CHAPTER 2
LIMITS

Limits

If f is the identity function $f(x) = x$, than for any value of c

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

$$\lim_{x \to 3} x = 3$$

If f is constant function $f(x) = k$, (function with the constant value k), then for any value of c.

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

$$\lim_{x \to 7} (4) = 4$$

$$\lim_{x \to c} \left(f(x).\, g(x)\right) = L.\, M$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \qquad , \qquad M \neq 0$$

$$\lim_{x \to c} [f(x)]^n = L^n$$

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$$

The Limit Laws

If L, M, c and k are real numbers and

$\lim\limits_{x\to c} f(x) = L$ **and** $\lim\limits_{x\to c} g(x) = M$

$$\lim_{x\to c}\big(f(x) + g(x)\big) = L + M$$

$$\lim_{x\to c}\big(f(x) - g(x)\big) = L - M$$

$$\lim_{x\to c}\big(k.f(x)\big) = k.L$$

Example

Find

$$\lim_{x\to c}\left(x^3 + 4x^2 - 3\right)$$

$$= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 \quad = \quad c^3 + 4c^2 - 3$$

$$\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$$

$$= \frac{\lim\limits_{x\to c}(x^4+x^2-1)}{\lim\limits_{x\to c}(x^2+5)} = \frac{\lim\limits_{x\to c} x^4+\lim\limits_{x\to c} x^2-\lim\limits_{x\to c} 1}{\lim\limits_{x\to c} x^2+\lim\limits_{x\to c} 5} = \frac{c^4+c^2-1}{c^2+5}$$

$$\lim_{x \to -2} \sqrt{4x^2 - 3}$$

$$= \sqrt{\lim_{x \to -2} (4x^2 - 3)} = \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$$

$$= \sqrt{4(-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$$

Example

Find

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \lim_{x \to -1} \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{x\,\cancel{(x-1)}} = \lim_{x \to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$$

$$=\lim_{x\to 0}\frac{\sqrt{x^2+100}\ -10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}$$

$$=\lim_{x\to 0}\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}=\lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}$$

$$=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}=\frac{1}{\sqrt{(0)^2+100}+10}=\frac{1}{10+10}=\frac{1}{20}$$

One Side Limits

$$\boxed{\lim_{x\to 0}\frac{\sin\theta}{\theta}=1}$$

Example

Show that

$$\lim_{h\to 0}\frac{\cos h\ -1}{h}=0$$

$$\cos h=1-2\sin^2\left(\frac{h}{2}\right)$$

$$\lim_{h\to 0}\frac{\cos h\,-1}{h}=\lim_{h\to 0}\frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h}$$

$$=\lim_{h\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{h}\quad,\qquad let\ \theta=\frac{h}{2}\quad\rightarrow\quad h=2\theta$$

$$=\lim_{h\to 0}\frac{-\cancel{2}\sin^2\theta}{\cancel{2}\theta}=-\lim_{h\to 0}\frac{\sin\theta}{\theta}\,.\sin\theta$$

$$=-(1)(0)=0$$

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\frac{2}{5}$$

$$\lim_{x\to 0}\frac{\sin 2x}{5x}=\lim_{x\to 0}\frac{\cancel{\left(\frac{2}{5}\right)}(\sin 2x)}{\cancel{\left(\frac{2}{5}\right)}(5x)}=\frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}\qquad,\ Let\ \theta=2x$$

$$=\frac{2}{5}\lim_{x\to 0}\frac{\sin\theta}{\theta}=\frac{2}{5}(1)=\frac{2}{5}$$

Example

Find $\lim\limits_{t\to 0} \dfrac{\tan t \sec 2t}{3t}$

$$\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} = \lim_{t\to 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3} \lim_{t\to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t}$$

$$= \frac{1}{3} (1)(1)(1) = \frac{1}{3}$$

Limits Involving Infinity

Example

Find

$$\lim_{x\to \infty} \left(5 + \frac{1}{x}\right) = \lim_{x\to \infty} 5 + \lim_{x\to \infty} \frac{1}{x} = 5 + 0 = 5$$

$$\lim_{x\to -\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to -\infty} \pi \cdot \sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x}$$

$$= \lim_{x\to -\infty} \pi\sqrt{3} \cdot \lim_{x\to -\infty} \frac{1}{x} \cdot \lim_{x\to -\infty} \frac{1}{x} = \pi\sqrt{3} \cdot (0) \cdot (0) = 0$$

Example

Find $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+\frac{8}{x}-\frac{3}{x^2}}{3+\frac{2}{x^2}}$$

$$= \frac{5+0+0}{3+0} = \frac{5}{3}$$

Example

Find $\lim_{x\to-\infty} \frac{11x+2}{3x^3-1}$

$$\lim_{x\to-\infty} \frac{11x+2}{3x^3-1} = \lim_{x\to-\infty} \frac{\frac{11}{x^2}+\frac{2}{x^3}}{2-\frac{1}{x^3}}$$

$$= \frac{0+0}{2-0} = 0$$

Example

Find $\lim\limits_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right)$

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right) = \lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right) \cdot \frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \to \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \to \infty} \frac{\cancel{x^2} - \cancel{x^2} - 16}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \to \infty} \frac{-16}{x + \sqrt{x^2 + 16}}$$

$$= \lim_{x \to \infty} \frac{-\frac{16}{x}}{\frac{x}{x} + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}} = \lim_{x \to \infty} \frac{-\frac{16}{x}}{1 + \sqrt{1 + \frac{16}{x^2}}}$$

$$= \frac{0}{1 + \sqrt{1 + 0}} = \frac{0}{2} = 0$$

Example

$$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)^{\cancel{2}}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2}\frac{x-2}{x+2} = \frac{2-2}{2+2} = 0$$

$$\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2}\frac{1}{x+2} = \frac{1}{2+2} = \frac{1}{4}$$

Example

Find $\lim\limits_{x\to 2}\frac{x^3-8}{x^2-4}$

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{\cancel{(x-2)}\left(x^2+2x+4\right)}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2}\frac{x^2+2x+4}{x+2}$$

$$= \frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4} = 3$$