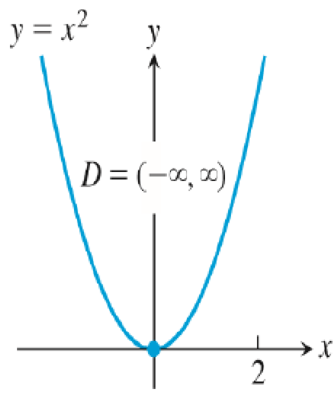
CHAPTER 4
APPLICATIONS OF DERIVATIVES

$y = x2$
Domain: (-∞,∞)
No absolute maximum

Absolute minimum of 0 $at\ x = 0$

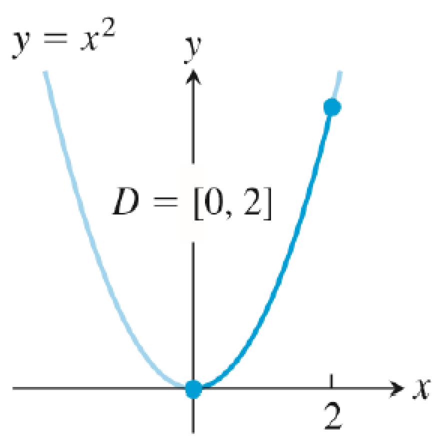


$y = x^2$

Domain: $[0, 2]$

Absolute maximum of 4 $at\ x = 2$

Absolute minimum of 0 $at\ x = 0$



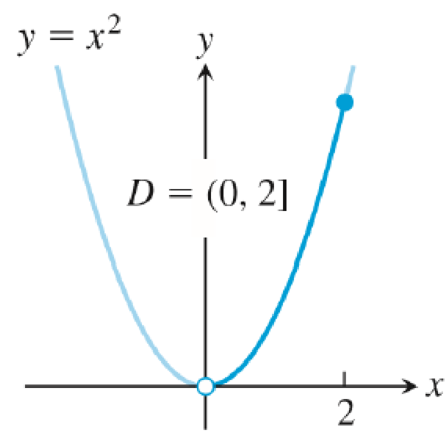
$y = x^2$

Domain: (0,2]

Absolute maximum

of 4 $at\ x = 2$

No absolute minimum



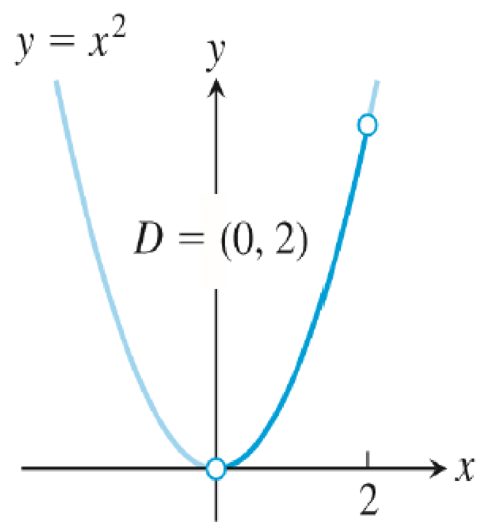
$y = x^2$

Domain: $(0, 2)$

No absolute extrema

(no maximum and no

minimum)



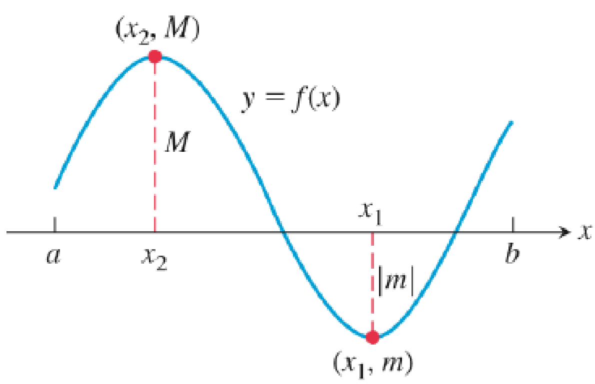
For the function on a closed interval $[a, b]$

Maximum = M = اعلى قيمة

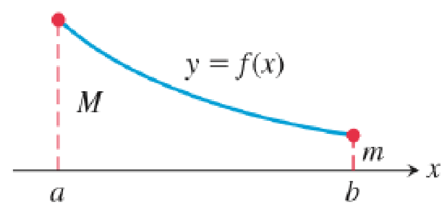
Minimum = m = اقل قيمة

Interior point نقطة داخلية

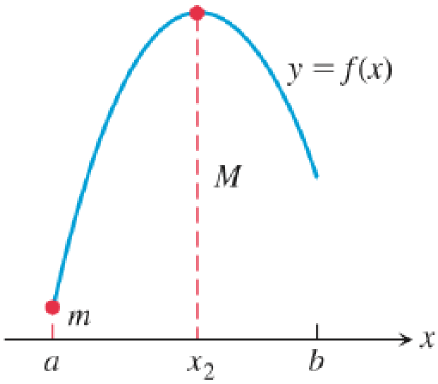
Endpoint نقطة النهاية



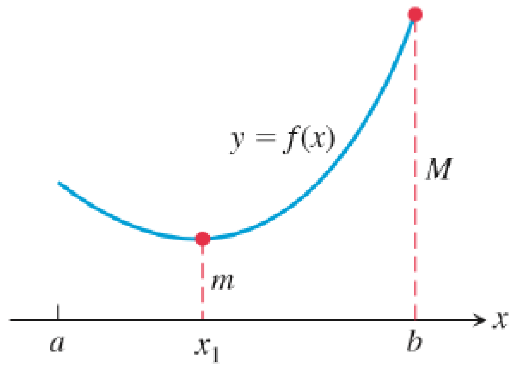
Maximum and minimum
at interior points



Maximum and minimum
at endpoints



Maximum at interior point,
minimum at endpoint



Minimum at interior point,
maximum at endpoint

Critical point نقطة حرجة او نقطة انقلاب

Critical point is the interior point of the domain of a function f *where* $\bar{f}$ *is zero*

To Find the Absolute Extrema of Continuous Function *f* on Closed Interval

1. **Evaluate *f* at all critical points and endpoints.**
2. **Take the largest and smallest of these values.**

Example

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$

$f(x) = x^2$

$\bar{f}(x) = 2x$, $0 = 2x$, $x = 0$

Critical point value: $f(0) = 0$

Endpoints value: $f(-2) = (-2)^2 = 4$

$f(1) = (1)^2 = 1$

Absolute maximum value of 4 at x= -2

Absolute minimum value of 0 at x= 0

Example

Find the absolute maximum and minimum values of $f(x) = 8x - x^4$ on $[-2, 1]$

$f(x) = 8x - x^4$

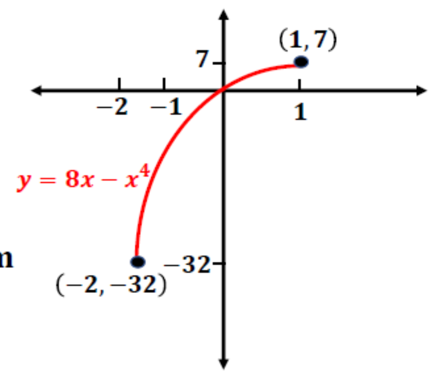
$\bar{f}(x) = 8 - 4x^3$

$0 = 8 - 4x^3$, $4x^3 = 8$, $x^3 = 2$

$x = \sqrt[3]{2}$ > 1 **, a point not in the domain**

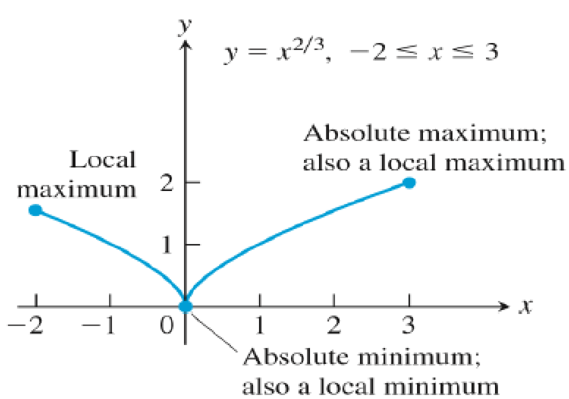
$f(-2) = 8(-2) - (-2)^4 = -32$ **, Absolute minimum**

$f(1) = 8(1) - (1)^4 = 7$ **, Absolute maximum**



Example

Find the absolute maximum and minimum values of $f(x) = x^{\frac{2}{3}}$ on $[-2, 3]$



$f(x) = x^{\frac{2}{3}}$

$\bar{f}(x) = \frac{2}{3} x^{\frac{-1}{3}} = \frac{2}{3\sqrt[3]{x}}$

Has no zero but is undefined at the interior point x=0

Critical point value: $f(0) = (0)^{\frac{2}{3}} = 0$

Endpoint's value: $f(-2) = (-2)^{\frac{2}{3}} = -1.587$

$f(3) = (3)^{\frac{2}{3}} = 2.08$

Absolute maximum value 2.08 at x= 3

Absolute minimum value 0 at occurs at the interior point x= 0

Example

Find the critical points of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$$f(x) = x^5 - 5x^4 + 5x^3 - 1$$
$$\bar{f}(x) = 5x^4 - 20x^3 + 15x^2$$
$$= 5x^2(x^2 - 4x + 3)$$
$$= 5x^2(x-1)(x-3)$$

The critical points at x=0 , x=1, and x=3

Example

Find the critical points of the function $f(x) = x^2 lnx$

$$f(x) = x^2 lnx$$

$$\bar{f}(x) = (x^2)\left(\frac{1}{x}\right) + (lnx)(2x)$$

$$= x + 2xlnx = x(2ln + 1)$$

$$0 = x(2ln + 1)$$

x=0 is not critical point because the function is defined only for $x > 0$

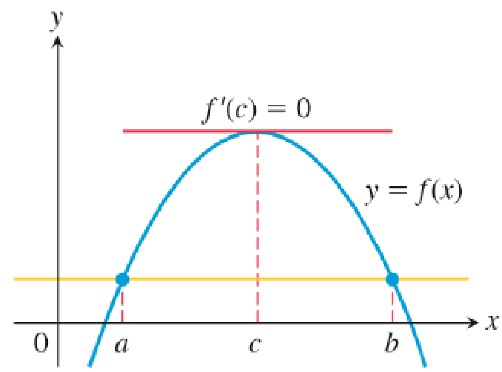
$2lnx + 1 = 0$, $lnx = \frac{-1}{2}$, $e^{lnx} = e^{\frac{-1}{2}}$

$x = e^{\frac{-1}{2}}$ is the critical point

The Mean Value

$$\bar{f}(x) = \frac{f(b)-f(a)}{b-a}$$

for closed interval $[a, b]$



Example

Find the mean value of $f(x) = x^2 - 4x + 1$, for

$$\bar{f}(x) = \frac{f(b)-f(a)}{b-a}$$

$$[a, b] = [1, 5]$$

$$f(x) = x^2 - 4x + 1$$

$$f(1) = (1)^2 - 4(1) + 1 = 1 - 4 + 1 = -2$$

$$f(5) = (5)^2 - 4(5) + 1 = 25 - 20 + 1 = 6$$

$\bar{f}(x) = 2x - 4$

$$2x - 4 = \frac{6 - (-2)}{5 - 1} \qquad \rightarrow \qquad 2x - 4 = \frac{8}{4}$$

$$2x - 4 = 2 \qquad \rightarrow \qquad 2x = 6 \qquad \rightarrow \qquad x = 3$$

$$The\ mean\ value = 3$$

Example

Find the mean value of $f(x) = x^{\frac{2}{3}}$, for [0 , 1]

$$\bar{f}(x) = \frac{f(b) - f(a)}{b - a}$$

$$[a, b] = [0, 1]$$

$$f(x) = x^{\frac{2}{3}}$$

$$f(a) = f(0) = (0)^{\frac{2}{3}} = 0$$

$$f(b) = f(1) = (1)^{\frac{2}{3}} = 1$$

$$\bar{f}(x) = \frac{2}{3}\, x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$$

$$\frac{2}{3x^{\frac{1}{3}}} = \frac{1-0}{1-0} \qquad \rightarrow \qquad \frac{2}{3x^{\frac{1}{3}}} = 1$$

$$3x^{\frac{1}{3}} = 2 \qquad \rightarrow \qquad x^{\frac{1}{3}} = \frac{2}{3} \qquad \rightarrow \qquad \sqrt[3]{x} = \frac{2}{3}$$

$$\left(\sqrt[3]{x}\right)^3 = \left(\frac{2}{3}\right)^3 \qquad \rightarrow \qquad x = \frac{8}{27}$$

Example

Find the mean value of $f(x) = \sqrt{x-4}$, for $[\,4\,,8\,]$

$$\bar{f}(x) = \frac{f(b)-f(a)}{b-a}$$

$$[a,b] = [0,1]$$

$$f(x) = \sqrt{x-4}$$

$$f(a) = f(4) = \sqrt{4-4} = 0$$

$$f(b) = f(8) = \sqrt{8-4} = 2$$

$$\bar{f}(x) = \frac{1}{2}(x-4)^{-\frac{1}{2}} = \frac{1}{2(x-4)^{\frac{1}{2}}}$$

$$\frac{1}{2(x-4)^{\frac{1}{2}}} = \frac{2-0}{8-4} = \frac{2}{4} \qquad \rightarrow \frac{1}{2(x-4)^{\frac{1}{2}}} = \frac{1}{2}$$

$$2(x-4)^{\frac{1}{2}} = 2 \qquad \rightarrow \qquad (x-4)^{\frac{1}{2}} = 1$$

$$\left[(x-4)^{\frac{1}{2}}\right]^2 = [1]^2$$

$$x - 4 = 1 \qquad \rightarrow \qquad x = 5$$

Increasing and Deceasing Intervals

Example

Find the intervals where $f(x) = x^3 + 3x^2 - 9x + 7$, is increasing or decreasing

$$\bar{f}(x) = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3\,(x+3)(x-1)$$

$x = -3 \quad , \qquad x = 1$

The critical points are $x = -3$ *and* $x = 1$

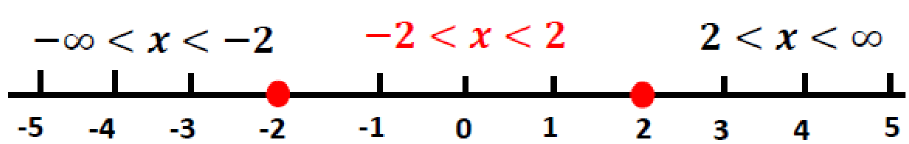
Interval	X- value	$\bar{f}(x)$	
$x < -3$	$x = -4$	$\bar{f}(-4) = 15 > 0$	f is increasing ↗
$-3 < x < 1$	$x = 0$	$\bar{f}(0) = -9 < 0$	f is decreasing ↘
$x > 1$	$x = 2$	$\bar{f}(2) = 15 > 0$	f is increasing ↗

Example

Find the critical points of $f(x) = x^3 - 12x - 5$, is and identify the open intervals on which f is increasing and on which f is decreasing

$$\bar{f}(x) = 3x^2 - 12 = 3(x^2 - 4) = 3\,(x+2)(x-2)$$

$x = 2$, $x = -2$

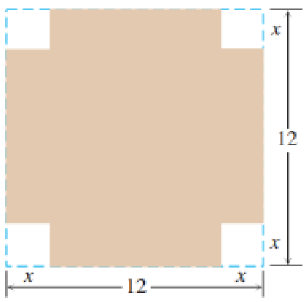


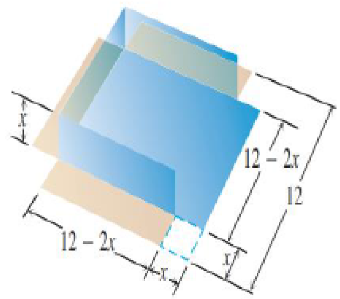
Interval	X- value	$\bar{f}(x)$	
$-2 < x < 2$	$x = 0$	$\bar{f}(0) = -12 < 0$	f is decreasing ↘
$-\infty < x < -2$	$x = -3$	$\bar{f}(-3) = 15 > 0$	f is increasing ↗
$2 < x < \infty$	$x = 3$	$\bar{f}(3) = 15 > 0$	f is increasing ↗

Applied Optimization

Example

An open top box is to be made by cutting small congruent squares from the corners of a 12 inches by 12 inch's sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible.





حجم الصندوق = الطول * العرض * الارتفاع

$$V(x) = x\,(12-2x)(12-2x)$$
$$= x\,(12-2x)^2$$
$$= x\left(144-48\,x+4x^2\right)$$
$$= 144x-48x^2+4x^3$$
$$\frac{dV}{dx} = 144-96\,x+12x^2$$
$$= 12\left(12-8x+x^2\right)$$
$$= 12\,(2-x)(6-x)$$

$if \quad 6-x=0 \quad \rightarrow \quad x=6$ ×

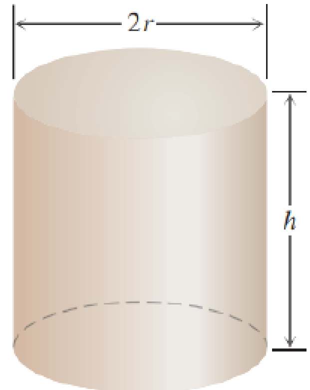
$if \quad 2-x=0 \quad \rightarrow \quad x=2$ ✓

$$The\ maximum\ volume = 144\,(2)-48\,(2)^2+4(2)^3$$
$$= 128\ inch^3$$

$$The\ cutout\ squares\ should\ be\ x = 2\ inch$$

Example

You have asked to design a one – liter can shape like a right circular cylinder. What dimensions will be using the material ?



R = radius of cylinder in *cm*

h = height of cylinder in *cm*

1 liter = 1000 cm^3

$volume = \pi r^2 h = 1000$

Surface area = A

$A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi r h}_{\text{cylindrical wall}}$

$$h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$0 = 4\pi r - \frac{2000}{r^2}$$

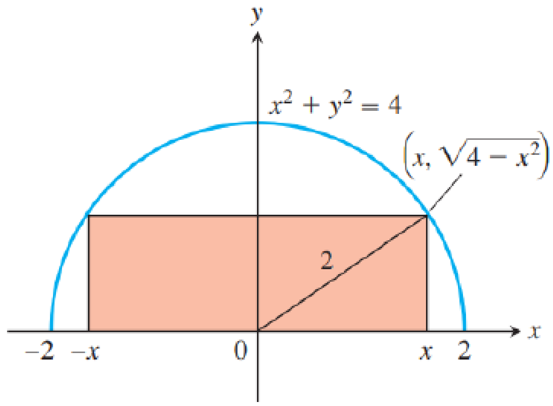
$$4\pi r^3 = 2000$$

$$r = \sqrt[3]{\frac{500}{\pi}} = 5.42 \quad cm$$

$$h = \frac{1000}{\pi r^2} = \frac{1000}{\pi (5.42)^2} = 10.82 \quad cm$$

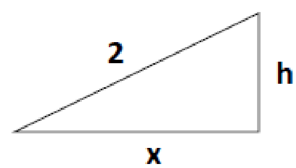
Example

A rectangle is to be inscribed in semicircle of radius 2 units. What is the largest area the rectangle can have, and what are its dimensions ?



$h = height$

$(2)^2 = x^2 + h^2 \quad \rightarrow \quad h = \sqrt{4 - x^2}$



$length = 2x$

$Area = 2x \, . \sqrt{4 - x^2}$

$A(x) = 2x \, . \sqrt{4 - x^2}$

$$\frac{dA}{dx} = \frac{-2x^2}{.\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

$$0 = \frac{-2x^2}{.\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

$$-2\,x^2 + 2\left(4 - \ x^2\right) = 0$$

$$-2\,x^2 + 8 - 2x^2 = 0 \qquad \rightarrow \qquad 8 - 4\,x^2 = 0$$

$$4\,x^2 = 8 \qquad \rightarrow \qquad x^2 = 2 \qquad \rightarrow \qquad x = \ \pm\sqrt{2}$$

$$x = \ \sqrt{2}$$

$$length = 2\,\sqrt{2}$$

$$height = \ \sqrt{4 - \left(\sqrt{2}\right)^2} = \ \sqrt{2}$$

$$area = 2\sqrt{2} \ \times \ \sqrt{2} = 4$$