CHAPTER 5
INTEGRATION

$$\int x^n = \frac{1}{n+1}\, x^{n+1} + c$$

$$\int \sin kx = \frac{-1}{k} \cos kx + c$$

$$\int \cos kx = \frac{1}{k} \sin kx + c$$

$$\int \sec^2 kx = \frac{1}{k} \tan kx + c$$

$$\int \csc^2 \ kx = \frac{-1}{k} \cot kx + c$$

$$\int \sec kx \,.\, \tan kx = \frac{1}{k} \sec kx + c$$

$$\int \csc kx \,.\, \cot kx = \frac{-1}{k} \csc kx + c$$

Example

Evaluate

$$\int (x^2 - 2x + 5)\, dx = \frac{x^3}{3} - x^2 + 5x + c$$

$$\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx = \int \left(x^{-2} - x^2 - \frac{1}{3}\right) dx$$

$$= \frac{-1}{x} - \frac{x^3}{3} - \frac{x}{3} + c$$

$$\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx = \int \left(\frac{1}{2} x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}\right) dx$$

$$= \frac{1}{2} \cdot \frac{2}{3}\, x^{\frac{3}{2}} + 2.2\, x^{\frac{1}{2}} + c$$

$$= \frac{1}{3}\, x^{\frac{3}{2}} + 4\, x^{\frac{1}{2}} + c$$

Example

Evaluate

$$\int 7 \sin \frac{\theta}{3}\, d\theta = -21 \cos \frac{\theta}{3} + c$$

$$\int -2 \cos t\; dt = -2 \sin t + c$$

$$\int (4 \sec x \tan x - 2 \sec^2 x)\, dx = 4 \sec x - 2 \tan x + c$$

$$\int \frac{1 + \cos 4t}{2}\, dt = \int \left(\frac{1}{2} + \frac{1}{2} \cos 4t\right) dt$$

$$= \frac{1}{2} t + \frac{1}{2} \cdot \frac{1}{4} \sin 4t + c$$

$$\int \cos\theta\, (\tan\theta + \sec\theta)\, d\theta =$$

$$\int \cos\theta \left(\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right) d\theta = \int \cos\theta \left(\frac{\sin\theta + 1}{\cos\theta}\right) d\theta =$$

$$\int (\sin\theta + 1)\, d\theta = -\cos\theta + \theta + c$$

$$\int (e^{3x} + 5e^{-x}) \, dx = \frac{e^{3x}}{3} - 5\, e^{x} + c$$

$$\int (2e^{x} - 3e^{-2x}) \, dx = 2e^{x} + \frac{3}{2}\, e^{-2x} + c$$

Example
Evaluate the integrals of the following

$$\int_{-2}^{2} \frac{3}{(x+3)^4} \, dx = 3 \int_{-2}^{2} (x+3)^{-4} \, dx$$

$$= 3 \, . \frac{(x+3)^{-3}}{-3} \Big]_{-2}^{2} = \frac{-1}{(x+3)^3} \Big]_{-2}^{2} = \frac{-1}{(2+3)^3} - \frac{-1}{(-2+3)^3}$$

$$= \frac{-1}{(5)^3} - \frac{-1}{(1)^3} = 1 - \frac{1}{125} = \frac{124}{125}$$

$$\int_{0}^{1} (x^2 + \sqrt{x}) \, dx = \frac{x^3}{3} + \frac{2}{3} \, . x^{\frac{3}{2}} \Big]_{0}^{1} = \left(\frac{1}{3} + \frac{2}{3}\right) - 0 = 1$$

$$\int_{0}^{\pi} (1 + \cos x) \, dx = x + \sin x]_{0}^{\pi} = (\pi + \sin \pi) - (0 - \sin 0)$$

$$= \pi + \sin \pi - \sin 0 = \pi$$

$$\int (1 - cot^2 x)\, dx$$

$$sin^2 x + cos^2 x = 1 \quad \rightarrow \quad 1 + cot^2 x = csc^2 x$$

$$\rightarrow \quad cot^2 x = csc^2 x - 1$$

$$\int \left(1 - (csc^2 x - 1)\right) dx = \int (2 - csc^2 x)\, dx$$

$$= 2x + cot\, x + c$$

$$\int \frac{\csc\theta}{\csc\theta - \sin\theta}\, d\theta$$

$$= \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta}\, d\theta = \int \frac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta} \cdot \frac{\sin\theta}{\sin\theta}\, d\theta$$

$$= \int \frac{1}{1 - sin^2\theta}\, d\theta = \int \frac{1}{cos^2\theta}\, d\theta$$

$$= \int sec^2\,\theta\; d\theta = tan\,\theta + c$$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} sin^2\, t\, dt$$

$$\boxed{cos\, 2t = 1 - 2\, sin^2 t} \rightarrow \quad 2\, sin^2 t = 1 - cos\, 2t \quad \rightarrow$$

$$sin^2 t = \frac{1 - cos\, 2t}{2}$$

$$\int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1 - cos\, 2t}{2}\right) dt = \int_{\frac{-\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{cos\, 2t}{2}\right) dt$$

$$= \frac{t}{2} - \frac{sin\, 2t}{4}\Bigg]_{\frac{-\pi}{3}}^{\frac{\pi}{3}} = \left[\frac{\pi}{6} - \frac{1}{4} sin\, 2\left(\frac{\pi}{3}\right)\right] - \left[\frac{-\pi}{6} - \frac{1}{4} sin\, 2\left(\frac{-\pi}{3}\right)\right]$$

$$= \left[\frac{\pi}{6} - \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{-\pi}{6} - \frac{1}{4}\left(\frac{-\sqrt{3}}{2}\right)\right]$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{8} + \frac{\pi}{6} - \frac{\sqrt{3}}{8} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

$$\int_{0}^{\frac{\pi}{4}} tan^2\, x\ dx$$

$$sin^2\, x + \ cos^2\, x = 1$$

$$\frac{sin^2\, x}{cos^2\, x} + \frac{cos^2\, x}{cos^2\, x} = \frac{1}{cos^2\, x}$$

$$tan^2\, x + 1 = \ sec^2\, x \quad \rightarrow \quad tan^2\, x = \ sec^2\, x - 1$$

$$\int_{0}^{\frac{\pi}{4}} \left(sec^2 x\ - 1\right)\ dx = \ tan\, x\ - x\Big]_{0}^{\frac{\pi}{4}} =$$

$$\left[tan\frac{\pi}{4}\ - \frac{\pi}{4}\right] - [tan\, 0\ - 0] = 1\ - \frac{\pi}{4}$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)\left(t^2+4\right) dt$$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \left(t^3+4t+\ t^2+4\right)\ dt = \frac{t^4}{4} + 2t^2 + \frac{t^3}{3} + 4t\Big]_{-\sqrt{3}}^{\sqrt{3}}$$

$$= \left[\frac{\left(\sqrt{3}\right)^4}{4} + \frac{\left(\sqrt{3}\right)^3}{3} + 2\left(\sqrt{3}\right)^2 + 4\sqrt{3}\right] - \left[\frac{\left(-\sqrt{3}\right)^4}{4} + \frac{\left(-\sqrt{3}\right)^3}{3} + 2\left(-\sqrt{3}\right)^2 + 4\left(-\sqrt{3}\right)\right]$$

$$= \frac{9}{4} + \frac{3\sqrt{3}}{3} + 6 + 4\sqrt{3}\ - \frac{9}{4} + \frac{3\sqrt{3}}{3}\ - 6 + 4\sqrt{3} = 2\sqrt{3} + 8\sqrt{3} = 10\sqrt{3}$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{sin\, 2x}{2\, sin\, x}\ dx$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{2\ sin\, x\, cos\, x}{2\ sin\, x}\ dx = \int_{\frac{\pi}{2}}^{\pi} cos\ x\ dx$$

$$sin\ x\Big]_{\frac{\pi}{2}}^{\pi} = sin\ \pi\ - sin\ \frac{\pi}{2} = 0\ - 1 = -1$$

$$\int_{0}^{\ln 2} e^{3x}\, dx$$

$$= \frac{1}{3}\, e^{3x}\Big]_{0}^{\ln 2} = \frac{1}{3}\, e^{3\ln 2} - \frac{1}{3}\, e^{0} = \frac{1}{3}\, e^{\ln (2)^3} - \frac{1}{3}$$

$$= \frac{1}{3}\,(8) - \frac{1}{3} = \frac{7}{3}$$

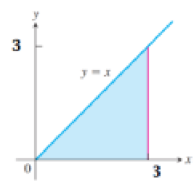
Example

Find the area under the $y = x$ for $0 \le x \le 3$

$$Area = \int_{0}^{3} x\, dx = \frac{x^2}{2}\Big]_{0}^{3}$$

$$= \frac{(3)^2}{2} - 0 = 4\,\frac{1}{2}$$

$$or \quad Area = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$$

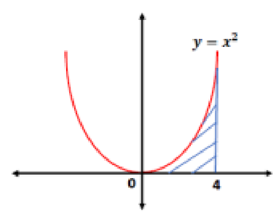


Example

Find the area under the curve $y = x^2$ $from$ $x = 0$ to $x = 4$

$$\int_{0}^{4} x^2\, dx = \frac{x^3}{3}\Big]_{0}^{4}$$

$$= \frac{(4)^3}{3} - 0 = \frac{64}{3}$$



Example

Find the area under the curve $y = -x^2 + 4x - 1$ $from$ $x = 1$ to $x = 3$

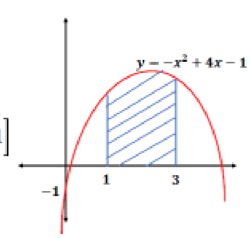
$$Area = \int_{1}^{3} (-x^2 + 4x - 1)\, dx$$

$$= \frac{-x^3}{3} + \frac{4x^2}{2} - x\Big]_{1}^{3}$$

$$= \left[\frac{-(3)^3}{3} + 2(3)^2 - 3\right] - \left[\frac{-(1)^3}{3} + 2(1)^2 - 1\right]$$

$$= \left[\frac{-27}{3} + 18 - 3\right] - \left[\frac{-1}{3} + 2 - 1\right]$$

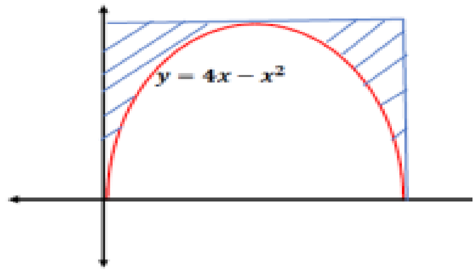
$$= 6 - \frac{2}{3} = 5\,\frac{1}{3}$$



Example

1) Find the points where the graph cross the $x-axis$

2) Find the area of shaded region



1) $0 = 4x - x^2 \quad \rightarrow \quad 0 = x(4-x)$

$x = 0 \quad , \quad x = 4$

2) $Area = \int_0^4 (4x - x^2)\,dx$

$= \frac{4x^2}{2} - \frac{x^3}{3}\Big]_0^4 = 2x^2 - \frac{x^3}{3}\Big]_0^4$

$= \left[2(4)^2 - \frac{(4)^3}{3}\right] - [0] = \frac{32}{3}$

$Total\ area = 4 \times 4 = 16$

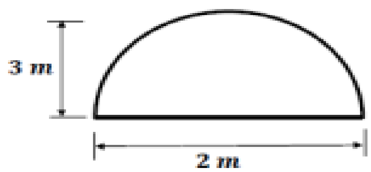
$Shaded\ area = 16 - \frac{32}{3} = \frac{16}{3}$

Example

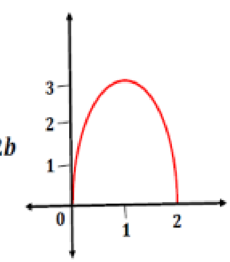
If the arch is $2m$ wide at bottom and is $3m$ high, Find

1) The equation of the parabola

2) The area under each arch, using integration



1) $General\ from\ parabola\ y = ax^2 + bx + c$



$at\ x = 0 \quad \rightarrow \quad y = 0$

$0 = 0 + 0 + c \quad \rightarrow \quad c = 0$

$at\ x = 1 \quad \rightarrow \quad y = 3$

$3 = a + b \quad \rightarrow \quad a = 3 - b$

$at\ x = 2 \quad \rightarrow \quad y = 0$

$0 = 4(3-b) + 2b \quad \rightarrow \quad 0 = 12 - 4b + 2b$

$0 = 12 - 2b \quad \rightarrow \quad b = \frac{12}{2} = 6$

$a = 3 - 6 = -3$

$y = -3x^2 + 6x$

2) $Area = \int_0^2 (-3x^2 + 6x)\ dx$

$= \frac{-3x^3}{3} + \frac{6x^2}{2}\Big]_0^2 = -x^3 + 3x^2\Big]_0^2$

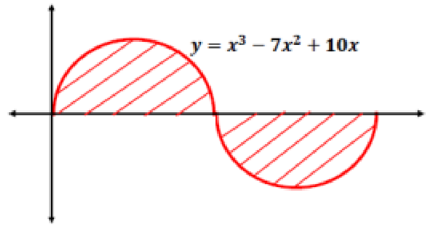
$= [-(2)^3 + 3(2)^2] - [0 + 0]$

$= -8 + 12 = 4\ m^2$

Example

1) Find the points where the graph cross the $x - axis$

2) Find the area of shaded raigon



1) $0 = x^3 - 7x^2 + 10x$

$0 = x(x^2 - 7x + 10) \quad \rightarrow \quad 0 = x(x-2)(x-5)$

$x = 0 \quad , \quad x = 2 \quad , \quad x = 5$

2) $Area = \int_0^2 (x^3 - 7x^2 + 10x)\,dx + \int_2^5 (x^3 - 7x^2 + 10x)\,dx$

$= \frac{x^4}{4} - \frac{7x^3}{3} + \frac{10x^2}{2}\Big]_0^2 + \frac{x^4}{4} - \frac{7x^3}{3} + \frac{10x^2}{2}\Big]_2^5$

$= \left[\frac{(2)^4}{4} - \frac{7(2)^3}{3} + 5(2)^2\right] + \left[\left\{\frac{(5)^4}{4} - \frac{7(5)^3}{3} + 5(5)^2\right\}\right.$

$\left.-\left\{\frac{(2)^4}{4} - \frac{7(2)^3}{3} + 5(2)^2\right\}\right] = \frac{253}{12}$

Integration by Substitution – Running the Chain Rule Backward

$$\boxed{\int u^n \, du = \frac{u^{n+1}}{n+1} + c}$$

Example

Evaluate $\int (x+2)^5 \, dx$

$u = x + 2 \quad , \quad du = dx$

$\int (x+2)^5 \, dx = \int u^5 \, du$

$= \frac{u^6}{6} + c = \frac{(x+2)^6}{6} + c$

Example

Evaluate $\int \sqrt{1+x^2}\,.2x\ dx$

$$u = 1 + x^2 \qquad , \qquad du = 2xdx$$

$$\int \sqrt{1+x^2}\,.2x\,dx = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c$$

Example

Evaluate $\int \sqrt{4x-1}\ dx$

$$u = 4x - 1 \quad , \quad du = 4dx \quad , \quad dx = \frac{du}{4}$$

$$\int \sqrt{4x-1}\ dx = \int u^{\frac{1}{2}}\frac{du}{4} = \frac{1}{4}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{4}\,.\frac{2}{3}u^{\frac{3}{2}} + c = \frac{1}{6}(4x-1)^{\frac{3}{2}} + c$$

Example

Evaluate $\int x\sqrt{2x+1}\ dx$

$$u = 2x + 1 \quad , \quad du = 2dx \quad , \quad dx = \frac{du}{2}$$

$$2x = u - 1 \quad \rightarrow \quad x = \frac{1}{2}(u-1)$$

$$\int x\sqrt{2x+1}\ dx = \int \frac{1}{2}(u-1)\,.\sqrt{u}\,.\frac{du}{2}$$

$$= \frac{1}{4}\int (u-1)\,.\,u^{\frac{1}{2}}du$$

$$= \frac{1}{4}\int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)\ du$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right] + c$$

$$= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + c$$

Example

Evaluate $\int sec^2(5x+1)\,.\,5\,dx$

$$u = 5x+1 \quad , \qquad du = 5dx$$

$$\int sec^2(5x+1)\,.\,5\,dx = \int sec^2u\,.\,du$$

$$= tan\,u + c = tan\,(5x+1) + c$$

Example

Evaluate $\int cos\,(7\theta+3)\,d\theta$

$$u = 7\theta+3 \quad , \qquad du = 7d\theta \quad , \qquad d\theta = \frac{1}{7}du$$

$$\int cos(7\theta+3)\,.\,d\theta = \int cos\,u\,.\,\frac{1}{7}.\,du$$

$$= \frac{1}{7}\int cos\,u\,du = \frac{1}{7}\,sin\,u + c$$

$$= \frac{1}{7}\,sin\,(7\theta+3) + c$$

Example

Evaluate $\int x^2 \cos x^3\, dx$

$$u = x^3 \quad , \qquad du = 3x^2 dx \quad \rightarrow \quad \frac{du}{3} = x^2 dx$$

$$\int x^2 \cos x^3\, dx = \int \cos u \,.\frac{du}{3} = \frac{1}{3}\int \cos u\, du$$

$$= \frac{1}{3} \sin u + c = \frac{1}{3} \sin x^3 + c$$

Example

Evaluate $\int \frac{4x^3}{(x^4+1)^2} dx$

$$u = x^4 + 1 \quad , \qquad du = 4x^3 dx \quad \rightarrow \quad \frac{du}{4} = x^3 dx$$

$$\int \frac{4x^3}{(x^4+1)^2} dx = \int 4\, x^3 \left(x^4+1\right)^{-2}\ dx$$

$$\int 4\, u^{-2} \,.\frac{1}{4}\, du = \int u^{-2}\, du = -\, u^{-1} + c$$

$$= \frac{-1}{x^4 + 1} + c$$

Example

Evaluate $\int (3x+2)\left(3x^2+4x\right)^4 dx$

$$u = 3x^2 + 4x \quad , \qquad du = (6x+4)dx = 2\,(3x+2)dx$$

$$\frac{du}{2} = (3x+2)dx$$

$$\int (3x+2)\left(3x^2+4x\right)^4 dx = \int u^4\, \frac{1}{2}\, du$$

$$= \frac{1}{2} \int u^4\ du = \frac{1}{2} \,.\frac{u^5}{5} + c$$

$$= \frac{1}{10} \,. u^5 + c$$

$$= \frac{1}{10} \left(3x^2+4x\right)^5 + c$$

Example

Evaluate $\int \sqrt{x}\, sin^2\left(x^{\frac{3}{2}} - 1\right) dx$

Let $u = x^{\frac{3}{2}} - 1$, $\quad du = \frac{3}{2} x^{\frac{1}{2}}\, dx$

$$\frac{2}{3} du = x^{\frac{1}{2}}\, dx$$

$$\boxed{sin^2\,\theta = \frac{1 - cos\,2\theta}{2}}$$

$$\boxed{cos^2\,\theta = \frac{1 + cos\,2\theta}{2}}$$

$$\int \sqrt{x}\, sin^2\left(x^{\frac{3}{2}} - 1\right) dx = \int \frac{2}{3}\, sin^2\, u\ du$$

$$= \frac{2}{3} \int \frac{1 - cos\,2u}{2}\ du$$

$$= \frac{2}{3} \int \left(\frac{1}{2} - \frac{cos\,2u}{2}\right) du$$

$$= \frac{2}{3} \left(\frac{u}{2} - \frac{1}{4}\, sin\,2u\right) + c$$

$$= \frac{1}{3}\, u - \frac{1}{6}\, sin\,2u + c$$

$$= \frac{1}{3}\left(x^{\frac{3}{2}} - 1\right) - \frac{1}{6}\, sin\left(2\, x^{\frac{3}{2}} - 2\right) + c$$