CHAPTER 3
DERIVATIVES

Differentiation Rules

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

$$\frac{d}{dx}(cx^n) = cnx^{n-1}$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$

Example

Find $\frac{dy}{dx}$

$y = x^4 + 12x$, $\frac{dy}{dx} = 4x^3 + 12$

$y = \frac{7x^2}{3} - 5$, $\frac{dy}{dx} = \frac{14}{3}x$

$y = x^3 + 3x^2 - 5x + 1$

$$\frac{dy}{dx} = 3x^2 + 6x - 5$$

Second and Higher Order Derivatives

$$\bar{y} = \frac{dy}{dx}$$

$$\bar{\bar{y}} = \frac{d\bar{y}}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

$$\bar{y}^{(n)} = \frac{d}{dx}y^{(n-1)}$$

Example

Find the four derivatives of $y = x^3 - 3x^2 + 2$

$$y = x^3 - 3x^2 + 2$$

$$\bar{y} = 3x^2 - 6x$$

$$\bar{\bar{y}} = 6x - 6$$

$$\bar{\bar{\bar{y}}} = 6$$

$$\bar{\bar{\bar{\bar{y}}}} = 0$$

$$\frac{d}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

Find the derivative of $y = (x^2+1)(x^3+3)$

$u = x^2 + 1$, $v = x^3 + 3$

$$\frac{d}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = (x^2+1)(3x^2) + (x^3+3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x$$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Example

Find the derivative of $y=\frac{x^2-1}{x^2+1}$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}=\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$$

$$=\frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}=\frac{4x}{(x^2+1)^2}$$

Example

Find the derivative of

a) $y=\frac{1}{x}$

$y=x^{-1}$, $\frac{dy}{dx}=-1x^{-1-1}=-x^{-2}=\frac{-1}{x^2}$

b) $y=x+\frac{2}{x}$

$y=x+2x^{-1}$, $\frac{dy}{dx}=1-2x^{-2}=1-\frac{2}{x^2}$

c) $y=\frac{4}{x^3}$

$y=4x^{-3}$, $\frac{dy}{dx}=-12x^{-4}=\frac{-12}{x^4}$

Example

Find an equation for the tangent to the curve $y=\sqrt{x}$, at $x=4$

$$y=\sqrt{x}=x^{\frac{1}{2}}$$

$$\bar{y}=\frac{1}{2}x^{\frac{-1}{2}}=\frac{1}{2\sqrt{x}}$$

at $x=4$, $y=\sqrt{x}=\sqrt{4}=2$

The tangent is the line through the point (4,2)

$$Slope = m = \bar{y} = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x - 1 + 2 \quad , \quad y = \frac{1}{4}x + 1$$

Example

Find an equation for the tangent to the curve $y = x + \frac{2}{x}$, at the point $(1, 3)$

$$\frac{dy}{dx} = 1 + 2\left(\frac{-1}{x^2}\right) = 1 - \frac{2}{x^2}$$

The slope at $x = 1$

$$m = \frac{dy}{dx} = 1 - \frac{2}{(1)^2} = 1 - 2 = -1$$

$$y - y_1 = m(x - x_1) \quad , \quad y - 3 = -1(x - 1)$$

$$y = -x + 1 + 3 \quad , \quad y = -x + 4$$

Example

Find the derivative of the functions

$$a)\quad y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$$

$$\bar{y} = (x^2 + 1)(1 - x^{-1}) + \left(x + 5 + \frac{1}{x}\right)(2x)$$

$$= x^2 - 1 + 1 - x^{-2} + 2x^2 + 10x + 2$$

$$= 3x^2 + 10x + 2 - \frac{1}{x^2}$$

$$b)\quad r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$$

$$\bar{r} = 2\left[\frac{\sqrt{\theta}(0) - 1\left(\frac{1}{2}\theta^{\frac{-1}{2}}\right)}{\theta} + \frac{1}{2}\theta^{\frac{-1}{2}}\right] + \left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)(0)$$

$$= 2\left[\frac{\frac{-1}{2\theta^{\frac{1}{2}}}}{\theta} + \frac{1}{2\sqrt{\theta}}\right] = 2\left[\frac{-1}{2\theta^{\frac{3}{2}}} + \frac{1}{2\sqrt{\theta}}\right]$$

$$= \frac{-1}{\theta^{\frac{3}{2}}} + \frac{1}{\sqrt{\theta}}$$

$$c)\quad y=\frac{1}{(x^2-1)(x^2+x+1)}$$

$$=\frac{[(x^2-1)(x^2+x+1)](0)-1[(x^2-1)(2x+1)+(x^2+x+1)(2x)]}{[(x^2-1)(x^2+x+1)]^2}$$

$$=\frac{-[2x^3+x^2-2x-1+2x^3+2x^2+2x]}{(x^2-1)^2(x^2+x+1)^2}$$

$$=\frac{-4x^3-3x^2+1}{(x^2-1)^2(x^2+x+1)^2}$$

Velocity and other Rate of Change

The average velocity $V_{av}=\frac{displacement}{travel\ time}=\frac{ds}{dt}$

Acceleration

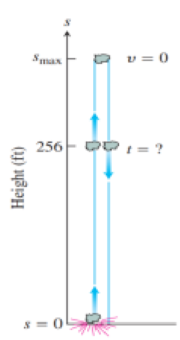
$$a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$$

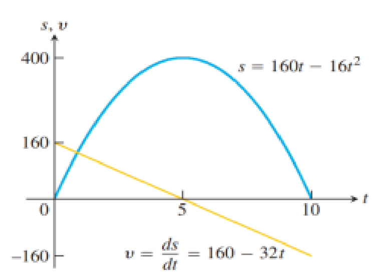
Example

A dynamic blast blows a heavy rock straight up with a velocity of 160 *ft/sec*. It reaches a height of *S=160t-16t2*, ft after *t* sec.

•How high dose the rock go?

•How fast is the rock traveling when it is 256 ft above the ground on the way up?





a) $V = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2)$

$= 160 - 32t$

$V = 0 \ \ when \ \ 160 - 32t = 0 \ \ , \qquad t = 5 \ \ sec$

The rock height at $t = 5 \ \ sec$, is

$S_{max} = S(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \ ft$

To find the rock velocity at 256 ft on the way up and again on the way down

$S(t) = 160t - 16t^2$

$256 = 160t - 16t^2$

$16t^2 - 160t + 256 = 0 \ \ , \ \ t^2 - 10t + 16 = 0$

$(t - 2)(t - 8) = 0$

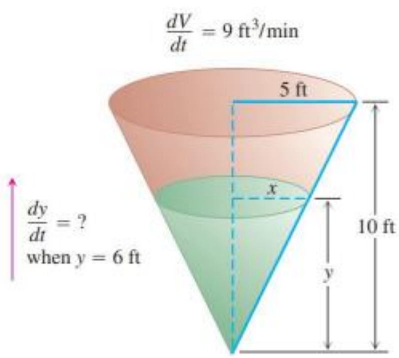
$t = 2 \ \ and \ \ t = 8$

$V(2) = 160 - 32(2) = 160 - 64 = 96 \ \ ft$

$V(8) = 160 - 32(8) = 160 - 256 = -96 \ \ ft$

Example

Water runs into a conical tank at the rate of *9 ft3/min.* The tank stands point down and has height of 10 *ft* and a base radius of 5*ft* . How fast in the water level rising when the water is 6 *ft* deep.



The variables in the problem are:

V=volume of water in the tank (ft^3) at time t (min.)

x=radius of the surface of water (ft) at time t

y=depth of the water in the tank (ft) at time t

Assume the V, x and y are differentiable function of t

We find $\frac{dy}{dt}$ when $y = 6\ ft\ and\ \frac{dV}{dt} = 9\ ft^3/min$.

$$V = \frac{1}{3}\pi x^2 y$$

$$\frac{x}{y} = \frac{5}{10} \quad , \qquad \frac{x}{y} = \frac{1}{2} \quad , \qquad x = \frac{y}{2}$$

$$V = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12} y^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}(3)y^2 \frac{dy}{dt} = \frac{\pi}{4} y^2 \frac{dy}{dt}$$

$$y = 6\ ft \ \ and \ \frac{dV}{dt} = 9\ ft^3/min.$$

$$9 = \frac{\pi}{4}(6)^2 \frac{dy}{dt} \quad , \qquad \frac{dy}{dt} = \frac{1}{\pi} = 0.32\, ft/min.$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx} \sin x = \cos x \quad , \qquad \frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x \quad , \qquad \frac{d}{dx} \sec x = secx tanx$$

$$\frac{d}{dx} \cot x = -csc^2 x \quad , \qquad \frac{d}{dx} \csc x = -cscx cotx$$

Example

Find the derivative of

$y = x^2 \sin x$

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

$y = \sin x \cos x$

$$\frac{dy}{dx} = \sin x\,(-\sin x) + \cos x\,(\cos x)$$
$$= \cos x^2 - \sin x^2$$

Example

Find $\frac{dy}{dx}$ **if** $y = \tan x$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$
$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Example

Find $\bar{\bar{y}}$ if $y = secx$

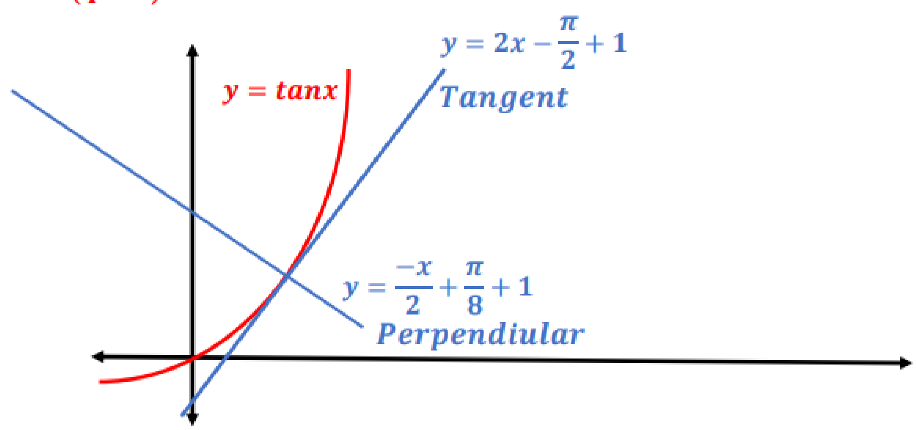
$$\bar{y} = secxtanx$$

$$\bar{\bar{y}} = secx\left(sec^2x\right) + tanx(secxtanx)$$

$$= sec^3x + secxtan^2x$$

Example

Find the lines that are tangent and perpendicular to the curve $y = tanx$, at the point $\left(\frac{\pi}{4}, 1\right)$



The slope of the curve at $\left(\frac{\pi}{4}, 1\right)$, is the value

$$\frac{dy}{dx} = sec^2 x \qquad at \quad x = \frac{\pi}{4}$$

$$m = \frac{dy}{dx} = sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2$$

The tangent is the line

$$y - 1 = 2\left(x - \frac{\pi}{4}\right) \qquad , \qquad y = 2x - \frac{\pi}{4} + 1$$

The perpendicular line, $m = \frac{-1}{2}$

$$y - 1 = \frac{-1}{2}\left(x - \frac{\pi}{4}\right) \qquad , \qquad y = \frac{-x}{2} + \frac{\pi}{8} + 1$$

Example

Find $\frac{ds}{dt}$ **for** $S = \frac{1+\csc t}{1-\csc t}$

$$\frac{ds}{dt} = \frac{(1-\csc t)(-\csc t\cot t) - (1+\csc t)(\csc t\cot t)}{(1-\csc t)^2}$$

$$= \frac{-\csc t\cot t + \csc^2 t\cot t - \csc t\cot t - \csc^2 t\cot t}{(1-\csc t)^2}$$

$$= \frac{-2\csc t\cot t}{(1-\csc t)^2}$$

Example

Find the derivative of $r = \left(\frac{\sin\theta}{\cos\theta - 1}\right)^2$

$$\frac{dr}{d\theta} = 2\left(\frac{\sin\theta}{\cos\theta - 1}\right)\left[\frac{(\cos\theta - 1)(\cos\theta) - \sin\theta\,(-\sin\theta)}{(\cos\theta - 1)^2}\right]$$

$$= 2\left(\frac{\sin\theta}{\cos\theta - 1}\right)\left[\frac{\cos^2\theta - \cos\theta + \sin^2\theta}{(\cos\theta - 1)^2}\right]$$

$$= 2\left(\frac{\sin\theta}{\cos\theta - 1}\right)\left[\frac{1 - \cos\theta}{(\cos\theta - 1)^2}\right]$$

$$= 2\left(\frac{\sin\theta}{\cos\theta - 1}\right)\left(\frac{-(\cos\theta - 1)}{(\cos\theta - 1)^2}\right) = \frac{-2\sin\theta}{(\cos\theta - 1)^2}$$

Example

Find the derivative of $p = \frac{\sin q + \cos q}{\cos q}$

$$\frac{dp}{dq} = \frac{\cos q\,(\cos q - \sin q) - (\sin q + \cos q)(-\sin q)}{\cos^2 q}$$

$$= \frac{\cos^2 q - \cos q \sin q + \sin^2 q + \cos q \sin q}{\cos^2 q}$$

$$= \frac{1}{\cos^2 q} = \sec^2 q$$

Example

Find the derivative of $r = \sin\theta\left(\theta + \sqrt{\theta + 1}\right)$

$$\frac{dr}{d\theta} = \cos\left(\theta + \sqrt{\theta + 1}\right)\left(1 + \frac{1}{2}(\theta + 1)^{-\frac{1}{2}}\right)$$

$$= \cos\left(\theta + \sqrt{\theta + 1}\right)\left(1 + \frac{1}{2\sqrt{\theta + 1}}\right)$$

$$= \frac{2\sqrt{\theta + 1} + 1}{2\sqrt{\theta + 1}}\cos\left(\theta + \sqrt{\theta + 1}\right)$$

Example

Find the derivative of $y = 20(3x-4)^{\frac{1}{4}}(3x-4)^{\frac{-1}{5}}$

$$y = 20(3x-4)^{\frac{1}{4}}(3x-4)^{\frac{-1}{5}} = 20(3x-4)^{\frac{1}{20}}$$

$$\frac{dy}{dx} = 20\left(\frac{1}{20}\right)(3x-4)^{\frac{-19}{20}}(3)$$

$$= \frac{3}{(3x-4)^{\frac{19}{20}}}$$

Example

Find the derivative of $y = x^{-2}sin^2(x^3)$

$$\frac{dy}{dx} = x^{-2}\left(2sin(x^3)\right)\left(cos(x^3)\right)(3x^2) + sin^2(x^3)(-2x^{-3})$$

$$= 6sin(x^3)cos(x^3) - 2x^{-3}sin^2(x^3)$$

Example

Find the derivative of $y = 4x\sqrt{x+\sqrt{x}}$

$$y = 4x\sqrt{x+\sqrt{x}} = 4x\left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4x\left(\frac{1}{2}\right)\left(x+x^{\frac{1}{2}}\right)^{\frac{-1}{2}}\left(1+\frac{1}{2}x^{\frac{-1}{2}}\right) + \left(x+x^{\frac{1}{2}}\right)^{\frac{1}{2}}(4)$$

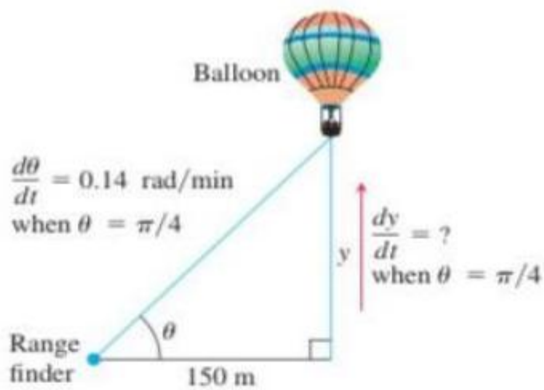
$$= \left(x + x^{\frac{1}{2}}\right)^{\frac{-1}{2}} \left[2x\left(1 + \frac{1}{2\sqrt{x}}\right) + 4\left(x + x^{\frac{1}{2}}\right)\right]$$

$$= \left(x + \sqrt{x}\right)^{\frac{-1}{2}} \left[2x + \sqrt{x} + 4x + 4\sqrt{x}\right]$$

$$= \frac{6x + 5\sqrt{x}}{\left(x + \sqrt{x}\right)^{\frac{1}{2}}} = \frac{6x + 5\sqrt{x}}{\sqrt{x + \sqrt{x}}}$$

Example

A hot air balloon rising straight up from level field is tracked by a range finder *150 m* from the life off point. At the moment the range finder's elevation angle $\frac{\pi}{4}$, the angle is increasing at the rate of *0.14 rad/min*. how fast is the balloon rising at that moment.



θ = the angle in radians the range finder with the ground.

y = the height in meters of the balloon above the ground.

$\frac{d\theta}{dt} = 0.14 \frac{rad}{min} \quad when \quad \theta = \frac{\pi}{4}$

$$\tan\theta = \frac{y}{150} \quad , \quad y = 150 \tan\theta$$

$$\frac{dy}{dt} = 150(\sec^2\theta)\frac{d\theta}{dt}$$

$$= 150\left(\sqrt{2}\right)^2(0.14) = 42 \ \frac{m}{min}$$

The balloon is rising at the rate of 42 m/min.

The Chain Rule

$$y = f[g(x)]$$

Let $u = g(x)$, $then \ y = f(x)$, and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example

Express $\frac{dy}{dx}$ in terms of x if $y = u^3$ and $u = x^2 - 1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = 3u^2 \quad , \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = (3u^2)(2x)$$

$$= 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2$$

Example

Find $\frac{dy}{dx}$ of $y = cosx^2$, Use Chain Rule

$$Let \;\; u = x^2 \quad , \qquad y = \cos u$$

$$\frac{du}{dx} = 2x \quad , \qquad \frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (-\sin u)(2x) = -2x \, sinx^2$$

Example

Use Chain Rule to find $\frac{dy}{dx}$ of $y = (2x - 5)^{10}$

$$Let\ u = 2x - 5 \quad , \quad y = u^{10}$$

$$\frac{du}{dx} = 2 \quad , \quad \frac{dy}{du} = 10u^9$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= (10u^9)(2) = 20(2x - 5)^9$$

Example

Use Chain Rule to find $\frac{dy}{dx}$ of $y = tan(x^2 + sinx)$

Let $u = x^2 + sinx \quad , \quad y = tanu$

$$\frac{du}{dx} = 2x + cosx \quad , \quad \frac{dy}{du} = sec^2u$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= (2x + cosx)(sec^2u)$$

$$= (2x + cosx)sec^2(x^2 + sinx)$$

Example

Use Chain Rule to find $\frac{dy}{dx}$ of $y = e^{x^3}$

Let $u = x^3$, $y = e^u$

$\frac{du}{dx} = 3x^2$, $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (e^u)(3x^2) = 3x^2 e^{x^3}$$

Example

Use Chain Rule to find $\frac{dy}{dx}$ of $y = e^{1+x^2}$

Let $u = 1 + x^2$, $y = e^u$

$\frac{du}{dx} = 2x$, $\frac{dy}{du} = e^u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (e^u)(2x) = 2xe^{1+x^2}$$

Example

Use Chain Rule to find $\frac{dy}{dx}$ of $y = ln(x + sinx)$

Let $u = x + sinx$, $y = lnu$

$$\frac{du}{dx} = 1 + cosx \quad , \quad \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= \left(\frac{1}{u}\right)(1 + cosx) = \frac{1 + cosx}{x + sinx}$$

Implicit Differentiation and Fractional Powers

Example

Find the slope of circle $x^2 + y^2 = 25$, at the point $(3, -4)$

$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} = \frac{-3}{-4} = \frac{3}{4}$$

Example

Find $\frac{dy}{dx}$ if $y^2 = x^2 + sinxy$

$$y^2 = x^2 + sinxy$$

$$2y\frac{dy}{dx} = 2x + cosxy\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} = 2x + ycosxy + xcosxy\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - xcosxy\frac{dy}{dx} = 2x + ycosxy$$

$$\frac{dy}{dx}(2y - xcosxy) = 2x + ycosxy$$

$$\frac{dy}{dx} = \frac{2x + ycosxy}{2y - xcosxy}$$

Example

Find $\frac{d^2y}{dx^2}$ **if** $2x^3 - 3y^2 = 8$

$$2x^3 - 3y^2 = 8$$

$$6x^2 - 6y\bar{y} = 0 \quad , \quad 6x^2 = 6y\bar{y} \quad , \quad \bar{y} = \frac{x^2}{y}$$

$$\bar{\bar{y}} = \frac{(y)(2x) - x^2\bar{y}}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\bar{y}$$

$$= \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}$$