MATHEMATICS II

Function of Two Variables

دالة المتغيرين

Lecturer	م.د .هدى عقيل الطيار
Department	قسم هندسة الاتصالات والأنظمة الرقمية الذكية
Stage	المرحلة الثانية
Academic Year	2026 - 2027

01 DEFINITIONS & CONCEPT

Functions of Two Variables — Foundations

Functions of One, Two, and More Variables

y = f(x)

Function of ONE variable

x: independent • y: dependent

z = f(x, y)

Function of TWO variables

x, y: independent • z: dependent

w = f(x, y, z)

Function of THREE variables

x, y, z: independent • w: dependent

Partial Derivative — الاشتقاق الجزئي

Ordinary Derivative — for y = f(x)

$$dy / dx = y'$$

Partial Derivative — for a function of two or more variables

$$\partial z / \partial x \quad , \quad \partial z / \partial y$$

Key Rule

All the ordinary differentiation rules used in single-variable calculus remain valid in partial differentiation.

كل القوانين المستخدمة في الاشتقاق الاعتيادي تستمر في الاشتقاق الجزئي

NOTATION & EXAMPLE

Partial Derivative Notation

let z = f(x, y)

1st ORDER	$\partial z/\partial x = \partial f/\partial x = f_x$
	$\partial z/\partial y = \partial f/\partial y = f_y$

2nd ORDER	$\partial^2 z/\partial x^2 = f_{xx}$
	$\partial^2 z/\partial y^2 = f_{yy}$

Example

let $z = y\,x^2 + \ln y + y\,e^x$, find $\partial z/\partial x$, $\partial z/\partial y$ at the point (0, 1)

$\partial z/\partial x = f_x = 2yx + 0 + y\,e^x$ → $f_x\,(0,1) = 1.0$

$\partial z/\partial y = f_y = x^2 + 1/y + e^x$ → $f_y\,(0,1) = 2.0$

Note y is held constant while differentiating with respect to x, and vice versa — then substitute the point (0, 1).

EXAMPLE

Second-Order Partial Derivatives

Ex. find $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$ for $z = y \sin x + x \cos y$

Solution :

$f_x = \partial z/\partial x = y \cos x + \cos y \quad \longrightarrow \quad f_{xx} = \partial^2 z/\partial x^2 = - y \sin x$

$f_y = \partial z/\partial y = \sin x - x \sin y \quad \longrightarrow \quad f_{yy} = \partial^2 z/\partial y^2 = - x \cos y$

Results $\partial^2 z/\partial x^2 = - y \sin x$ $\partial^2 z/\partial y^2 = - x \cos y$

EXAMPLE — 3 VARIABLES

Function of Three Variables

Ex. let $w = z\,e^{x} + \ln y + y^{2}\,x$
find all first & second order partials at the point (0, 1, 1)

1st ORDER

$\partial w/\partial x = f_x = z\,e^{x} + y^{2}$

$f_x = 2.0$

$\partial w/\partial y = f_y = 1/y + 2yx$

$f_y = 1.0$

$\partial w/\partial z = f_z = e^{x}$

$f_z = 1.0$

2nd ORDER

$\partial^2 w/\partial x^2 = f_{xx} = z\,e^{x}$

$f_{xx} = 1.0$

$\partial^2 w/\partial y^2 = f_{yy} = -1/y^2 + 2x$

$f_{yy} = -1.0$

$\partial^2 w/\partial z^2 = f_{zz} = 0$

$f_{zz} = 0$

Point of evaluation: (x, y, z) = (0, 1, 1)

MIXED DERIVATIVES

Mixed (Cross) Partial Derivatives

الاشتقاق الجزئي المتقاطع

Let z = f(x, y)

$$\frac{\partial^2 z}{\partial x \partial y} = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

In general (for m + n order derivatives):

$$\frac{\partial^{m+n} f}{\partial y^n \partial x^m} = \frac{\partial^{m+n} f}{\partial x^m \partial y^n} \qquad \text{e.g.} \quad \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial y^2 \partial x}$$

EXAMPLE

Verifying Mixed Partial Derivatives

Ex. find $\partial^2 f/\partial x \partial y$, $\partial^2 f/\partial y \partial x$ for $z = y\,x^2 + x\,y^2 + x^2$

Solution :

$$\partial^2 f/\partial x \partial y = f_{yx} = \partial/\partial x(\partial f/\partial y) = \partial/\partial x(x^2 + 2xy) = 2x + 2y$$

$$\partial^2 f/\partial y \partial x = f_{xy} = \partial/\partial y(\partial f/\partial x) = \partial/\partial y(2xy + y^2 + 2x) = 2x + 2y$$

$$f_{yx} = f_{xy} = 2x + 2y \quad \checkmark$$

Equality confirmed — consistent with Clairaut's rule on mixed partials.

EXAMPLE

Application Example — Voltage Divider

A voltage divider has R1 and R2 in series with a constant supply Vin.
$V_{out} = V_{in} \cdot R2 / (R1 + R2)$, find $\partial V_{out}/\partial R1$ and $\partial V_{out}/\partial R2$.

Solution :

$\partial V_{out}/\partial R1$: $u = V_{in} R2$ (const.), $v = R1+R2 \rightarrow u'=0$, $v'=1$

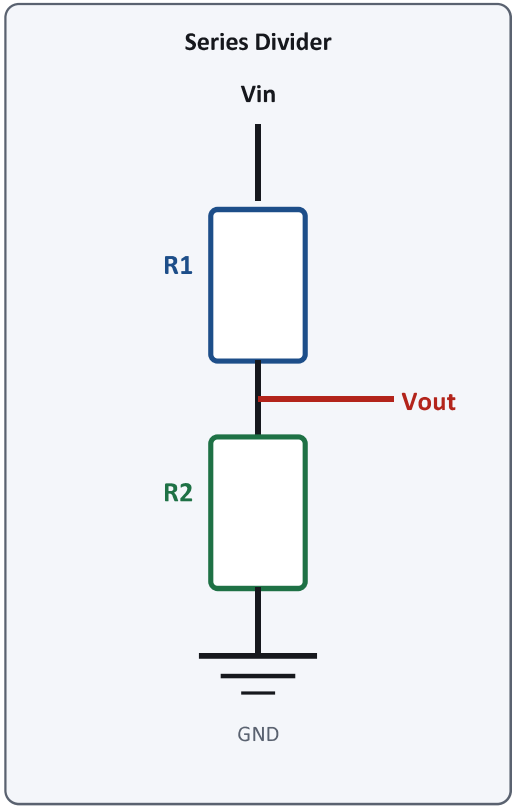
$$\partial V_{out}/\partial R1 = (u'v - uv')/v^2 = -V_{in} R2 / (R1+R2)^2$$

$$\boxed{\partial V_{out}/\partial R1 = - V_{in} R2 / (R1+R2)^2}$$

$\partial V_{out}/\partial R2$: $u = V_{in} R2$, $v = R1+R2 \rightarrow u'=V_{in}$, $v'=1$

$$\partial V_{out}/\partial R2 = (u'v - uv')/v^2 = V_{in} R1 / (R1+R2)^2$$

$$\boxed{\partial V_{out}/\partial R2 = V_{in} R1 / (R1+R2)^2}$$



EXAMPLE

Voltage Divider — Second-Order Derivative

Find $\partial^2 \text{Vout}/\partial \text{R2}^2$ by differentiating $\partial \text{Vout}/\partial \text{R2} = \text{Vin R1}/(\text{R1}+\text{R2})^2$ again, w.r.t. R2

Solution :

$u = \text{Vin R1 (const.)} \rightarrow u' = 0$

$v = (R1+R2)^2 \rightarrow v' = 2(R1+R2)$ (chain: same idea as $d/dx(x^2) = 2x$)

$\partial^2 \text{Vout}/\partial \text{R2}^2 = (u'v - uv')/v^2 = -\text{Vin R1} \cdot 2(R1+R2)/(R1+R2)^4$

$$\partial^2 \text{Vout}/\partial \text{R2}^2 = -2\,\text{Vin R1}/(R1+R2)^3$$

Evaluated at R1 = 100 Ω , R2 = 200 Ω , Vin = 12 V

$\partial \text{Vout}/\partial \text{R1}$	$\partial \text{Vout}/\partial \text{R2}$	$\partial^2 \text{Vout}/\partial \text{R2}^2$
-0.0267 V/Ω	0.0133 V/Ω	-8.9×10^{-5} V/Ω²

Physical meaning: Vout falls as R1 grows (negative sign) and rises as R2 grows (positive sign); the negative second derivative shows Vout's sensitivity to R2 decreases as R2 gets larger (the curve bends downward).

H.W. HOMEWORK

Homework — Equivalent Resistance

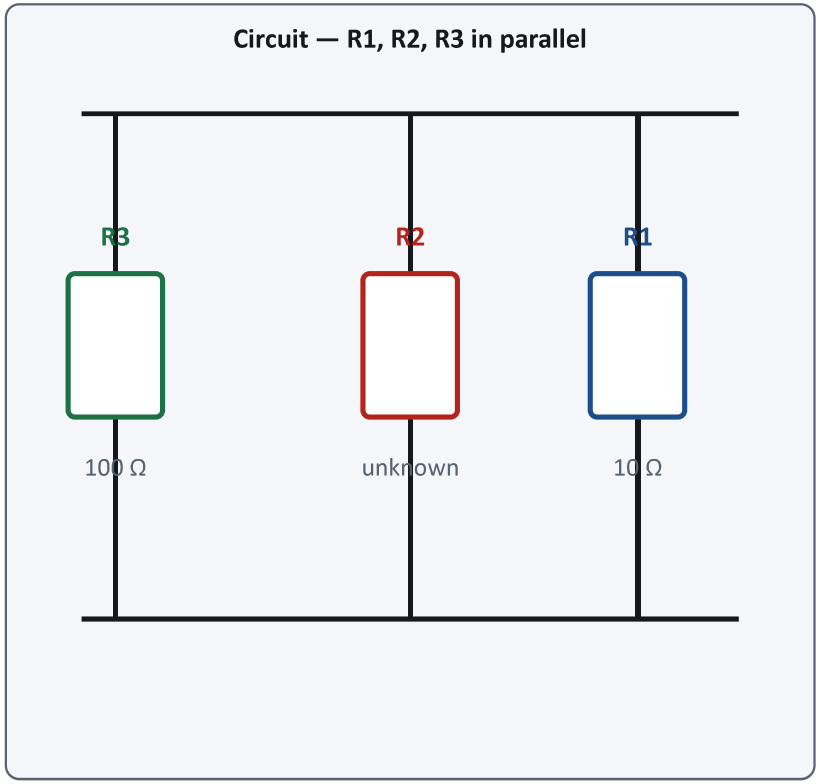
Problem

Find $\partial Req / \partial R2$, when three resistors R1, R2, R3 are connected in parallel.

$$\frac{1}{Req} = \frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3}$$

Given values:

R1	10 Ω
R3	100 Ω
R2	find as variable



MATHEMATICS II

Maximum , Minimum & Saddle Points of the Function

القيم العظمى والصغرى ونقطة السرج

Lecturer	م.د .هدى عقيل الطيار
Department	قسم هندسة الاتصالات والأنظمة الرقمية الذكية
Stage	المرحلة الثانية
Academic Year	2026 - 2027

01 CONCEPT

Maxima, Minima and Saddle Points

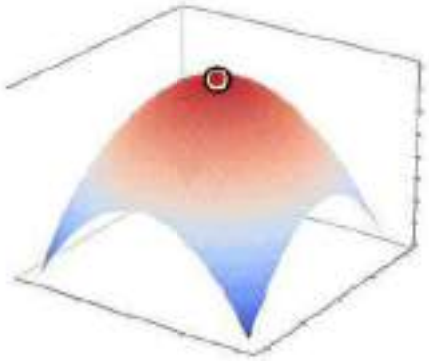
القيم العظمى، القيم الصغرى، ونقطة السرج لدوال متغيرين

One Variable — let y = f(x)

$$\frac{dy}{dx} = 0 \quad \text{at a maximum or minimum point}$$

Two Variables — let z = f(x, y)

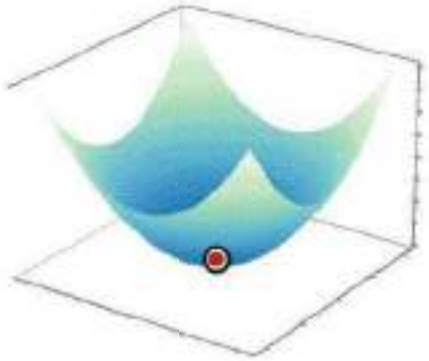
$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$



Local Maximum

قيمة عظمى

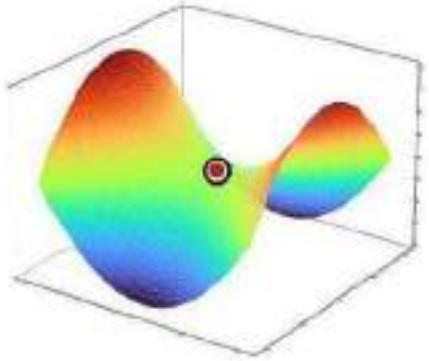
$f_{xx} < 0 \ , \ D > 0$



Local Minimum

قيمة صغرى

$f_{xx} > 0 \ , \ D > 0$



Saddle Point

نقطة السرج

$D < 0$

02 PROCEDURE

How to Find Maximum and Minimum Points

of z = f(x, y)

1 Differentiate z = f(x, y) w.r.t. x and y, obtain ∂f/∂x, ∂f/∂y, and equate them to zero.

$$\frac{\partial f}{\partial x} = 0 \quad ① \qquad \frac{\partial f}{\partial y} = 0 \quad ②$$

2 Solve equations ① and ② together to obtain all the critical points (a, b).

3 Use the Second-Order Partial Derivative Test at each critical point (a, b):

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

$f_{xx} < 0$ and $D > 0$ → **Maximum Point**	$f_{xx} > 0$ and $D > 0$ → **Minimum Point**
$D < 0$ → **Saddle Point**	$D = 0$ → **Test Failed (inconclusive)**

EXAMPLE 1

Finding the Extreme Values of a Function

Ex. Find the extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2y + 4$

STEP 1 & 2 — Critical Point

$f_x = \partial f/\partial x = y - 2x - 2 = 0$ ①

$f_y = \partial f/\partial y = x - 2y - 2 = 0$ ②

Solve ① and ② together:

$y = -2$, $x = -2$

$(a, b) = (-2, -2)$

STEP 3 — Second-Order Test

$f_{xx} = -2$ $f_{yy} = -2$

$f_{xy} = \partial/\partial y(\partial f/\partial x) = 1$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-2)(-2) - 1 = 3$

$f_{xx} < 0$, $D > 0$

∴ The point (−2, −2) is a MAXIMUM point for the given function.

EXAMPLE 2

Finding the Extreme Values of a Function

Ex. Find the extreme values of the function $f(x, y) = xy$

STEP 1 & 2 — Critical Point

$f_x = \partial f/\partial x = y = 0$

$f_y = \partial f/\partial y = x = 0$

$\therefore \quad x = 0 \ , \ y = 0$

$(a, b) = (0, 0)$

STEP 3 — Second-Order Test

$f_{xx} = \partial/\partial x(y) = 0 \qquad f_{yy} = \partial/\partial y(x) = 0$

$f_{xy} = \partial/\partial y(\partial f/\partial x) = \partial/\partial y(y) = 1$

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (0)(0) - 1 = -1$

$D < 0$

$\therefore$ The point (0, 0) is a SADDLE POINT for the given function.

EXAMPLE 3

Classifying Multiple Critical Points

Ex. Find and classify all the critical points of the function $f(x,y) = x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y = 0 \quad \Longrightarrow \quad y = x^2$

$f_y = 3y^2 - 3x = 0 \quad \Longrightarrow \quad x = y^2$

Substitute: $x = (x^2)^2 = x^4 \quad \Longrightarrow \quad x^4 - x = 0 \quad \Longrightarrow \quad x(x^3-1) = 0 \quad \Longrightarrow \quad x = 0 \text{ or } x = 1$

Critical points: (0, 0) and (1, 1)

$f_{xx} = 6x \;,\quad f_{yy} = 6y \;,\quad f_{xy} = -3 \quad \rightarrow \quad D = f_{xx} \cdot f_{yy} - f_{xy}^2$

Point	fxx	fyy	D	fxx sign	Conclusion
(0, 0)	0	0	−9	−	Saddle Point
(1, 1)	6	6	27	+	Minimum

f(1,1) = 1 + 1 − 3 = −1 (the local minimum value)

EXAMPLE 4

Applied Example — Minimum-Material Box

Ex. **An *open-top* rectangular box must hold a volume of 32 m^3. Find the base dimensions x, y that minimize the amount of material (surface area) used.**



Setup & Critical Point

$V = xyh = 32 \implies h = 32/(xy)$

$S(x,y) = xy + 2xh + 2yh = xy + 64/x + 64/y$

$S_x = y - 64/x^2 = 0 \quad , \quad S_y = x - 64/y^2 = 0$

Solving: $x = y = 4$ (then $h = 32/16 = 2$)

Second-Order Test

$S_{xx} = 128/x^3 \ , \ S_{yy} = 128/y^3 \ , \ S_{xy} = 1$

at (4,4): $S_{xx} = 2 \ , \ S_{yy} = 2 \ , \ S_{xy} = 1$

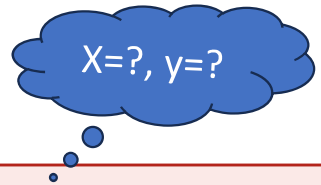
$D = S_{xx} \cdot S_{yy} - (S_{xy})^2 = 4 - 1 = 3 > 0$

$S_{xx} = 2 > 0 \implies$ Minimum

Minimum material at $x = y = 4$ m , $h = 2$ m → minimum surface area $S = 48$ m^2

EXAMPLE 5

Applied Example — Maximizing Profit



Ex. A firm's profit (in thousands of $) from producing x units of product A and y units of product B is $P(x,y) = 50x - x^2 - xy - 2y^2 + 60y$. Find the production levels that maximize profit.

Critical Point

$P_x = 50 - 2x - y = 0 \quad \Longrightarrow \quad 2x + y = 50$

$P_y = 60 - x - 4y = 0 \quad \Longrightarrow \quad x + 4y = 60$

Solving the system:

$x = 20 , \quad y = 10$

Second-Order Test

$P_{xx} = -2 , \; P_{yy} = -4 , \; P_{xy} = -1$

$D = P_{xx} \cdot P_{yy} - P_{xy}^2 = 8 - 1 = 7 > 0$

$P_{xx} = -2 < 0 \quad \Longrightarrow \quad$ Maximum

Produce 20 units of A and 10 units of B → maximum profit
P(20,10) = 800 (thousand $)

EXAMPLE 6

Applied Example — Closest Point on a Plane

Ex. **Find the point on the plane x + y + z = 3 that is closest to the origin.**

Minimize the distance between 2 points (x,y,z) and (0,0,0): $d = \sqrt{x^2 + y^2 + z^2}$

$d^2 = x^2 + y^2 + z^2$ but, the plane is: $x + y + z = 3$ ⟹ $z = 3 - x - y$

$f(x,y) = x^2 + y^2 + (3-x-y)^2$ ... To have function of 2 variables only

$f_x = 4x + 2y - 6 = 0$, $f_y = 2x + 4y - 6 = 0$

Solving the system: $x = 1$, $y = 1$ ⟹ $z = 3 - 1 - 1 = 1$

$f_{xx} = 4$, $f_{yy} = 4$, $f_{xy} = 2$ → $D = 16 - 4 = 12 > 0$, $f_{xx} = 4 > 0$ ⟹ Minimum

Closest point: (1, 1, 1) → minimum distance = √(1+1+1) = √3

H.W

Find and classify all critical points of:

f(x,y)=$x^2 - 2xy + 2y^2 + 4x - 6y$

For what value of a does the function $f(x, y) = x^2 + y^2 + axy$ have a local minimum at the origin?

Find the point on the plane $2x + y + 2z = 9$ that is closest to the origin.

Find the point on the plane $x + y + z = 9$ that is closest to the point P(1,2,3)

Thank You



MATHEMATICS II

Vectors

المتجهات

Lecturer	د. هدى عقيل الطيار
Department	هندسة الاتصالات والأنظمة الرقمية الذكية
Class	المرحلة الثانية
University	كلية الهندسة/جامعة الموصل

01 FORMATION

Formation of Vectors — the xy-Plane

1. Scalar Functions
2. Vector Functions

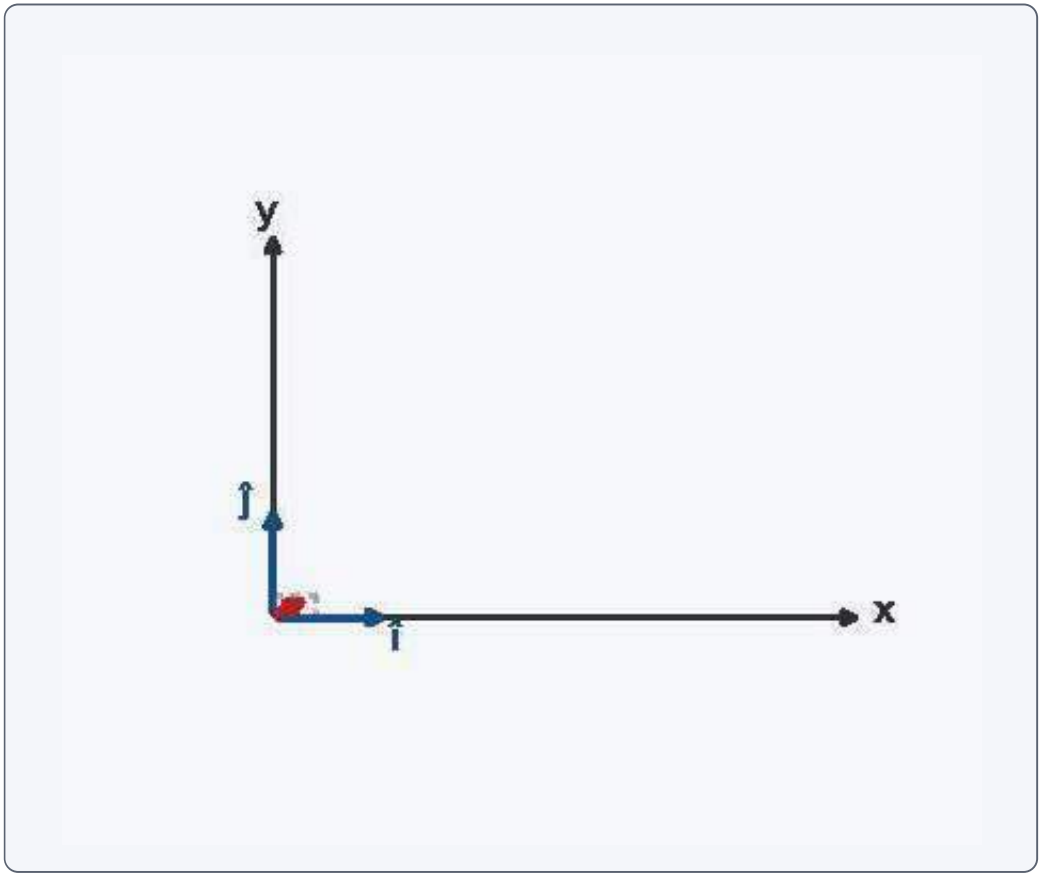
Vector in the xy-Plane

$$V = a\,\hat{i} + b\,\hat{j}$$

$\hat{i}, \hat{j}$ are unit vectors along the x and y axes

$\hat{i} = (1, 0)$ $\hat{j} = (0, 1)$

a is the component along x
b is the component along y

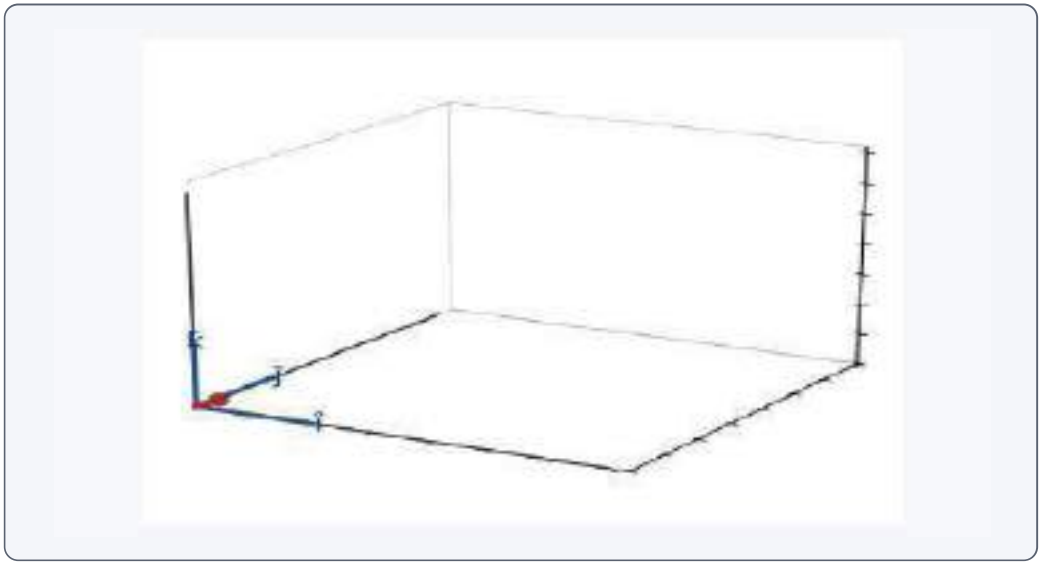


01 FORMATION

Vectors in Space & Between Two Points

$$V = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$$

$\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors (x, y, z axes)



Vector Between Two Points

A(x_1, y_1, z_1) and B(x_2, y_2, z_2)

$$\overrightarrow{AB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$

(2D case: $\overrightarrow{AB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j}$)

EXAMPLE 1

Vector Between Two Points

Ex. A = (2, 4, −1) , B = (−3, 0, 5) , find $\vec{AB}$

$$\vec{AB} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$

$$\overrightarrow{AB} = -5\,\hat{i} - 4\,\hat{j} + 6\,\hat{k}$$

Magnitude and Direction of a Vector (2D)

Magnitude (length) |V|

$$V = a\,\hat{i} + b\,\hat{j} \rightarrow |V| = \sqrt{(a^2 + b^2)}$$

Direction θ

$$\theta = \tan^{-1}(Vy / Vx) = \tan^{-1}(b/a)$$

Magnitude & Direction Cosines in Space

In space: $V = V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}$

$$|V| = \sqrt{(V_1^2 + V_2^2 + V_3^2)}$$

V_1 = component of V along the positive x-axis

V_2 = component of V along the positive y-axis

V_3 = component of V along the positive z-axis

α, β, γ are the angles between V and the x, y, z axes respectively

$$V_1 = |V| \cos\alpha \quad , \quad V_2 = |V| \cos\beta \quad , \quad V_3 = |V| \cos\gamma$$

$$\cos\alpha = V_1 / |V| \;,\; \cos\beta = V_2 / |V| \;,\; \cos\gamma = V_3 / |V|$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

(Direction Cosine Law)

03 UNIT VECTOR

The Unit Vector

$$\hat{u} = V / |V|$$

û : unit vector V : vector |V| : length (magnitude) of the vector

Continuing Example 1: A = (2, 4, −1) , B = (−3, 0, 5)

$$\vec{AB} = -5\,\hat{i} - 4\,\hat{j} + 6\,\hat{k}$$

$$|AB| = \sqrt{25 + 16 + 36} = \sqrt{77}$$

$$\hat{u}_{AB} = \vec{AB} / |AB| = (-5\,\hat{i} - 4\,\hat{j} + 6\,\hat{k}) / \sqrt{77}$$

04 OPERATIONS

Addition, Subtraction & Scalar Multiplication

Let $A = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $B = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$

Addition

$$A+B = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$$

Subtraction

$$A-B = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$$

Multiplication of a Vector by a Scalar c

let $V = V_1\,\hat{i} + V_2\,\hat{j} + V_3\,\hat{k}$, c is a scalar value

$$c\,V = c\,V_1\,\hat{i} + c\,V_2\,\hat{j} + c\,V_3\,\hat{k}$$

$$|\,c\,V\,| = c\,\sqrt{(V_1^2 + V_2^2 + V_3^2)}$$

05 DOT PRODUCT

The Dot Product of Two Vectors

(Scalar Product)

$$A \cdot B = |A|\,|B| \cos\theta = B \cdot A \qquad (\theta : \text{angle between A and B})$$

Properties

$A \cdot B = B \cdot A$

$A \cdot (B+C) = A \cdot B + A \cdot C$

$A \cdot B = 0 \quad \Rightarrow \quad A \perp B$

$A \cdot A = |A|^2 \quad , \quad A \cdot B > 0 \Leftrightarrow \theta < \pi/2$



Unit-Vector Dot Products

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \qquad (\text{angle } 0°)$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \qquad (\text{angle } 90°)$

$$A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3$$