Applications of Differential Equation تطبيقات على المعادلات التفاضلية

Example

The acceleration and velocity of a body falling in the air approximately satisfy the equation:

$$Acceleration = g - k\,V^2$$

Where

$V \rightarrow$ **the velocity of body at any time** (t)

g **and** $k \rightarrow$ **constants**

Find the distance traversed as a function of the time , if the body falls from the rest.

$$Acceleration = g - k\,V^2 \qquad \rightarrow \qquad \frac{dV}{dt} = g - k\,V^2$$

$$\frac{dV}{g - k\,V^2} = dt \qquad \rightarrow \qquad \left[\frac{1}{\left(\sqrt{g} + \sqrt{k}\,V\right)\left(\sqrt{g} - \sqrt{k}\,V\right)}\right] dV = dt$$

$$\frac{1}{2\sqrt{g}}\left[\frac{1}{\sqrt{g} + \sqrt{k}\,V} + \frac{1}{\sqrt{g} - \sqrt{k}\,V}\right] dV = dt$$

$$\frac{1}{2\sqrt{g}}\int \frac{dV}{\sqrt{g} + \sqrt{k}\,V} + \frac{1}{2\sqrt{g}}\int \frac{dV}{\sqrt{g} - \sqrt{k}\,V} = \int dt$$

$$\frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} + \sqrt{k}\,V\right) - \frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} - \sqrt{k}\,V\right)$$

$$= t + c$$

$when \quad t = 0 \quad and \quad V = 0$

$$\frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} + 0\right) - \frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} - 0\right) = 0 + c$$

$$\frac{1}{2\sqrt{gk}}\left[ln\sqrt{g} - ln\sqrt{g}\right] = c$$

$$\frac{1}{2\sqrt{gk}} \cdot ln\frac{\sqrt{g}}{\sqrt{g}} = c \quad \rightarrow \quad \frac{1}{2\sqrt{gk}} \cdot ln\,1 = c \quad \rightarrow \quad c = 0$$

$$\frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} + \sqrt{k}\,V\right) - \frac{1}{2\sqrt{g}} \cdot \frac{1}{\sqrt{k}} \cdot ln\left(\sqrt{g} - \sqrt{k}\,V\right) = t$$

$$\frac{1}{2\sqrt{gk}}\left[ln\frac{\sqrt{g} + \sqrt{k}\,V}{\sqrt{g} - \sqrt{k}V}\right] = t$$

$$ln\frac{\sqrt{g} + \sqrt{k}\,V}{\sqrt{g} - \sqrt{k}V} = 2\sqrt{gk}\,t$$

$$e^{ln\frac{\sqrt{g} + \sqrt{k}\,V}{\sqrt{g} - \sqrt{k}V}} = e^{2\sqrt{gk}\,t}$$

$$\frac{\sqrt{g} + \sqrt{k}\,V}{\sqrt{g} - \sqrt{k}V} = e^{2\sqrt{gk}\,t} \quad \rightarrow \sqrt{g} + \sqrt{k}\,V = e^{2\sqrt{gk}\,t}\left(\sqrt{g} - \sqrt{k}\,V\right)$$

$$\sqrt{g} + \sqrt{k}\,V = \sqrt{g}\,.\,e^{2\sqrt{gk}\,t} - \sqrt{k}\,.\,V\,.\,e^{2\sqrt{gk}\,t}$$

$$\sqrt{k}\,.\,V + \sqrt{k}\,.\,V.\,e^{2\sqrt{gk}\,t} = \sqrt{g}\,.\,e^{2\sqrt{gk}\,t} - \sqrt{g}$$

$$\sqrt{k}\,.\,V\left[1 + e^{2\sqrt{gk}\,t}\right] = \sqrt{g}\left[e^{2\sqrt{gk}\,t} - 1\right]$$

$$\frac{\sqrt{k}\,V}{\sqrt{g}} = \frac{e^{2\sqrt{gk}\,t} - 1}{e^{2\sqrt{gk}\,t} + 1} \times \frac{e^{-2\sqrt{gk}\,t}}{e^{-2\sqrt{gk}\,t}}$$

$$\frac{\sqrt{k}\,V}{\sqrt{g}} = \frac{e^{\sqrt{gk}\,t} - e^{-\sqrt{gk}\,t}}{e^{\sqrt{gk}\,t} + e^{-\sqrt{gk}\,t}} = tanh\,\sqrt{gk}\,t$$

$$\sqrt{k}\,V = \sqrt{g}\,.\,tanh\,\sqrt{gk}\,t$$

$$tanh\,\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

$$V = \sqrt{\frac{g}{k}}\,.\,tanh\,\sqrt{gk}\,t$$

$$\frac{dx}{dt} = \sqrt{\frac{g}{k}}\,.\,tanh\,\sqrt{gk}\,t$$

$$\int dx = \sqrt{\frac{g}{k}}\int tanh\,\sqrt{gk}\,t\,dt$$

$$x = \sqrt{\frac{g}{k}} \cdot \frac{1}{\sqrt{gk}} \cdot ln\, cosh \sqrt{gk}\, t + c$$

$$x = \frac{1}{k} \cdot ln\, cosh \sqrt{gk}\, t + c$$

$$at \quad t = 0 \quad and \quad x = 0 \quad \rightarrow \quad c = 0$$

$$x = \frac{1}{k} \cdot ln\, cosh \sqrt{gk}\, t$$

Example

The initial temperature of body is 130 ℃ , it put in media of 10 ℃ temperature. Find the temperature of the body after 30 min. , if the temperature of the body became 80 ℃ , after put it in this media for 15 min.

$$Ues\ Newtens\ law\ of\ cooling: \ \frac{dT}{dt} = -k\,(T - T_a)$$

Where

$k \rightarrow constant$

$T \rightarrow temperature\ of\ body$

$T_a \rightarrow media\ temperature$

$$\frac{dT}{dt} = -k\,(T - T_a) \quad \rightarrow \frac{dT}{dt} = -k\,(T - 10)$$

$$\int \frac{dT}{T-10} = -k \int dt$$

$$ln\,(T-10) = -kt + c \quad \rightarrow \quad e^{ln\,(T-10)} = e^{-kt+c} \quad \rightarrow \quad T-10$$

$$= e^{-kt} + e^{c}$$

$$T - 10 = e^{-kt}.\, e^{c}$$

$$let\ e^{c} = A$$

$$T - 10 = e^{-kt} . A \quad \rightarrow \quad T = A\, e^{-kt} + 10$$

$$at \quad t = 0 \quad and \quad T = 130 \,℃$$

$$130 = A\, e^{0} + 10 \quad \rightarrow \quad A = 120$$

$$T = 120\, e^{-kt} + 10$$

$$at \quad t = 15\, min. \quad and \quad T = 80 \,℃$$

$$80 = 120\, e^{-k\,(15)} + 10 \quad \rightarrow \quad e^{-k(15)} = \frac{80 - 10}{120} = 0.5833$$

$$ln\left(e^{-k(15)}\right) = ln\,(0.5833) \quad \rightarrow \quad -k\,(15)$$

$$= ln\,(0.5833) \quad \rightarrow \quad k = 0.036$$

$$T = 120\, e^{-0.036t} + 10$$

$$at \quad t = 30\, min.$$

$$T = 120\, e^{-0.036\,(30)} + 10 = 50.75$$