1R_2 + R_1R_3 + R_1R_L + R_2R_3 + R_2R_L}$ ---------- (4)

Thevenin's voltage $V_{Th} = \dfrac{ER_2}{R_1 + R_2}$ ---------- (5)

Thevenin's resistance $R_{Th} = R_3 + \dfrac{R_1R_2}{R_1 + R_2}$ ---------- (6)

Consider the equivalent circuit (circuit (2))

The current I'' in the equivalent circuit is $I'' = \dfrac{V_{Th}}{R_{Th} + R_L}$ ---------- (7)

Substituting for V_{Th} and R_{Th} from (5) and (6) in (7)

$$I'' = \frac{E R_2}{R_1 + R_2} \times \frac{1}{R_3 + \frac{R_1 R_2}{R_1 + R_2} + R_L} = \frac{E R_2}{(R_1 + R_2)\frac{R_3 R_1 + R_3 R_2 + R_1 R_L + R_2 R_L + R_1 R_2}{(R_1 + R_2)}}$$

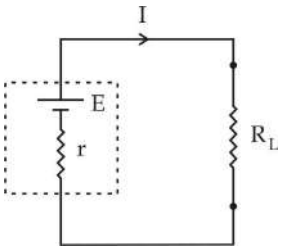
or $$I'' = \frac{E R_2}{R_1 R_2 + R_1 R_3 + R_1 R_L + R_2 R_3 + R_2 R_L} \text{ ---------- (8)}$$

From equations (4) and (8), it is observed that $I' = I''$.

Hence Thevenin's theorem is verified.

2. Maximum Power Transfer Theorem

Statement: *The power transferred by a source to the load resistance in a network is maximum when the load resistance is equal to the internal resistance of the source.*



Proof of Maximum power transfer theorem:

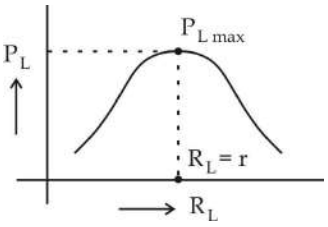
Consider a network with a source of emf E and internal resistance r connected to a load resistance R_L. The current I in the circuit is

$$I = \frac{E}{R_L + r} \text{ ---------- (1)}$$

The power delivered to load resistance R_L is $P_L = I^2 R_L$ or $P_L = \left(\frac{E}{R_L + r}\right)^2 R_L$

(from equation (1))

$$P_L = \frac{E^2}{(R_L + r)^2} R_L \text{ ---------- (2)}$$



The variation of P_L with R_L is as shown.

P_L is found to be maximum for a particular value of R_L when P_L is maximum, $\frac{dP_L}{dR_L} = 0$

[$\because$ No variation of P_L with R_L at $P_{L\,max}$]

i.e., $\frac{d}{dR_L}\left(\frac{E^2 R_L}{(R_L + r)^2}\right) = 0$ or $\frac{d}{dR_L}\left[E^2 R_L (R_L + r)^{-2}\right] = 0$

Differentiating $E^2\left[R_L(-2)(R_L + r)^{-3} + (R_L + r)^{-2}\right] = 0$ or $\frac{-2R_L}{(R_L + r)^3} + \frac{1}{(R_L + r)^2} = 0$ or

$$\frac{2R_L}{(R_L + r)^3} = \frac{1}{(R_L + r)^2}$$

Thus, $\frac{2R_L}{R_L + r} = 1 \quad \Rightarrow \quad 2R_L = R_L + r$ or $\boxed{R_L = r}$

Thus the power delivered to the load resistance is maximum when the load resistance is equal to the internal resistance of the source.

To show that the maximum power transfer efficiency of a circuit is 50%:

The power across the load $P_L = I^2 R_L = \dfrac{E^2}{(R_L + r)^2} R_L$ ---------- (1)

From the maximum power transfer theorem, P_L is maximum when $R_L = r$. Putting this condition in equation (1),

$$P_{L\max} = \frac{E^2}{(2R_L)^2} R_L \Rightarrow P_{L\max} = \frac{E^2}{4R_L} \quad \text{---------- (2)}$$

The power that is taken from the voltage source is (or power generated by the source),

$P = I^2 (R_L + r) = \dfrac{E^2}{(R_L + r)^2}(R_L + r)$ or $P = \dfrac{E^2}{R_L + r}$. When $R_L = r$, $P = \dfrac{E^2}{2R_L}$ -------- (3)

Dividing equation (2) by (3)

$$\frac{P_{L\max}}{P} = \frac{E^2}{4R_L} \times \frac{2R_L}{E^2} = \frac{1}{2} \text{ or } P_{L\max} = \frac{P}{2}$$

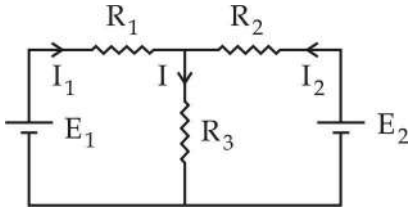
Thus the maximum power delivered to the load is only half the power generated by the source or the maximum power transfer efficiency is 50%. The remaining 50% power is lost across the internal resistance of the source.

3. Superposition theorem

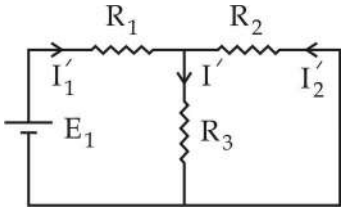
Statement: *In a linear network having number of voltage or current sources and resistances, the current through any branch of the network is the algebraic sum of the currents due to each of the sources when acting independently.*

Explanation: By mesh current analysis.

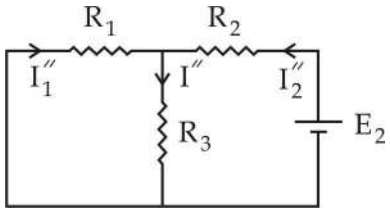
1. Consider the network as shown. The currents in different branches of the network are I_1, I_2 and I as shown. Also $I_1 + I_2 = I$.



2. [Let the internal resistance r of the cells be negligible].
   The cell E_2 is removed and the terminals are short as shown. Now the currents in the branches are I_1', I_2' and I'. Also $I' = I_1' + I_2'$.



3. The E_1 is removed and the terminals are short as shown. The currents are I_1'', I_2'' and I''. Also $I'' = I_1'' + I_2''$.



According to superposition theorem $I_1 = I_1' + I_1''$. $I_2 = I_2' + I_2''$ and $I = I' + I''$

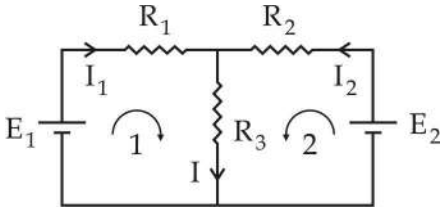
$\boxed{I = I_1 + I_2}$

Verification of superposition theorem:

1. Consider the network shown. Applying Kirchhoff's voltage to the loop 1.

$I_1R_1 + I_1R_3 + I_2R_3 = E_1 \quad [\because I_1R_1 + I(R_3) = E_1 \quad I = I_1 + I_2]$

or $I_1 = \dfrac{E_1 - I_2R_3}{R_1 + R_3}$ ---------- (1)



Considering loop 2, $I_2R_2 + I\,R_3 = E_2$.

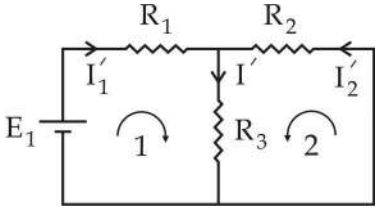
$I_2R_2 + I_1R_3 + I_2R_3 = E_2$

$I_2 = \dfrac{E_2 - I_1R_3}{R_2 + R_3}$ ---------- (2)

Thus, $I = I_1 + I_2$

$I = \dfrac{E_1 - I_2R_3}{R_1 + R_3} + \dfrac{E_2 - I_1R_3}{R_2 + R_3}$ ---------- (3)

2. Consider the circuit shown with E_2 removed and terminals short. Applying Kirchhoff's law to loop 1.



$I_1'R_1 + I'R_3 = E_1 \qquad$ As $I' = I_1' + I_2'$,

$I_1R_1 + I_1'R_3 + I_2'R_3 = E_1 \Rightarrow \quad I_1' = \dfrac{E_1 - I_2'R_3}{R_1 + R_3}$ ---------- (4)

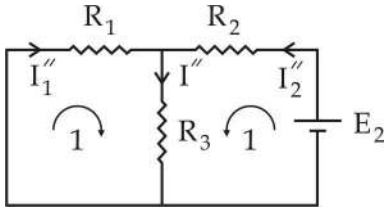
Similarly for loop 2,

$I_2'R_2 + I'R_3 = 0 \quad \Rightarrow \qquad I_2'R_2 + I_1'R_3 + I_2'R_3 = 0$

$I_2' = -\dfrac{I_1'R_3}{R_2 + R_3}$ ---------- (5)

$I' = I_1' + I_2' = \dfrac{E_1 - I_2'R_3}{R_1 + R_3} - \dfrac{I_1'R_3}{R_2 + R_3}$ .....(6)

3. Consider the circuit with E_1 removed and terminals short.



For loop (1) $I_1''R_1 + I''R_3 = 0$

As $I'' = I_1'' + I_2''$

$$I_1''R_1 + I_1''R_3 + I_2''R_3 = 0 \quad \Rightarrow \quad I_1'' = \frac{-I_2''R_3}{R_1 + R_3} \text{ ---------- (7)}$$

For loop (2)

$$I_2''R_2 + I''R_3 = E_2 \quad \Rightarrow \quad I_2''R_2 + I_1''R_3 + I_2''R_3 = E_2$$

or $$I_2'' = \frac{E_2 - I_1''R_3}{R_2 + R_3} \text{ ---------- (8)}$$

$$I'' = I_1'' + I_2'' = \frac{-I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3} \text{ ---------- (9)}$$

Adding equations (6) and (9)

$$I' + I'' = \frac{E_1 - I_2'R_3}{R_1 + R_3} - \frac{I_1'R_3}{R_2 + R_3} - \frac{I_2''R_3}{R_1 + R_3} + \frac{E_2 - I_1''R_3}{R_2 + R_3}$$

$$= \frac{1}{R_1 + R_3}\left[E_1 - I_2'R_3 - I_2''R_3\right] + \frac{1}{R_2 + R_3}\left[E_2 - I_1''R_3 - I_1'R_3\right]$$

$$I' + I'' = \frac{1}{R_1 + R_3}\left[E_1 - R_3\left(I_2' + I_2''\right)\right] + \frac{1}{R_2 + R_3}\left[E_2 - R_3\left(I_1' + I_1''\right)\right] \text{ .....(10)}$$

Comparing equations (3) and (10) it is observed that

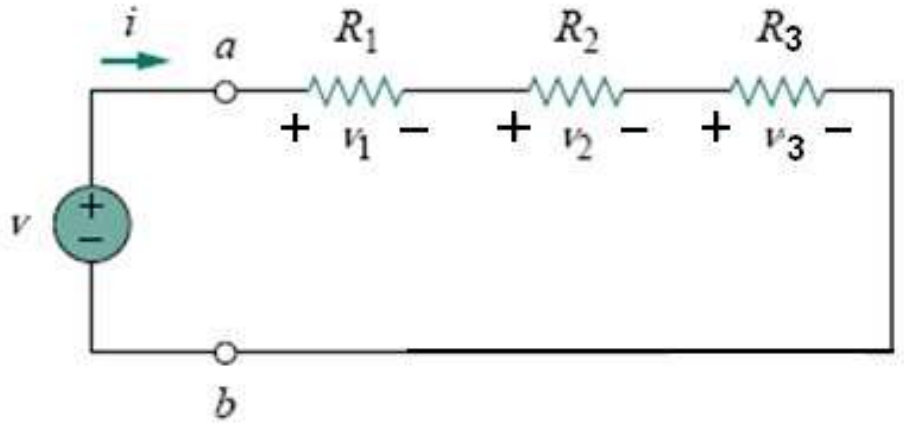
$$I_1 = I_1' + I_1''$$

$$I_2 = I_2' + I_2''$$

$$I = I' + I''$$

Hence the proof of the theorem.

Voltage Division



For example, we know

$i = V_{TOTAL} / (R_1 + R_2 + R_3)$

so the voltage over the **first resistor** is

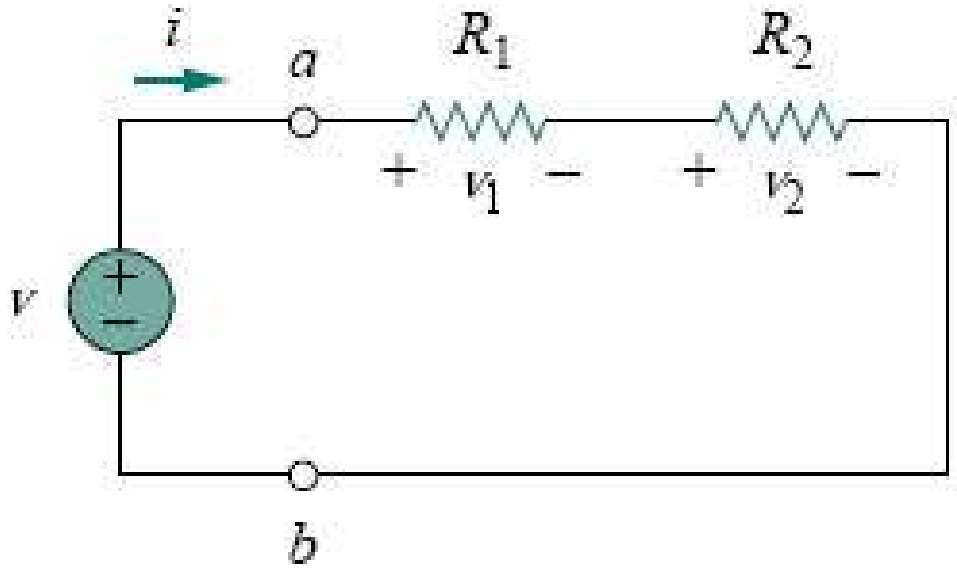
$V_1 = i R_1 = R_1 V_{TOTAL} / (R_1 + R_2 + R_3)$

$$v_1 = V \frac{R_1}{R_1 + R_2 + R_3}$$

To find the voltage over an individual resistance in series, take the total series voltage and multiply by the individual resistance over the total resistance.

Example

determine the voltage across each resistor



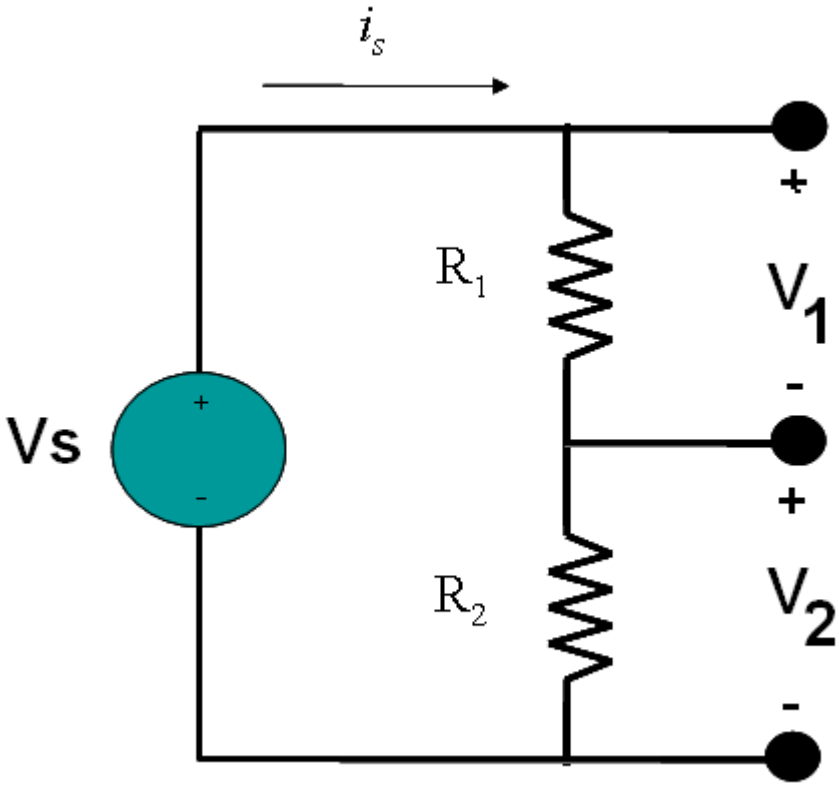
$$v_1 = \frac{R_1}{R_1 + R_2} v \qquad v_2 = \frac{R_2}{R_1 + R_2} v$$

Example
determine the voltage acros
each resistor

$$i_s = \frac{v_s}{R_1 + R_2}$$

$$v_1 = \frac{R_1}{R_1 + R_2} v_s$$

$$v_2 = \frac{R_2}{R_1 + R_2} v_s$$



Summary

•The source voltage *v* is divided among the resistors in direct proportion to their resistances

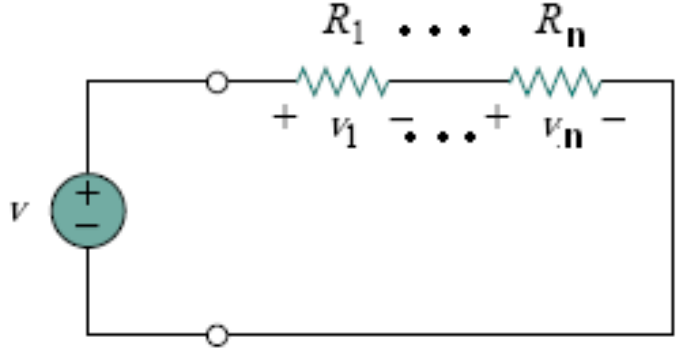
•the larger the resistance, the larger the voltage drop.

•This is called the *principle of voltage division*,

•In general

If a voltage divider has *N* resistors ($R_1, R_2, \ldots, R_N$) in series with the source voltage *v*, the n^{th} resistor (R_n) will have a voltage drop of

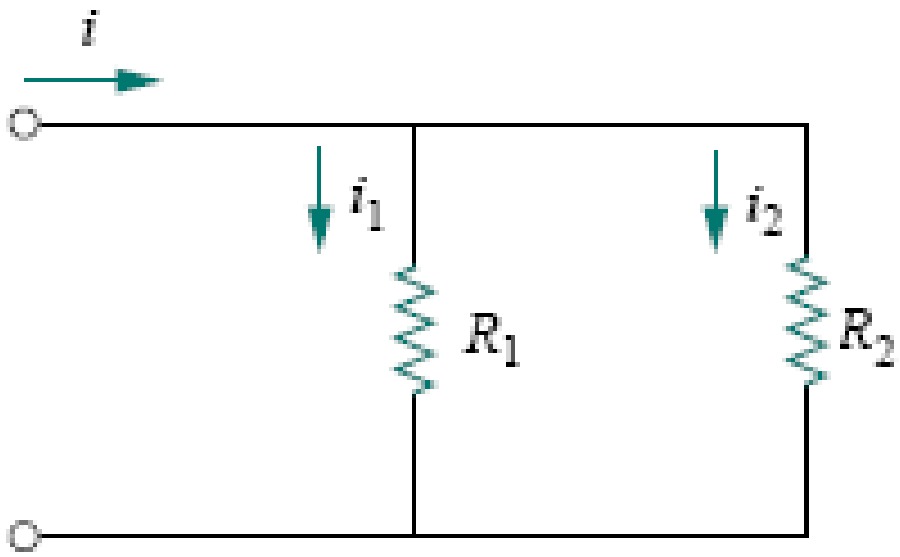
$$v_n = \frac{R_n}{R_1 + R_2 + \cdots + R_N} v$$



Current Division

If we know the current flowing into **two parallel resistors,** we can find out how the current will divide up in one step.

$$v = i\frac{R_1 R_2}{R_1 + R_2} \leftarrow R_{Eq}$$

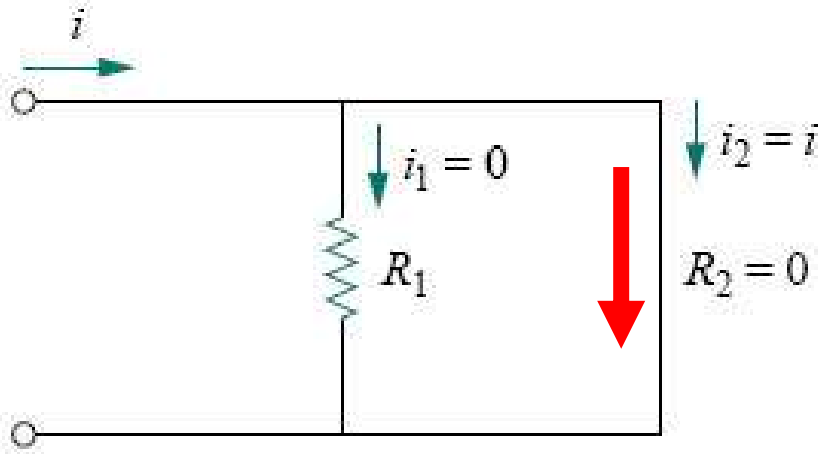


$$i_1 = \frac{v}{R_1}$$

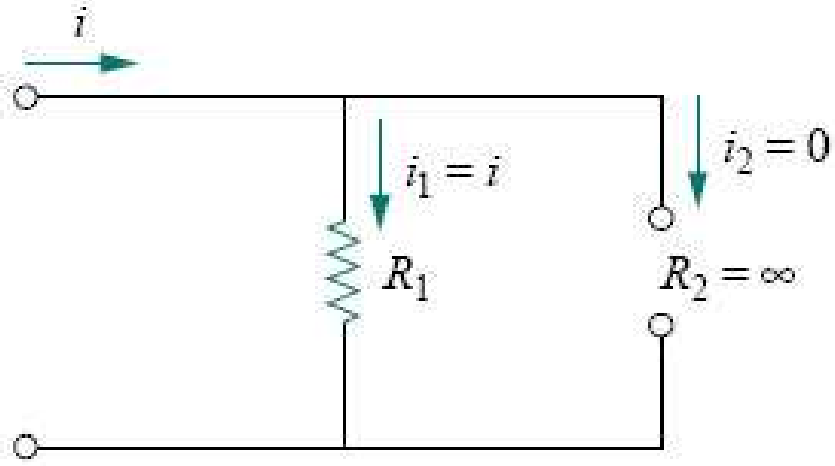
$$i_1 = \frac{1}{R_1}\left[i\frac{R_1 R_2}{R_1 + R_2} \right] \Rightarrow \boxed{i_1 = i\frac{R_2}{R_1 + R_2}}$$

$$i_1 = \frac{R_2\, i}{R_1 + R_2} \qquad i_2 = \frac{R_1\, i}{R_1 + R_2}$$

Note that this differs slightly from the voltage division formula for series resistors



$R_2 = 0$ implies that $i_1 = 0$, $i_2 = i$.

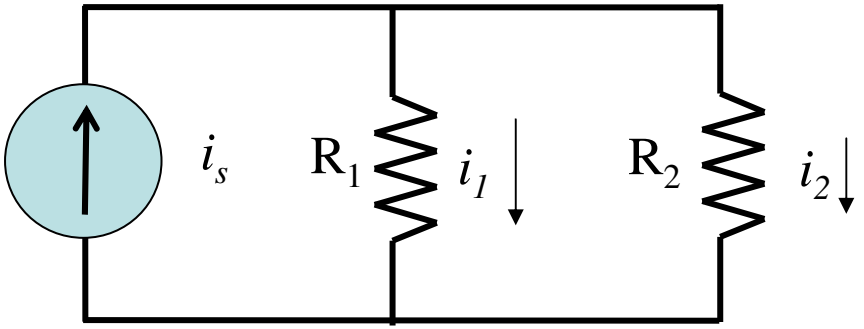


$R_2 = \infty$, that is, R_2 is an open

$R_2 \rightarrow \infty$, then
$R_{eq} = R_1$

Example

Find i_1 and i_2 in terms of i_s



$$i_s = i_1 + i_2$$

$v_1 = v_2$

$$i_1 R_1 = i_2 R_2$$

$$i_2 = i_1 \frac{R_1}{R_2} \Rightarrow i_s = i_1 + i_1 \frac{R_1}{R_2} = i_1 \left(\frac{R_2 + R_1}{R_2} \right)$$

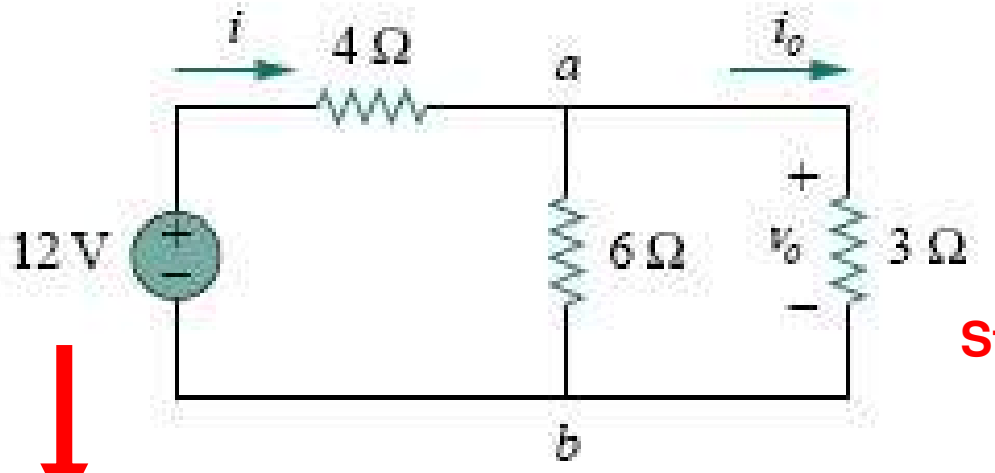
$$i_1 = i_s \left(\frac{R_2}{R_1 + R_2} \right)$$

$$i_2 = i_s \left(\frac{R_1}{R_1 + R_2} \right)$$

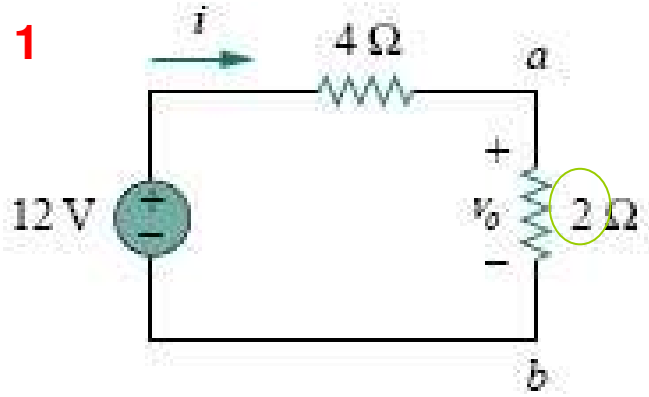
Example

Use Voltage and current division rules to find V_o and i_o

$$\frac{6 \cdot 3}{6+3} = 2$$



Step 1



Step 2

Step 1

$$i = \frac{12}{4+2} = 2\ \text{A}$$

$$v_o = \frac{2}{2+4}(12\ \text{V}) = 4\ \text{V}$$

$$i_o = \frac{6}{6+3} i = \frac{2}{3}(2\ \text{A}) = \frac{4}{3}\ \text{A}$$

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Real Analog Chapter 3: Nodal & Mesh Analysis

3 Introduction and Chapter Objectives

In Chapters 1 & 2, we introduced several tools in circuit analysis:

- Ohm's Law
- Kirchhoff's laws
- Circuit reduction

Circuit reduction, it should be noted, is not fundamentally different from direct application of Ohm's and Kirchhoff's laws - it is simply a convenient re-statement of these laws for specific combinations of circuit elements.

In Chapter 1, we saw that direct application of Ohm's law and Kirchhoff's laws to a specific circuit using the *exhaustive method* often results in a large number of unknowns - even if the circuit is relatively simple. A correspondingly large number of equations must be solved to determine these unknowns. Circuit reduction allows us, in some cases, to simplify the circuit to reduce the number of unknowns in the system. Unfortunately, not all circuits are reducible and even analysis of circuits that are reducible depends upon the engineer "noticing" certain resistance combinations and combining them appropriately.

In cases where circuit reduction is not feasible, approaches are still available to reduce the total number of unknowns in the system. *Nodal analysis* and *mesh analysis* are two of these. Nodal and mesh analysis approaches still rely upon application of Ohm's law and Kirchhoff's laws - we are just applying these laws in a very specific way in order to simplify the analysis of the circuit. One attractive aspect of nodal and mesh analysis is that the approaches are relatively rigorous - we are assured of identifying a reduced set of variables, if we apply the analysis rules correctly. Nodal and mesh analysis are also more general than circuit reduction methods - virtually any circuit can be analyzed using nodal or mesh analysis.

Since nodal and mesh analysis approaches are fairly closely related, section 3.1 introduces the basic ideas and terminology associated with both approaches. Section 3.2 provides details of nodal analysis, and mesh analysis is presented in section 3.3.

After Completing this Chapter, You Should be Able to:

- Use nodal analysis techniques to analyze electrical circuits
- Use mesh analysis techniques to analyze electrical circuits

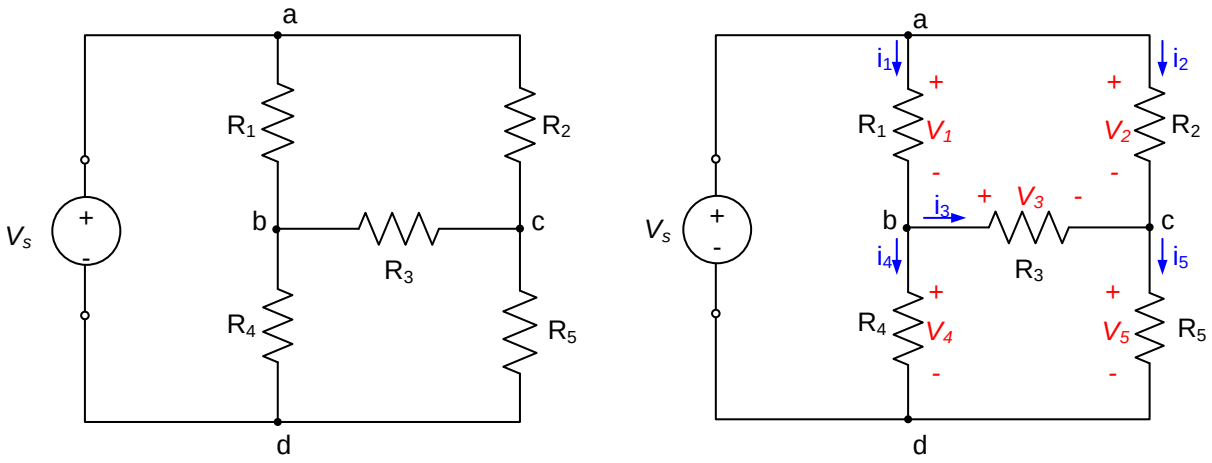
3.1 Introduction and Terminology

As noted in the introduction, both nodal and mesh analysis involve identification of a "minimum" number of unknowns, which completely describe the circuit behavior. That is, the unknowns themselves may not directly provide the parameter of interest, but any voltage or current in the circuit can be determined from these

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unknowns. In nodal analysis, the unknowns are the *node voltages*. In mesh analysis, the unknowns are the *mesh currents*. We introduce the concept of these unknowns via an example below.

Consider the circuit shown in Fig. 3.1(a). The circuit nodes are labeled in Fig. 3.1(a), for later convenience. The circuit is not readily analyzed by circuit reduction methods. If the exhaustive approach toward applying KCL and KVL is taken, the circuit has 10 unknowns (the voltages and currents of each of the five resistors), as shown in Fig. 3.1(b). Ten circuit equations must be written to solve for the ten unknowns. Nodal analysis and mesh analysis provide approaches for defining a reduced number of unknowns and solving for these unknowns. If desired, any other desired circuit parameters can subsequently be determined from the reduced set of unknowns.



(a) Circuit schematic *(b) Complete set of unknowns*

Figure 3.1. Non-reducible circuit.

In nodal analysis, the unknowns will be *node voltages*. Node voltages, in this context, are the *independent voltages* in the circuit. It will be seen later that the circuit of Fig. 3.1 contains only two independent voltages - the voltages at nodes b and c[1]. Only two equations need be written and solved to determine these voltages! Any other circuit parameters can be determined from these two voltages.

Basic Idea

In nodal analysis, Kirchhoff's current law is written at each independent voltage node; Ohm's law is used to write the currents in terms of the node voltages in the circuit.

In mesh analysis, the unknowns will be *mesh currents*. Mesh currents are defined only for *planar circuits*; planar circuits are circuits which can be drawn in a single plane such that no elements overlap one another. When a circuit is drawn in a single plane, the circuit will be divided into a number of distinct areas; the boundary of each area is a *mesh* of the circuit. A *mesh current* is the current flowing around a mesh of the circuit. The circuit of Fig. 3.1 has three meshes:

- The mesh bounded by V_s, node a, and node d
- The mesh bounded by node a, node c, and node b
- The mesh bounded by node b, node c, and node d

[1] The voltages at nodes a and d are not independent; the voltage source v_s *constrains* the voltage at node a relative to the voltage at node d (KVL around the leftmost loop indicates the $v_{ab} = V_S$

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These three meshes are illustrated schematically in Fig. 3.2. Thus, in a mesh analysis of the circuit of Fig. 3.1, three equations must be solved in three unknowns (the mesh currents). Any other desired circuit parameters can be determined from the mesh currents.

Basic Idea

In mesh analysis, Kirchhoff's voltage law is written around each mesh loop; Ohm's law is used to write the voltage in terms of the mesh currents in the circuit. Since KVL is written around closed loops in the circuit, mesh analysis is sometimes known as *loop analysis*.

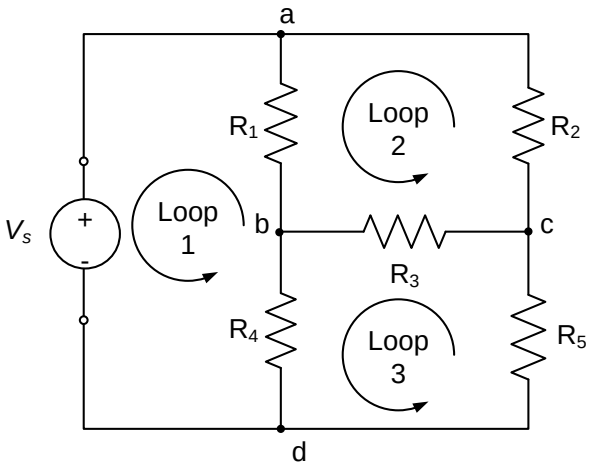


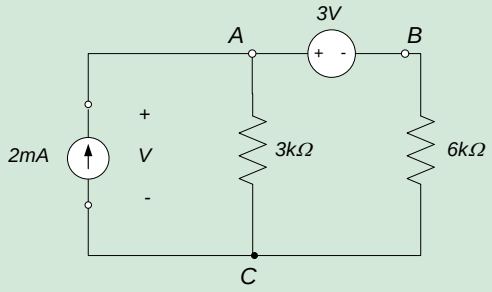
Figure 3.2. Meshes for circuit of Figure 3.1.

Section Summary

1. In nodal analysis:
   - Unknowns in the analysis are called the *node voltages*
   - Node voltages are the voltages at the independent nodes in the circuit
   - Two nodes connected by a voltage source are not independent. The voltage source constrains the voltages at the nodes relative to one another. A node which is not independent is also called dependent.
2. In mesh analysis:
   - Unknowns in the analysis are called *mesh currents*.
   - Mesh currents are defined as flowing through the circuit elements which form the perimeter of the circuit meshes. A mesh is any enclosed, non-overlapping region in the circuit (when the circuit schematic is drawn on a piece of paper).

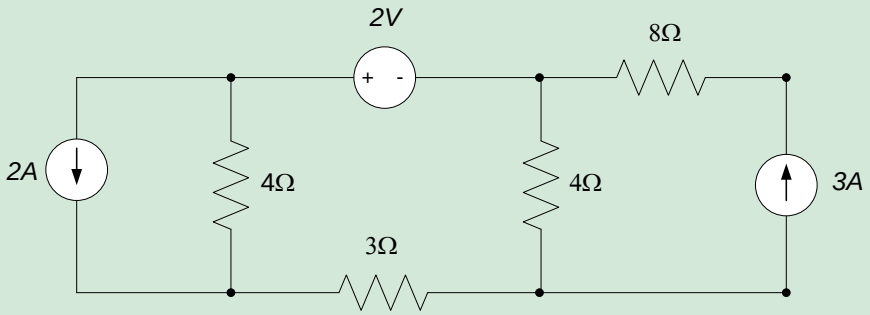
Exercises

1. The circuit below has three nodes, A, B, and C. Which two nodes are dependent? Why?



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2. Identify meshes in the circuit below.



3.2 Nodal Analysis

As noted in section 3.1, in nodal analysis we will define a set of node voltages and use Ohm's law to write Kirchhoff's current law in terms of these voltages. The resulting set of equations can be solved to determine the node voltages; and other circuit parameters (e.g. currents) can be determined from these voltages.

The steps used in nodal analysis are provided below. The steps are illustrated in terms of the circuit of Fig. 3.3.

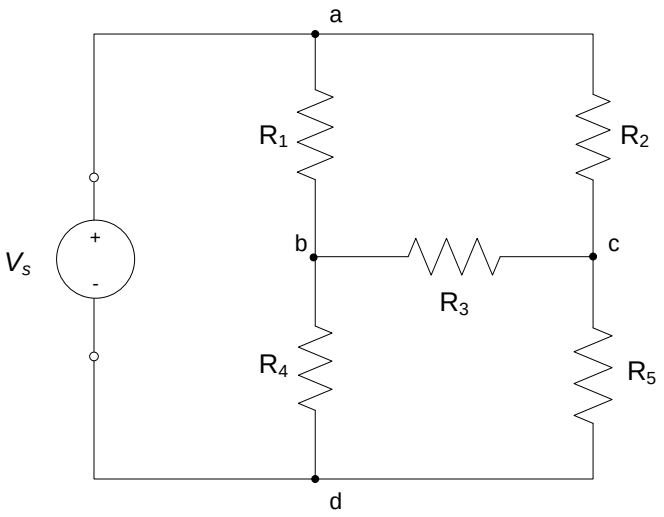


Figure 3.3. Example circuit.

Step 1: Define Reference Voltage

One node will be arbitrarily as a *reference node* or *datum node*. The voltages of all other nodes in the circuit will be defined to be relative to the voltage of this node. Thus, for convenience, it will be assumed that the reference node voltage is zero volts. It should be emphasized that this definition is arbitrary - since voltages are actually potential differences, choosing the reference voltage as zero is primarily a convenience.

For our example circuit, we will choose node d as our reference node and define the voltage at this node to be 0V, as shown in Fig. 3.4.

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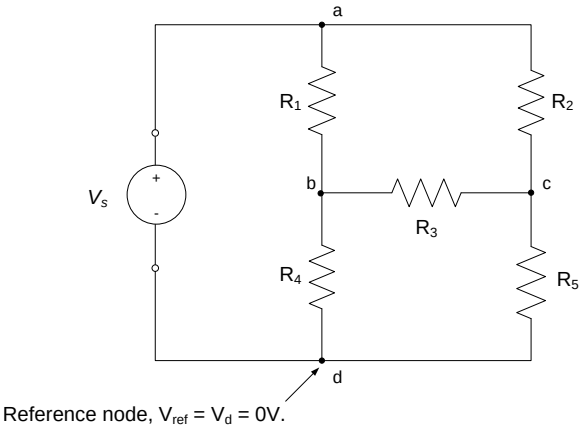


Figure 3.4. Definition of reference node and reference voltage.

Step 2: Determine Independent Nodes

We now define the voltages at the *independent* nodes. These voltages will be the unknowns in our circuit equations. In order to define independent nodes:

- "Short-circuit" all voltage sources
- "Open-circuit" all current sources

After removal of the sources, the remaining nodes (with the exception of the reference node) are defined as *independent* nodes (the nodes which were removed in this process are *dependent* nodes. The voltages at these nodes are sometimes said to be *constrained)*. Label the voltages at these nodes - they are the unknowns for which we will solve.

For our example circuit of Fig. 1, removal of the voltage source (replacing it with a short circuit) results in nodes remaining only at nodes b and c. This is illustrated in Fig. 3.

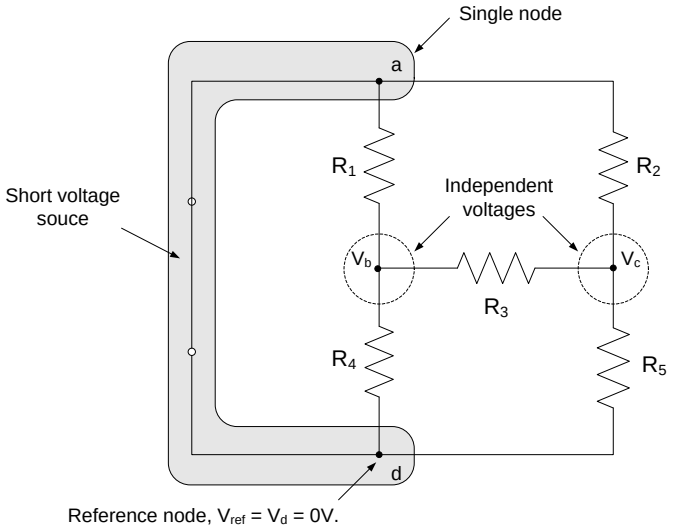


Figure 3.5. Independent voltages V_b and V_c.

Step 3: Replace Sources in the Circuit and Identify Constrained Voltages

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With the independent voltages defined as in Step 2, replace the sources and define the voltages at the dependent nodes in terms of the independent voltages and the known voltage difference.

For our example, the voltage at node a can be written as a known voltage V_s above the reference voltage, as shown in Fig. 3.6.

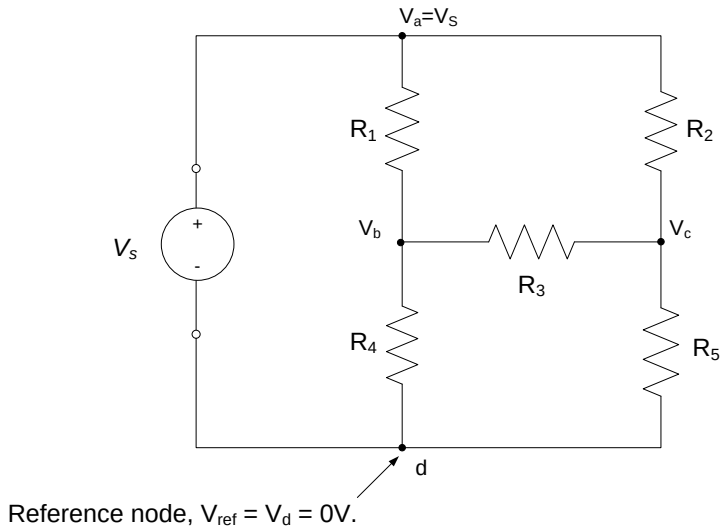


Figure 3.6. Dependent voltages defined.

Step 4: Applying KCL at Independent Nodes

Define currents and write Kirchhoff's current law at all independent nodes. Currents for our example are shown in Fig. 3.7 below. The defined currents include the assumed direction of positive current - this defines the sign convention for our currents. To avoid confusion, these currents are defined consistently with those shown in Fig. 3.1(a). The resulting equations are (assuming that currents leaving the node are defined as positive):

Node b:

$$-i_1 + i_3 + i_4 = 0 \qquad \text{Eq. 3.1}$$

Node c:

$$-i_2 + i_3 + i_5 = 0 \qquad \text{Eq. 3.2}$$

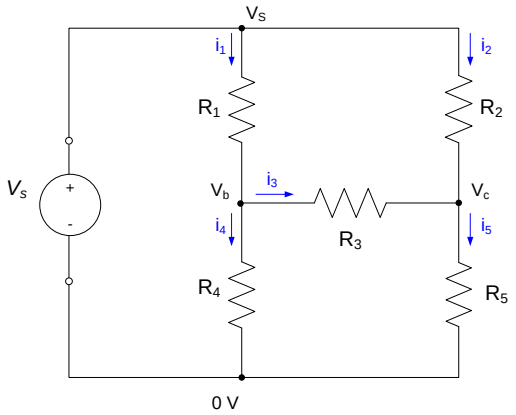


Figure 3.7. Current definitions and sign conventions.

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Step 5: Use Ohm's Law to Write the Equations From Step 4 in Terms of Voltages

The currents defined in Step 4 can be written in terms of the node voltages defined previously. For example, from Fig. 3.7: $i_1 = \frac{V_S - V - b}{R_1}$, $i_3 = \frac{V_b - V_c}{R_3}$, and $i_4 = \frac{V_b - 0}{R_4}$, so equation (3.1) can be written as:

So the KCL equation for node b becomes:

$$\left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right) V_b - \frac{1}{R_3} V_c = \frac{1}{R_1} V_S \qquad \text{Eq. 3.3}$$

Likewise, the KCL equation for node c can be written as:

$$-\frac{1}{R_3} V_b + \left(\frac{1}{R_3} + \frac{1}{R_2} + \frac{1}{R_5}\right) V_c = \frac{1}{R_2} V_S \qquad \text{Eq. 3.3}$$

Double-checking Results

If the circuit being analyzed contains only independent sources, and the sign convention used in KCL equations is the same as used above (currents leaving nodes are assumed positive), the equations written at each node will have the following form:

- The term multiplying the voltage at that node will be the sum of the conductances connected to that node. For the example above, the term multiplying V_b in the equation for node b is $\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}$ while the term multiplying V_c in the equation for node c is $\frac{1}{R_3} + \frac{1}{R_2} + \frac{1}{R_5}$.
- The term multiplying the voltages adjacent to the node will be the negative of the conductance connecting the two nodes. For the example above, the term multiplying V_c in the equation for node b is $-\frac{1}{R_3}$, and the term multiplying V_b in the equation for node c is $-\frac{1}{R_3}$.

If the circuit contains dependent sources, or a different sign convention is used when writing the KCL equations, the resulting equations will not necessarily have the above form.

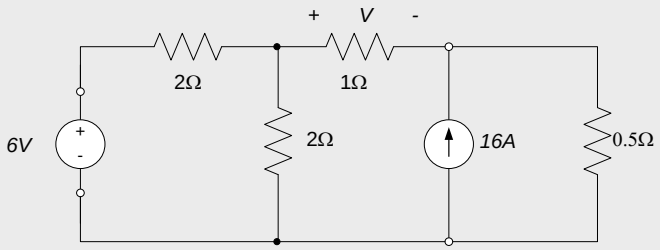
Step 6: Solve the System of Equations Resulting from Step 5

Step 5 will always result in N equations in N unknowns, where N is the number of independent nodes identified in Step 2. These equations can be solved for the independent voltages. Any other desired circuit parameters can be determined from these voltages.

The example below illustrates the above approach.

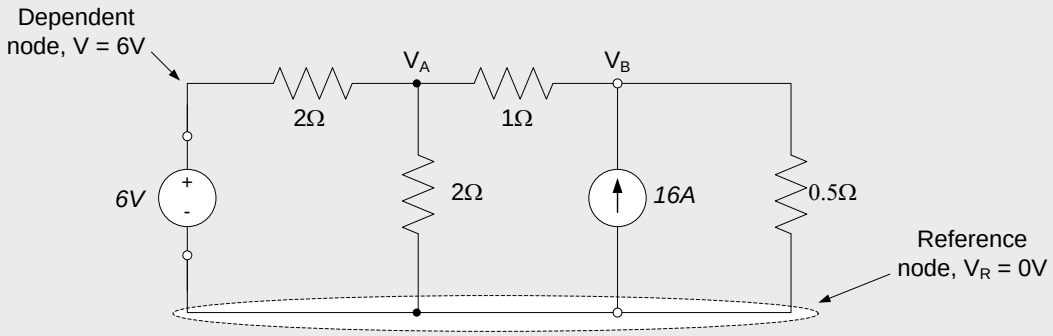
Example 3.1

Find the voltage V for the circuit shown below:



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Steps 1, 2, and 3: Choosing the reference voltage as shown below, identifying voltages at dependent nodes, and defining voltages V_A and V_B at the independent nodes results in the circuit schematic shown below:



Steps 4 and 5: Writing KCL at nodes A and B and converting currents to voltages using Ohm's law results in the following two equations:

Node A:

$$\frac{V_A-6}{2}+\frac{V_A-V_B}{2}+\frac{V_A-V_B}{1}=0 \Rightarrow \left(\frac{1}{2}+\frac{1}{2}+\frac{1}{1}\right)V_A-V_B=3 \Rightarrow 2V_A-V_B=3$$

Node B:

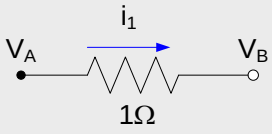
$$\frac{V_B-V_A}{1}+\frac{V_B-0}{0.5}+16=0 \Rightarrow \left(\frac{1}{1}+\frac{1}{0.5}\right)V_B-V_A=-16 \Rightarrow 3V_B=16$$

Step 6: Solving the above equations results in $V_A=5V$ and $V_B=7V$. The voltage V is:

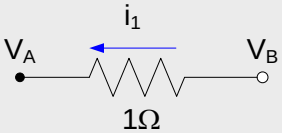
$$V=V_A-V_B=-2V$$

Several comments should be made relative to the above example:

1. Steps 4 and 5 (applying KCL at each independent node and using Ohm's law to write these equations in terms of voltages) have been combined into a single step. This approach is fairly common, and can provide a significant savings in time.
2. There may be a perceived inconsistency between the two node equations, in the assumption of positive current direction in the 1Ω resistor. In the equation for node A, the current is apparently assumed to be positive from node A to node B, as shown below:



This leads to the corresponding term in the equation for node A becoming: $\frac{V_A-V_B}{1}$. In the equation for node B, however, the positive current direction appears to be from node B to node A, as shown below:



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This definition leads to the corresponding term in the equation for node B becoming: $\frac{V_B - V_A}{1}$.

The above inconsistency in sign is, however, insignificant. Suppose that we had assumed (consistently with the equation for node A) that the direction of positive current for the node B equation is from Node A to B. Then, the corresponding term in the equation for node B would have been: $-\frac{V_A - V_B}{1}$ (note that a negative sign has been applied to this term to accommodate our assumption that currents flowing into nodes are negative). This is equal to $\frac{V_A - V_B}{1}$, which is exactly what our original result was.

3. The current source appears directly in the nodal equations.

Note

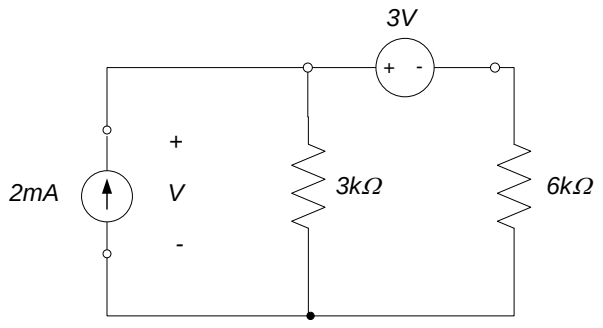
When we write nodal equations in these chapters, we will generally assume that any unknown currents are flowing away from the node for which we are writing the equation, regardless of any previous assumptions we have made for the direction of that current. The signs will work out, as long as we are consistent in our sign convention between assumed voltage polarity and current direction and our sign convention relative to positive currents flowing out of nodes.

The sign applied to currents induced by current sources must be consistent with the current direction assigned by the source.

3.2.1 Supernodes

In the previous examples, we identified dependent nodes and determined constrained voltages. Kirchhoff's current law was then only written at independent nodes. Many readers find this somewhat confusing, especially if the dependent voltages are not relative to the reference voltage. We will thus discuss these steps in more detail here in the context of an example, introducing the concept of a *supernode* in the process.

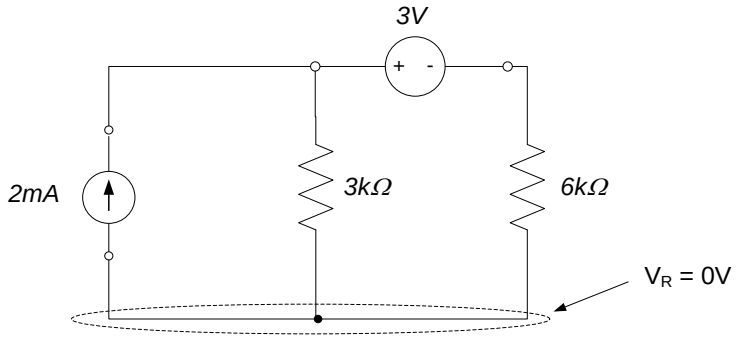
Example: for the circuit below, determine the voltage difference, V, across the 2mA source.



Step 1: define Reference Node

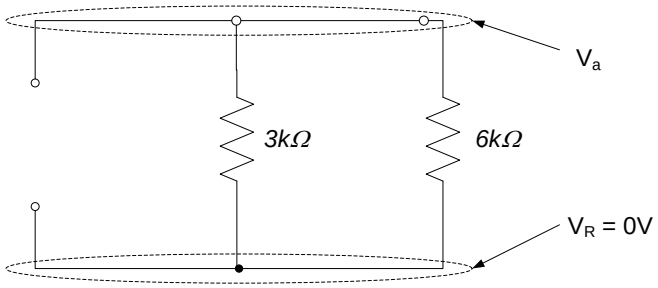
Choose reference node (somewhat arbitrarily) as shown below; label the reference node voltage, V_R, as zero volts.

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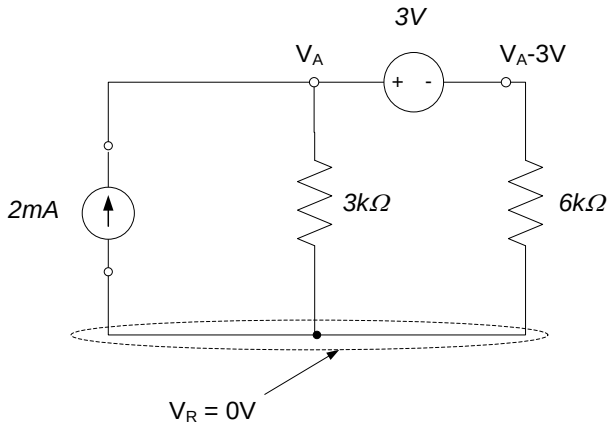
Step 2: Define Independent Nodes

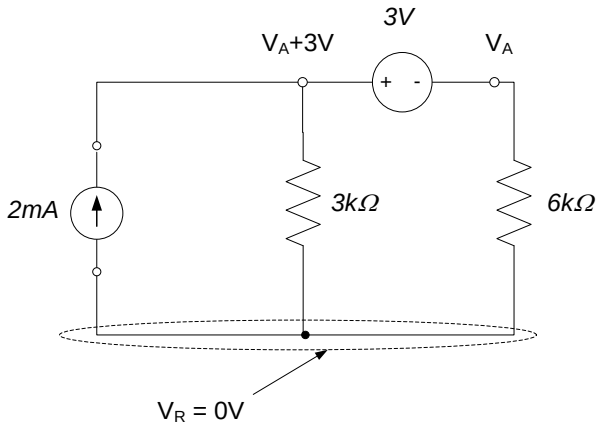
Short circuit voltage sources, open circuit current sources as shown below and identify independent nodes/voltages. For our example, this result in only one independent voltage, labeled as V_A below.



Step 3: Replace Sources and Label any Known Voltages

The known voltages are written in terms of node voltages identified above. There is some ambiguity in this step. For example, either of the representations below will work equally well - either side of the voltage source can be chosen as the node voltage, and the voltage on the other side of the source written in terms of this node voltage. Make sure, however, that the correct polarity of the voltage source is preserved. In our example, the left side of the source has a potential that is three volts higher than the potential of the right side of the source. This fact is represented correctly by both of the choices below.





Step 4: Apply KCL at the Independent Nodes

It is this step that sometimes causes confusion among readers, particularly when voltage sources are present in the circuit. Conceptually, it is possible to think of two nodes connected by an ideal voltage source as forming a single *supernode* (some authors use the term *generalized node* rather than supernode). A node is rigorously

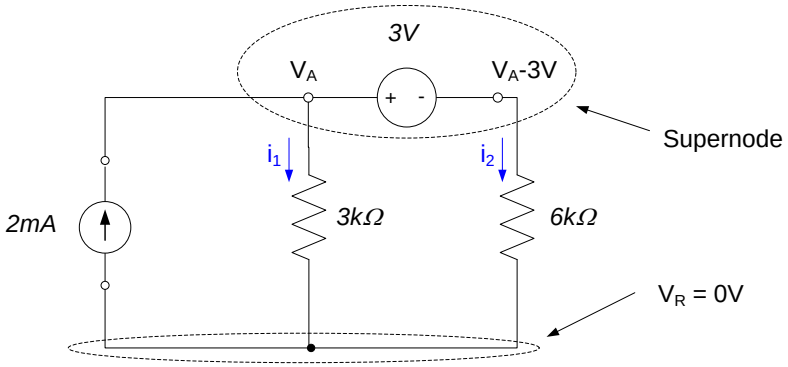
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defined as having a single, unique voltage. However, although the two nodes connected by a voltage source do not share the same voltage, they are not entirely independent - the two voltages are *constrained* by one another. This allows us to simplify the analysis somewhat.

For our example, we will arbitrarily choose the circuit to the left above to illustrate this approach. The supernode is chosen to include the voltage source and both nodes to which it is connected, as shown below. We define two currents leaving the supernode, i_1 and i_2, as shown. KCL, applied at the supernode, results in:

$$-2mA + i_1 + i_2 = 0$$

As before, currents leaving the node are assumed to be positive. This approach allows us to account for the current flowing through the voltage source without ever explicitly solving for it.



Step 5: Use Ohm's Law to Write the KCL Equations in Terms of Voltages

For the single KCL equation written above, this results in:

$$-2mA + \frac{V_A - 0}{3k\Omega} + \frac{(V_A - 3) - 0}{6k\Omega} = 0$$

Step 6: Solve the System of Equations to Determine the Nodal Voltages

Solution of the equation above results in $V_A = 5V$. Thus, the voltage difference across the current source is $V = 5V$.

3.2.2 Alternate Approach: Constraint Equations

The use of supernodes can be convenient, but is not a necessity. An alternate approach, for those who do not wish to identify supernodes, is to restrain separate nodes on either side of the voltage source and then write a constraint equation relating these voltages. Thus, in cases where the reader does not recognize a supernode, the analysis can proceed correctly. We now revisit the previous example, but use constraint equations rather than the previous supernode technique.

In this approach, Steps 2 and 3 (identification of independent nodes) are not necessary. One simply writes Kirchhoff's current law at all nodes and then writes constraints equations for the voltage sources. A disadvantage of this approach is that currents through voltage sources must be accounted for explicitly; this result in a greater number of unknowns (and equations to be solved) than the supernode technique.

Example (revisited): for the circuit below, determine the voltage difference, V, across the 2mA source.

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