# Lecture (1)

## Techniques of Integration

## (التكامل بالتجزئة) Integration by Parts

تستخدم هذه الطريقة لتبسيط التكامل لحالة النموذج التالي

$$\int f(x)g(x)\,dx.$$

حيث تكون الدالة f قابلة للاشتقاق و الدالة g قابلة للتكامل

### Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

**EXAMPLE 1** Find

$$\int x\cos x\,dx.$$

**Solution** We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$

$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$

**EXAMPLE 2** Find

$$\int \ln x\,dx.$$

**Solution** Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

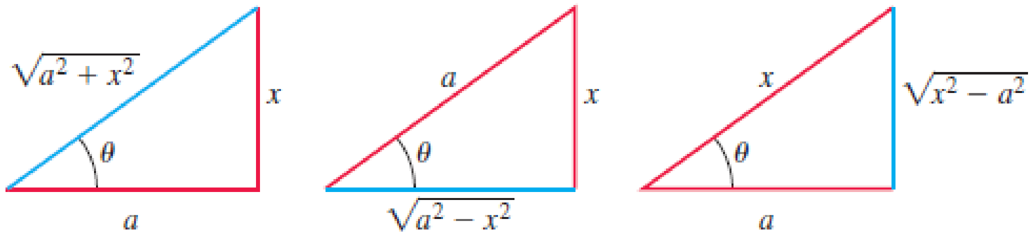
$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}dx,$ $\qquad v = x.$ Simplest antiderivative

Then from Equation (2),

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C. \quad \blacksquare$$

# Trigonometric Substitutions




$x = a \tan \theta$ $\qquad$ $x = a \sin \theta$ $\qquad$ $x = a \sec \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$ $\qquad$ $\sqrt{a^2 - x^2} = a|\cos \theta|$ $\qquad$ $\sqrt{x^2 - a^2} = a|\tan \theta|$

**EXAMPLE 1** Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

**Solution** We set

$$x = 2 \tan \theta, \qquad dx = 2 \sec^2 \theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

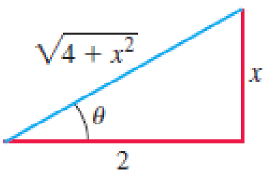



**FIGURE 8.4** Reference triangle for $x = 2 \tan \theta$ (Example 1):

$$\tan \theta = \frac{x}{2}$$

and

$$\sec \theta = \frac{\sqrt{4 + x^2}}{2}.$$

Then

$$\begin{aligned} \int \frac{dx}{\sqrt{4 + x^2}} &= \int \frac{2 \sec^2 \theta \, d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{|\sec \theta|} \\ &= \int \sec \theta \, d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C. \end{aligned}$$

**EXAMPLE 4** Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

**Solution** We first rewrite the radical as

$$\begin{aligned} \sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\ &= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2} \end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$\begin{aligned} x^2 - \left(\frac{2}{5}\right)^2 &= \frac{4}{25}\sec^2\theta - \frac{4}{25} \\ &= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta \end{aligned}$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for $0 < \theta < \pi/2$

With these substitutions, we have

$$\begin{aligned} \int \frac{dx}{\sqrt{25x^2 - 4}} &= \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta} \\ &= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \end{aligned}$$

From Fig. 8.6

# Integration of Rational Functions by Partial Fractions

**EXAMPLE 1** Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx.$$

**Solution** The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients $A$, $B$, and $C$, we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of $x$, obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: & A + B + C &= 1 \\ &\text{Coefficient of } x^1: & 4A + 2B \phantom{{}-C} &= 4 \\ &\text{Coefficient of } x^0: & 3A - 3B - C &= 1 \end{aligned}$$

There are several ways of solving such a system of linear equations for the unknowns $A$, $B$, and $C$, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K, \end{aligned}$$

where $K$ is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as $C$). ■

**EXAMPLE 2** Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2}\, dx.$$

**Solution** First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

$$6x + 7 = A(x + 2) + B \qquad \text{Multiply both sides by } (x + 2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of $x$ gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x + 7}{(x + 2)^2}\, dx = \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2}\right) dx$$

$$= 6\int \frac{dx}{x + 2} - 5\int (x + 2)^{-2}\, dx$$

$$= 6 \ln |x + 2| + 5(x + 2)^{-1} + C.$$

■

**EXAMPLE 4** Use partial fractions to evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx.$$

**Solution** The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 \\ &\quad + (A-2B+C)x + (B-C+D). \end{aligned}$$

Equating coefficients of like terms gives

| | |
|---|---|
| Coefficients of $x^3$: | $0 = A + C$ |
| Coefficients of $x^2$: | $0 = -2A + B - C + D$ |
| Coefficients of $x^1$: | $-2 = A - 2B + C$ |
| Coefficients of $x^0$: | $4 = B - C + D$ |

We solve these equations simultaneously to find the values of $A$, $B$, $C$, and $D$:

| | |
|---|---|
| $-4 = -2A, \quad A = 2$ | Subtract fourth equation from second. |
| $C = -A = -2$ | From the first equation |
| $B = (A + C + 2)/2 = 1$ | From the third equation and $C = -A$ |
| $D = 4 - B + C = 1.$ | From the fourth equation. |

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned} \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \end{aligned}$$ ■

**EXAMPLE 5** Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

**Solution** The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}
1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\
&= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\
&= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A.
\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus,

$$\begin{aligned}
\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\
&= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\
&= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} && u = x^2+1, \; du = 2x\,dx \\
&= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\
&= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\
&= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K.
\end{aligned}$$

■