NUMERICAL ANALYSIS

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CHAPETER TWO

LINEAR DIFERENTIAL EQUATIONS OF SECEND ORDER

معادلات تفاضلية خطية من المرتبة الثانية

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Linear differential Equations of Second Order معادلات تفاضلية خطية من المرتبة الثانية

The general form of linear differential of second order in:

$$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Q\,y = R$$

Where P and Q are constants and R is a function of x or constant.

Differential operator. Symbol D stands for the operation of differential i.e.

$$D_y = \frac{dy}{dx} \quad , \quad D_y^2 = \frac{d^2y}{dx^2}$$

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$$\frac{1}{D} \text{ stands for the integration}$$

$$\frac{1}{D^2} \text{ stands for the integration twice}$$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q\,y = R \quad , \qquad \text{can be written from:}$$

$$D_y^2 + P\,D_y + Q\,y = R$$

$$or$$

$$\left(D^2 + P\,D + Q\right)y = R$$

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Complete Solution = Complementary Function + Particular Integral

$$y = C.F. + P.I.$$

Method for Finding the Complementary Function

1- In finding the complementary function of the given equation is replaced by zero

2- $let \quad \frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Q\,y = 0$

$$D_y^2 + P\,D_y + Q\,y = 0$$

$$\left(D^2 + P\,D + Q\right)_y = 0$$

$m^2 + P\,m + Q = 0$ *is called* ***Auxiliary equation***

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3- Solve the Auxiliary equation

Case I: **Roots, Real and Different.** If m_1 and m_2 are the roots, then C.F. is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case II: **Roots, Real and Equal.** If both the roots are m_1, m_2 are the roots, then C.F. is

$$y = (C_1 + C_2 x) e^{m_1 x}$$

Case III: **Roots, Imaginary. If the roots are $\alpha \pm i\beta$,** then the solution will be

$$y = e^{\alpha x} [A \cos \beta x + B \sin \beta x]$$

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Example 1: **Solve,** $\frac{d^2y}{dx^2} - 8\,\frac{dy}{dx} + 15\,y = 0$

Given equation van be written as

$$\left(D^2 - 8\,D + 15\right) y = 0$$

Here auxiliary equation, $m^2 - 8\,m + 15 = 0$

$(m-3)(m-5) = 0 \quad \rightarrow \quad m_1 = 3\ , \quad m_2 = 5$

The required solution is $y = C_1\,e^{3x} + C_2\,e^{5x}$

Example 2: **Solve** $\frac{d^2y}{dx^2} - 8\,\frac{dy}{dx} + 16\,y = 0$

$\left(D^2 - 8\,D + 16\right) y = 0 \quad \rightarrow \quad m^2 - 8\,m + 16 = 0$

$(m-4)(m-4) = 0 \quad \rightarrow \quad m_1 = m_2 = 4$

The required solution is $y = (C_1 + x\,C_2)\,e^{4x}$

Example 3: Solve, $\frac{d^2y}{dx^2} + 4\,\frac{dy}{dx} + 5\,y = 0$

$$(D^2 + 4\,D + 5)\,y = 0 \qquad \rightarrow \qquad m^2 + 4\,m + 5 = 0$$

$$m = \frac{-b\ \pm\ \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1\ , \qquad b = 4\ , \qquad c = 5$$

$$m = \frac{-4\ \pm\ \sqrt{16 - 4 \times 1 \times 5}}{2 \times 1} = \frac{-4 \pm\ \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

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The complementary function is $y = e^{-2x}(A\cos x + B\sin x)$ ……………………(1)
On putting $\boldsymbol{y}$ = 2 and $\boldsymbol{x}$ = 0 in (1) , we get 2 = A
On putting A= 2 in (1), we have

$$\boldsymbol{y = e^{\alpha x}\,[A\,cos\,\beta\,x + B\,sin\,\beta\,x]}$$

$\boldsymbol{\alpha = -2}$, $\boldsymbol{\beta = 1}$

$\boldsymbol{y = e^{-2x}\,[2\,cos\,x + B\,sin\,x]}$

divided all by 2

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Example 4: **Solve** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 5\,y = 0$

$$(D^2 + D - 1)\,y = 0 \quad \rightarrow \quad m^2 + m - 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}\,i$$

$$y = e^{\frac{-1}{2}x}\left[A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right]$$

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Rules to Find Particular Integral (P. I.)

1) $$\frac{1}{f(D)}\, e^{ax} = \frac{1}{f(a)}\, e^{ax}$$

$$If \quad f(a) = 0 \;, \qquad then \;\; \frac{1}{f(D)} \,.\, e^{ax} = x \,.\, \frac{1}{\bar{f}\,(a)} \,.\, e^{ax}$$

$$If \quad \bar{f}(a) = 0 \;, \qquad then \;\; \frac{1}{f(D)} \,.\, e^{ax} = x^2 \,.\, \frac{1}{\bar{\bar{f}}(a)} \,.\, e^{ax}$$

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Example 5: **Solve** $\frac{d^2y}{dx^2} + 6\,\frac{dy}{dx} + 9\,y = 5\,e^{3x}$

$$\left(D^2 + 6\,D + 9\right)y = 5\,e^{3x}$$
$$m^2 + 6\,m + 9 = 0$$
$$(m+3)(m+3) = 0 \qquad \rightarrow \qquad m_1 = m_2 = -3$$
$$C.F. = \ (c_1 + x\,c_2)\,e^{-3x}$$

$$P.I. = \frac{1}{f\,(D)}\,.5\,e^{3x} = \frac{1}{D^2 + 6\,D + 9}\,.5\,e^{3x} \quad = 5\,.\frac{e^{3x}}{(3)^2 + 6\,(3) + 9} = \frac{5\,e^{3x}}{36}$$

$$y = C.F. + P.I.$$

$$y = (c_1 + x\,c_2)\,e^{-3x} + \frac{5\,e^{3x}}{36}$$

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Example 6: Solve $\bar{\bar{y}} + 4\,\bar{y} + 5\,y = 2\,e^{x}$

$$(D^2 + 4\,D + 5)_y = 2\,e^{x}$$

$$m^2 + 4\,m + 5 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2 \times 1} = -2 \pm i$$

$$C.F. = e^{-2x}\,[A\cos x + B\sin x]$$

$$P.I. = \frac{1}{D^2 + 4D + 5}\,.\,2\,e^{x} \qquad = \frac{1}{(1)^2 + 4\,(1) + 5}\,.\,2\,e^{x} = \frac{2}{10}\,e^{x} = \frac{1}{5}\,e^{x}$$

$$y = e^{-2x}\,[A\cos x + B\sin x] + \frac{1}{5}\,e^{x}$$

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Example 7: Solve $(D^2 - 1)_y = 5\,e^x$

$$m^2 - 1 = 0 \quad \rightarrow \quad m = \pm 1$$

$$C.F. = c_1\,e^x + c_2\,e^{-x}$$

$$P.I. = \frac{1}{D^2 - 1}\,.5e^x = \frac{1}{1-1}\,.5e^x = \frac{1}{2D}\,.x\,.5e^x = \frac{5}{2}\,.x\,.e^x$$

0

$$y = c_1\,e^x + c_2\,e^{-x} + \frac{5}{2}\,.x\,.e^x$$

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Example 8: Solve $\frac{d^2y}{dx^2} - 6\,\frac{dy}{dx} + 9\,y = 6\,e^{3x} + 7\,e^{-2x} - ln\,2$

$$\left(D^2 - 6\,D + 9\right)_y = 6\,e^{3x} + 7\,e^{-2x} - ln\,2$$

$$m^2 - 6\,m + 9 = 0$$

$$(m-3)(m-3) = 0 \qquad \rightarrow \qquad m_1 = m_2 = 3$$

$$C.F. = \left(c_1 + x\,c_2\right) e^{3x}$$

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$$P.I. = \frac{1}{D^2 - 6\,D + 9}\,6\,e^{3x} + \frac{1}{D^2 - 6\,D + 9}\,7\,e^{-2x} - \frac{1}{D^2 - 6\,D + 9}\,ln\,2$$

$$= \frac{1}{(3)^2 - 6\,(3) + 9}\,6\,e^{3x} + \frac{1}{(-2)^2 - 6\,(-2) + 9}\,7\,e^{-2x} - ln\,2\,\frac{1}{D^2 - 6\,D + 9}\,e^{0x}$$

0

$$= x\,\frac{1}{2\,D - 6}\,6\,e^{3x} + \frac{1}{4 + 12 + 9}\,7\,e^{-2x} - ln\,2\,\frac{1}{0 - 0 + 9}$$

0

$$= x^2 \cdot \frac{1}{2} \cdot 6\,e^{3x} + \frac{7}{25}\,e^{-2x} - ln\,2 \cdot \frac{1}{9} = 3\,x^2\,e^{3x} + \frac{7}{25}\,e^{-2x} - \frac{1}{9}\,ln2$$

$$y = (c_1 + x\,c_2)\,e^{3x} + 3\,x^2\,e^{3x} + \frac{7}{25}\,e^{-2x} - \frac{1}{9}\,ln2$$

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