Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure illustrates the standard polar coordinate tests for symmetry.

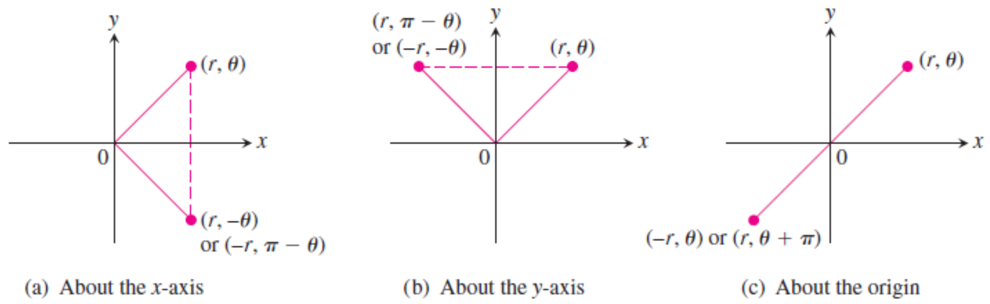


FIGURE Three tests for symmetry in polar coordinates.

Symmetry Tests for Polar Graphs

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure c).

EXAMPLE A Cardioid

Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\ &\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph. The curve is called a *cardioid* because of its heart shape. Cardioid shapes appear in the cams that direct the even layering of thread on bobbins and reels, and in the signal-strength pattern of certain radio antennas.

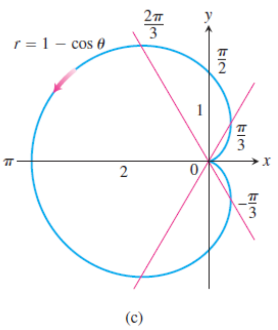


FIGURE The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.

EXAMPLE

Graph the Curve $r^2 = 4\cos\theta$.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\ &\Rightarrow (r, -\theta) \text{ on the graph.}\end{aligned}$$

The curve is also symmetric about the origin because

$$\begin{aligned}(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\ &\Rightarrow (-r)^2 = 4\cos\theta \\ &\Rightarrow (-r, \theta) \text{ on the graph.}\end{aligned}$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

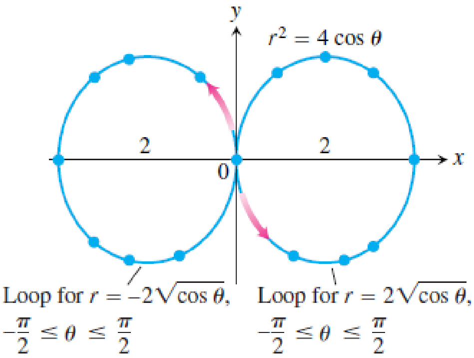
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded

EXAMPLE

Find the points of intersection of the curves

$$r^2 = 4\cos\theta \qquad \text{and} \qquad r = 1 - \cos\theta.$$

Solution In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing.

If we substitute $\cos\theta = r^2/4$ in the equation $r = 1 - \cos\theta$, we get

$$r = 1 - \cos\theta = 1 - \frac{r^2}{4}$$

$$4r = 4 - r^2$$

$$r^2 + 4r - 4 = 0$$

$$r = -2 \pm 2\sqrt{2}. \qquad \text{Quadratic formula}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\theta = \cos^{-1}(1 - r) \qquad \text{From } r = 1 - \cos\theta$$

$$= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) \qquad \text{Set } r = 2\sqrt{2} - 2.$$

$$= \cos^{-1}\left(3 - 2\sqrt{2}\right)$$

$$= \pm 80°. \qquad \text{Rounded to the nearest degree}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80°)$.

If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together , as we can now do by combining the graphs in Figures 10.44 and 10.45, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points $(0, 0)$ and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$.

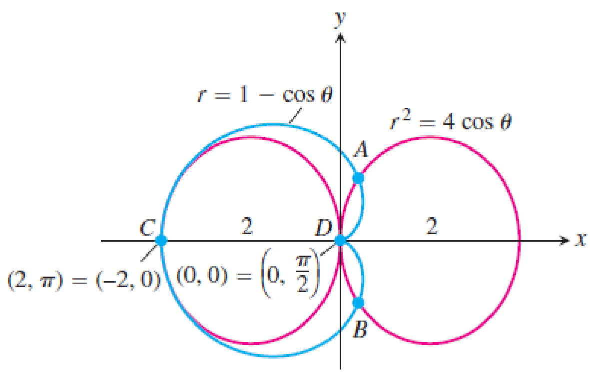


FIGURE The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$
Only A and B were found by simultaneous solution. The other two were disclosed by graphing.

Areas and Lengths in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta .$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} (f(\theta))^2 \, d\theta .$$

EXAMPLE Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \\
&= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi .
\end{aligned}$$

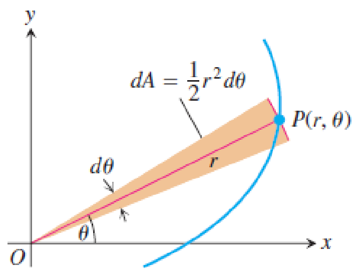
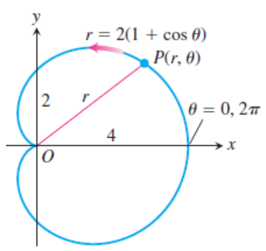


FIGURE The area differential dA for the curve $n = f(\theta)$.



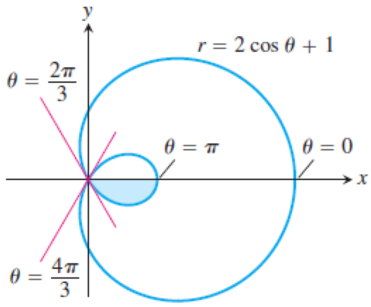
EXAMPLE Finding Area

Find the area inside the smaller loop of the limaçon

$$r = 2\cos\theta + 1 .$$

Solution After sketching the curve we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{\pi} r^2 \, d\theta .$$



Since

$$r^2 = (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1$$
$$= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1$$
$$= 2 + 2\cos 2\theta + 4\cos\theta + 1$$
$$= 3 + 2\cos 2\theta + 4\cos\theta,$$

we have

$$A = \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta$$
$$= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi}$$
$$= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right)$$
$$= \pi - \frac{3\sqrt{3}}{2}.$$

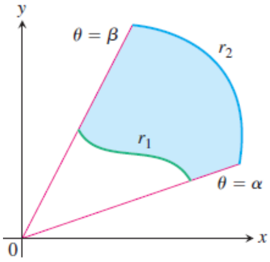
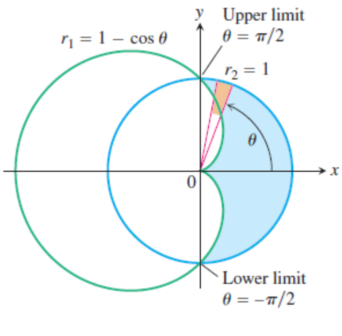


FIGURE The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta), \quad \alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2} r_2^2\, d\theta - \int_\alpha^\beta \frac{1}{2} r_1^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

Solution We sketch the region to determine its boundaries and find the limits of integration . The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$.



$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$
$$= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$
$$= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta$$
$$= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2} \left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta$$
$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.$$

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta.$$

The parametric length formula,

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Length of a Polar Curve

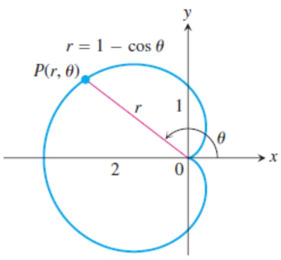
If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

Solution We sketch the cardioid to determine the limits of integration
The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.



With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$
$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$
$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi$$
$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$,

---

Area of a Surface of Revolution of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis ($y \ge 0$):

$$S = \int_{\alpha}^{\beta} 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

2. Revolution about the y-axis ($x \ge 0$):

$$S = \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

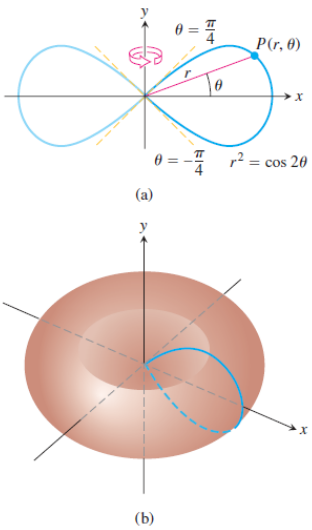


FIGURE The right-hand half of a lemniscate (a) is revolved about the y-axis to generate a surface (b),

EXAMPLE Finding Surface Area

Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution We sketch the loop to determine the limits of integration . The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand

$$2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi \cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}.$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$

Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$.

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.$$

All together, we have

$$\begin{aligned} S &= \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_{-\pi/4}^{\pi/4} 2\pi \cos\theta \cdot (1)\, d\theta \\ &= 2\pi\Big[\sin\theta\Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

CHAPTER 17

THOMAS' CALCULUS

Twelfth Edition

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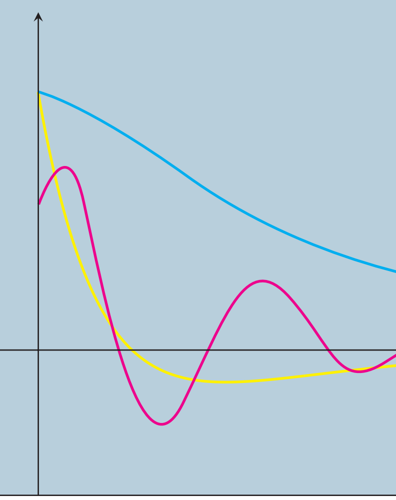


Chapter 17

SECOND-ORDER DIFFERENTIAL EQUATIONS

OVERVIEW In this chapter we extend our study of differential equations to those of *second order*. Second-order differential equations arise in many applications in the sciences and engineering. For instance, they can be applied to the study of vibrating springs and electric circuits. You will learn how to solve such differential equations by several methods in this chapter.

17.1 Second-Order Linear Equations

An equation of the form

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x), \tag{1}$$

which is linear in y and its derivatives, is called a **second-order linear differential equation**. We assume that the functions P, Q, R, and G are continuous throughout some open interval I. If $G(x)$ is identically zero on I, the equation is said to be **homogeneous**; otherwise it is called **nonhomogeneous**. Therefore, the form of a second-order linear homogeneous differential equation is

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{2}$$

We also assume that $P(x)$ is never zero for any $x \in I$.

Two fundamental results are important to solving Equation (2). The first of these says that if we know two solutions y_1 and y_2 of the linear homogeneous equation, then any **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

THEOREM 1—The Superposition Principle If $y_1(x)$ and $y_2(x)$ are two solutions to the linear homogeneous equation (2), then for any constants c_1 and c_2, the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution to Equation (2).

Proof Substituting y into Equation (2), we have

$$\begin{aligned}P(x)y'' + Q(x)y' + R(x)y &= P(x)(c_1y_1 + c_2y_2)'' + Q(x)(c_1y_1 + c_2y_2)' + R(x)(c_1y_1 + c_2y_2) \\ &= P(x)(c_1y_1'' + c_2y_2'') + Q(x)(c_1y_1' + c_2y_2') + R(x)(c_1y_1 + c_2y_2) \\ &= c_1\underbrace{(P(x)y_1'' + Q(x)y_1' + R(x)y_1)}_{=\,0,\ y_1 \text{ is a solution}} + c_2\underbrace{(P(x)y_2'' + Q(x)y_2' + R(x)y_2)}_{=\,0,\ y_2 \text{ is a solution}} \\ &= c_1(0) + c_2(0) = 0.\end{aligned}$$

Therefore, $y = c_1y_1 + c_2y_2$ is a solution of Equation (2). ■

Theorem 1 immediately establishes the following facts concerning solutions to the linear homogeneous equation.

1. A sum of two solutions $y_1 + y_2$ to Equation (2) is also a solution. (Choose $c_1 = c_2 = 1$.)
2. A constant multiple ky_1 of any solution y_1 to Equation (2) is also a solution. (Choose $c_1 = k$ and $c_2 = 0$.)
3. The **trivial solution** $y(x) \equiv 0$ is always a solution to the linear homogeneous equation. (Choose $c_1 = c_2 = 0$.)

The second fundamental result about solutions to the linear homogeneous equation concerns its **general solution** or solution containing all solutions. This result says that there are two solutions y_1 and y_2 such that any solution is some linear combination of them for suitable values of the constants c_1 and c_2. However, not just any pair of solutions will do. The solutions must be **linearly independent**, which means that neither y_1 nor y_2 is a constant multiple of the other. For example, the functions $f(x) = e^x$ and $g(x) = xe^x$ are linearly independent, whereas $f(x) = x^2$ and $g(x) = 7x^2$ are not (so they are linearly dependent). These results on linear independence and the following theorem are proved in more advanced courses.

THEOREM 2 If P, Q, and R are continuous over the open interval I and $P(x)$ is never zero on I, then the linear homogeneous equation (2) has two linearly independent solutions y_1 and y_2 on I. Moreover, if y_1 and y_2 are *any* two linearly independent solutions of Equation (2), then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants.

We now turn our attention to finding two linearly independent solutions to the special case of Equation (2), where P, Q, and R are constant functions.

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0, \tag{3}$$

where a, b, and c are constants. To solve Equation (3), we seek a function which when multiplied by a constant and added to a constant times its first derivative plus a constant times its second derivative sums identically to zero. One function that behaves this way is the exponential function $y = e^{rx}$, when r is a constant. Two differentiations of this exponential function give $y' = re^{rx}$ and $y'' = r^2e^{rx}$, which are just constant multiples of the original exponential. If we substitute $y = e^{rx}$ into Equation (3), we obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Since the exponential function is never zero, we can divide this last equation through by e^{rx}. Thus, $y = e^{rx}$ is a solution to Equation (3) if and only if r is a solution to the algebraic equation

$$ar^2 + br + c = 0. \tag{4}$$

Equation (4) is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. The auxiliary equation is a quadratic equation with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider which depend on the value of the discriminant $b^2 - 4ac$.

Case 1: $b^2 - 4ac > 0$. In this case the auxiliary equation has two real and unequal roots r_1 and r_2. Then $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ are two linearly independent solutions to Equation (3) because e^{r_2x} is not a constant multiple of e^{r_1x} (see Exercise 61). From Theorem 2 we conclude the following result.

THEOREM 3 If r_1 and r_2 are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{r_1x} + c_2e^{r_2x}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1e^{3x} + c_2e^{-2x}.$$

■

Case 2: $b^2 - 4ac = 0$. In this case $r_1 = r_2 = -b/2a$. To simplify the notation, let $r = -b/2a$. Then we have one solution $y_1 = e^{rx}$ with $2ar + b = 0$. Since multiplication of e^{rx} by a constant fails to produce a second linearly independent solution, suppose we try multiplying by a *function* instead. The simplest such function would be $u(x) = x$, so let's see if $y_2 = xe^{rx}$ is also a solution. Substituting y_2 into the differential equation gives

$$\begin{aligned} ay_2'' + by_2' + cy_2 &= a(2re^{rx} + r^2xe^{rx}) + b(e^{rx} + rxe^{rx}) + cxe^{rx} \\ &= (2ar + b)e^{rx} + (ar^2 + br + c)xe^{rx} \\ &= 0(e^{rx}) + (0)xe^{rx} = 0. \end{aligned}$$

The first term is zero because $r = -b/2a$; the second term is zero because r solves the auxiliary equation. The functions $y_1 = e^{rx}$ and $y_2 = xe^{rx}$ are linearly independent (see Exercise 62). From Theorem 2 we conclude the following result.

THEOREM 4 If r is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1e^{rx} + c_2xe^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1e^{-2x} + c_2xe^{-2x}.$$ ■

Case 3: $b^2 - 4ac < 0$. In this case the auxiliary equation has two complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real numbers and $i^2 = -1$. (These real numbers are $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.) These two complex roots then give rise to two linearly independent solutions

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos\beta x + i\sin\beta x) \quad \text{and} \quad y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos\beta x - i\sin\beta x).$$

(The expressions involving the sine and cosine terms follow from Euler's identity in Section 9.9.) However, the solutions y_1 and y_2 are *complex valued* rather than real valued. Nevertheless, because of the superposition principle (Theorem 1), we can obtain from them the two real-valued solutions

$$y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2 = e^{\alpha x}\cos\beta x \qquad \text{and} \qquad y_4 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 = e^{\alpha x}\sin\beta x.$$

The functions y_3 and y_4 are linearly independent (see Exercise 63). From Theorem 2 we conclude the following result.

THEOREM 5 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$ or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$ ■

Initial Value and Boundary Value Problems

To determine a unique solution to a first-order linear differential equation, it was sufficient to specify the value of the solution at a single point. Since the general solution to a second-order equation contains two arbitrary constants, it is necessary to specify two conditions. One way of doing this is to specify the value of the solution function and the value of its derivative at a single point: $y(x_0) = y_0$ and $y'(x_0) = y_1$. These conditions are called **initial conditions**. The following result is proved in more advanced texts and guarantees the existence of a unique solution for both homogeneous and nonhomogeneous second-order linear initial value problems.

THEOREM 6 If P, Q, R, and G are continuous throughout an open interval I, then there exists one and only one function $y(x)$ satisfying both the differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = G(x)$$

on the interval I, and the initial conditions

$$y(x_0) = y_0 \qquad \text{and} \qquad y'(x_0) = y_1$$

at the specified point $x_0 \in I$.

It is important to realize that any real values can be assigned to y_0 and y_1 and Theorem 6 applies. Here is an example of an initial value problem for a homogeneous equation.

EXAMPLE 4 Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$

Solution The auxiliary equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

The repeated real root is $r = 1$, giving the general solution

$$y = c_1 e^x + c_2 xe^x.$$

Then,

$$y' = c_1 e^x + c_2(x + 1)e^x.$$

From the initial conditions we have

$$1 = c_1 + c_2 \cdot 0 \qquad \text{and} \qquad -1 = c_1 + c_2 \cdot 1.$$

Thus, $c_1 = 1$ and $c_2 = -2$. The unique solution satisfying the initial conditions is

$$y = e^x - 2xe^x.$$

The solution curve is shown in Figure 17.1. ■

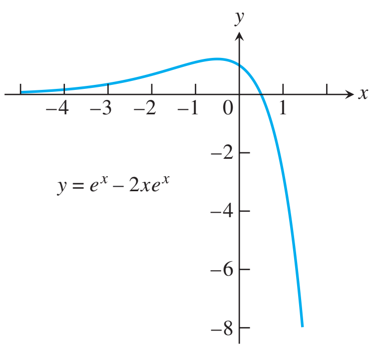


FIGURE 17.1 Particular solution curve for Example 4.

Another approach to determine the values of the two arbitrary constants in the general solution to a second-order differential equation is to specify the values of the solution function at *two different points* in the interval I. That is, we solve the differential equation subject to the **boundary values**

$$y(x_1) = y_1 \qquad \text{and} \qquad y(x_2) = y_2,$$

where x_1 and x_2 both belong to I. Here again the values for y_1 and y_2 can be any real numbers. The differential equation together with specified boundary values is called a **boundary value problem**. Unlike the result stated in Theorem 6, boundary value problems do not always possess a solution or more than one solution may exist (see Exercise 65). These problems are studied in more advanced texts, but here is an example for which there is a unique solution.

EXAMPLE 5 Solve the boundary value problem

$$y'' + 4y = 0, \qquad y(0) = 0, \quad y\left(\frac{\pi}{12}\right) = 1.$$

Solution The auxiliary equation is $r^2 + 4 = 0$, which has the complex roots $r = \pm 2i$. The general solution to the differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x.$$

The boundary conditions are satisfied if

$$y(0) = c_1 \cdot 1 + c_2 \cdot 0 = 0$$

$$y\left(\frac{\pi}{12}\right) = c_1 \cos\left(\frac{\pi}{6}\right) + c_2 \sin\left(\frac{\pi}{6}\right) = 1.$$

It follows that $c_1 = 0$ and $c_2 = 2$. The solution to the boundary value problem is

$$y = 2 \sin 2x.$$ ■

EXERCISES 17.1

In Exercises 1–30, find the general solution of the given equation.

1. $y'' - y' - 12y = 0$

2. $3y'' - y' = 0$

3. $y'' + 3y' - 4y = 0$

4. $y'' - 9y = 0$

5. $y'' - 4y = 0$

6. $y'' - 64y = 0$

7. $2y'' - y' - 3y = 0$

8. $9y'' - y = 0$

9. $8y'' - 10y' - 3y = 0$

10. $3y'' - 20y' + 12y = 0$

11. $y'' + 9y = 0$

12. $y'' + 4y' + 5y = 0$

13. $y'' + 25y = 0$

14. $y'' + y = 0$

15. $y'' - 2y' + 5y = 0$

16. $y'' + 16y = 0$

17. $y'' + 2y' + 4y = 0$

18. $y'' - 2y' + 3y = 0$

19. $y'' + 4y' + 9y = 0$

20. $4y'' - 4y' + 13y = 0$

21. $y'' = 0$

22. $y'' + 8y' + 16y = 0$

23. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

24. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

25. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$

26. $4\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 9y = 0$

27. $4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$

28. $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

29. $9\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + y = 0$

30. $9\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0$

In Exercises 31–40, find the unique solution of the second-order initial value problem.

31. $y'' + 6y' + 5y = 0, \quad y(0) = 0, \; y'(0) = 3$

32. $y'' + 16y = 0, \quad y(0) = 2, \; y'(0) = -2$

33. $y'' + 12y = 0, \quad y(0) = 0, \; y'(0) = 1$

34. $12y'' + 5y' - 2y = 0, \quad y(0) = 1, \; y'(0) = -1$

35. $y'' + 8y = 0, \quad y(0) = -1, \; y'(0) = 2$

36. $y'' + 4y' + 4y = 0, \quad y(0) = 0, \; y'(0) = 1$

37. $y'' - 4y' + 4y = 0, \quad y(0) = 1, \; y'(0) = 0$

38. $4y'' - 4y' + y = 0, \quad y(0) = 4, \; y'(0) = 4$

39. $4\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 9y = 0, \quad y(0) = 2, \; \dfrac{dy}{dx}(0) = 1$

40. $9\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0, \quad y(0) = -1, \; \dfrac{dy}{dx}(0) = 1$

In Exercises 41–55, find the general solution.

41. $y'' - 2y' - 3y = 0$

42. $6y'' - y' - y = 0$

43. $4y'' + 4y' + y = 0$

44. $9y'' + 12y' + 4y = 0$

45. $4y'' + 20y = 0$

46. $y'' + 2y' + 2y = 0$

47. $25y'' + 10y' + y = 0$

48. $6y'' + 13y' - 5y = 0$

49. $4y'' + 4y' + 5y = 0$

50. $y'' + 4y' + 6y = 0$

51. $16y'' - 24y' + 9y = 0$

52. $6y'' - 5y' - 6y = 0$

53. $9y'' + 24y' + 16y = 0$

54. $4y'' + 16y' + 52y = 0$

55. $6y'' - 5y' - 4y = 0$

In Exercises 56–60, solve the initial value problem.

56. $y'' - 2y' + 2y = 0, \quad y(0) = 0, \; y'(0) = 2$

57. $y'' + 2y' + y = 0, \quad y(0) = 1, \; y'(0) = 1$

58. $4y'' - 4y' + y = 0, \quad y(0) = -1, \; y'(0) = 2$

59. $3y'' + y' - 14y = 0, \quad y(0) = 2, \; y'(0) = -1$

60. $4y'' + 4y' + 5y = 0, \quad y(\pi) = 1, \; y'(\pi) = 0$

61. Prove that the two solution functions in Theorem 3 are linearly independent.

62. Prove that the two solution functions in Theorem 4 are linearly independent.

63. Prove that the two solution functions in Theorem 5 are linearly independent.

64. Prove that if y_1 and y_2 are linearly independent solutions to the homogeneous equation (2), then the functions $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ are also linearly independent solutions.

65. a. Show that there is no solution to the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, \; y(\pi) = 1.$$

b. Show that there are infinitely many solutions to the boundary value problem

$$y'' + 4y = 0, \quad y(0) = 0, \; y(\pi) = 0.$$

66. Show that if a, b, and c are positive constants, then all solutions of the homogeneous differential equation

$$ay'' + by' + cy = 0$$

approach zero as $x \to \infty$.

17.2 Nonhomogeneous Linear Equations

In this section we study two methods for solving second-order linear nonhomogeneous differential equations with constant coefficients. These are the methods of *undetermined coefficients* and *variation of parameters*. We begin by considering the form of the general solution.

Form of the General Solution

Suppose we wish to solve the nonhomogeneous equation

$$ay'' + by' + cy = G(x), \tag{1}$$

where a, b, and c are constants and G is continuous over some open interval I. Let $y_c = c_1y_1 + c_2y_2$ be the general solution to the associated **complementary equation**

$$ay'' + by' + cy = 0. \tag{2}$$

(We learned how to find y_c in Section 17.1.) Now suppose we could somehow come up with a particular function y_p that solves the nonhomogeneous equation (1). Then the sum

$$y = y_c + y_p \tag{3}$$

also solves the nonhomogeneous equation (1) because

$$\begin{aligned} a(y_c + y_p)'' &+ b(y_c + y_p)' + c(y_c + y_p) \\ &= (ay_c'' + by_c' + cy_c) + (ay_p'' + by_p' + cy_p) \\ &= 0 + G(x) \qquad \text{y_c solves Eq. (2) and y_p solves Eq. (1)} \\ &= G(x). \end{aligned}$$

Moreover, if $y = y(x)$ is the general solution to the nonhomogeneous equation (1), it must have the form of Equation (3). The reason for this last statement follows from the observation that for any function y_p satisfying Equation (1), we have

$$\begin{aligned} a(y - y_p)'' &+ b(y - y_p)' + c(y - y_p) \\ &= (ay'' + by' + cy) - (ay_p'' + by_p' + cy_p) \\ &= G(x) - G(x) = 0. \end{aligned}$$

Thus, $y_c = y - y_p$ is the general solution to the homogeneous equation (2). We have established the following result.

THEOREM 7 The general solution $y = y(x)$ to the nonhomogeneous differential equation (1) has the form

$$y = y_c + y_p,$$

where the **complementary solution** y_c is the general solution to the associated homogeneous equation (2) and y_p is any **particular solution** to the nonhomogeneous equation (1).

The Method of Undetermined Coefficients

This method for finding a particular solution y_p to the nonhomogeneous equation (1) applies to special cases for which $G(x)$ is a sum of terms of various polynomials $p(x)$ multiplying an exponential with possibly sine or cosine factors. That is, $G(x)$ is a sum of terms of the following forms:

$$p_1(x)e^{rx}, \qquad p_2(x)e^{\alpha x}\cos\beta x, \qquad p_3(x)e^{\alpha x}\sin\beta x.$$

For instance, $1 - x$, e^{2x}, xe^x, $\cos x$, and $5e^x - \sin 2x$ represent functions in this category. (Essentially these are functions solving homogeneous linear differential equations with constant coefficients, but the equations may be of order higher than two.) We now present several examples illustrating the method.

EXAMPLE 1 Solve the nonhomogeneous equation $y'' - 2y' - 3y = 1 - x^2$.

Solution The auxiliary equation for the complementary equation $y'' - 2y' - 3y = 0$ is

$$r^2 - 2r - 3 = (r + 1)(r - 3) = 0.$$

It has the roots $r = -1$ and $r = 3$ giving the complementary solution

$$y_c = c_1e^{-x} + c_2e^{3x}.$$

Now $G(x) = 1 - x^2$ is a polynomial of degree 2. It would be reasonable to assume that a particular solution to the given nonhomogeneous equation is also a polynomial of degree 2 because if y is a polynomial of degree 2, then $y'' - 2y' - 3y$ is also a polynomial of degree 2. So we seek a particular solution of the form

$$y_p = Ax^2 + Bx + C.$$

We need to determine the unknown coefficients A, B, and C. When we substitute the polynomial y_p and its derivatives into the given nonhomogeneous equation, we obtain

$$2A - 2(2Ax + B) - 3(Ax^2 + Bx + C) = 1 - x^2$$

or, collecting terms with like powers of x,

$$-3Ax^2 + (-4A - 3B)x + (2A - 2B - 3C) = 1 - x^2.$$

This last equation holds for all values of x if its two sides are identical polynomials of degree 2. Thus, we equate corresponding powers of x to get

$$-3A = -1, \qquad -4A - 3B = 0, \qquad \text{and} \qquad 2A - 2B - 3C = 1.$$

These equations imply in turn that $A = 1/3$, $B = -4/9$, and $C = 5/27$. Substituting these values into the quadratic expression for our particular solution gives

$$y_p = \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$

By Theorem 7, the general solution to the nonhomogeneous equation is

$$y = y_c + y_p = c_1e^{-x} + c_2e^{3x} + \frac{1}{3}x^2 - \frac{4}{9}x + \frac{5}{27}.$$ ■

EXAMPLE 2 Find a particular solution of $y'' - y' = 2 \sin x$.

Solution If we try to find a particular solution of the form

$$y_p = A \sin x$$

and substitute the derivatives of y_p in the given equation, we find that A must satisfy the equation

$$-A \sin x + A \cos x = 2 \sin x$$

for all values of x. Since this requires A to equal both -2 and 0 at the same time, we conclude that the nonhomogeneous differential equation has no solution of the form $A \sin x$.

It turns out that the required form is the sum

$$y_p = A \sin x + B \cos x.$$

The result of substituting the derivatives of this new trial solution into the differential equation is

$$-A \sin x - B \cos x - (A \cos x - B \sin x) = 2 \sin x$$

or

$$(B - A) \sin x - (A + B) \cos x = 2 \sin x.$$

This last equation must be an identity. Equating the coefficients for like terms on each side then gives

$$B - A = 2 \qquad \text{and} \qquad A + B = 0.$$

Simultaneous solution of these two equations gives $A = -1$ and $B = 1$. Our particular solution is

$$y_p = \cos x - \sin x.$$

■

EXAMPLE 3 Find a particular solution of $y'' - 3y' + 2y = 5e^x$.

Solution If we substitute

$$y_p = Ae^x$$

and its derivatives in the differential equation, we find that

$$Ae^x - 3Ae^x + 2Ae^x = 5e^x$$

or

$$0 = 5e^x.$$

However, the exponential function is never zero. The trouble can be traced to the fact that $y = e^x$ is already a solution of the related homogeneous equation

$$y'' - 3y' + 2y = 0.$$

The auxiliary equation is

$$r^2 - 3r + 2 = (r - 1)(r - 2) = 0,$$

which has $r = 1$ as a root. So we would expect Ae^x to become zero when substituted into the left-hand side of the differential equation.

The appropriate way to modify the trial solution in this case is to multiply Ae^x by x. Thus, our new trial solution is

$$y_p = Axe^x.$$

The result of substituting the derivatives of this new candidate into the differential equation is

$$(Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2Axe^x = 5e^x$$

or

$$-Ae^x = 5e^x.$$

Thus, $A = -5$ gives our sought-after particular solution

$$y_p = -5xe^x.$$ ■

EXAMPLE 4 Find a particular solution of $y'' - 6y' + 9y = e^{3x}$.

Solution The auxiliary equation for the complementary equation

$$r^2 - 6r + 9 = (r - 3)^2 = 0$$

has $r = 3$ as a repeated root. The appropriate choice for y_p in this case is neither Ae^{3x} nor Axe^{3x} because the complementary solution contains both of those terms already. Thus, we choose a term containing the next higher power of x as a factor. When we substitute

$$y_p = Ax^2e^{3x}$$

and its derivatives in the given differential equation, we get

$$(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}) - 6(3Ax^2e^{3x} + 2Axe^{3x}) + 9Ax^2e^{3x} = e^{3x}$$

or

$$2Ae^{3x} = e^{3x}.$$

Thus, $A = 1/2$, and the particular solution is

$$y_p = \frac{1}{2}x^2e^{3x}.$$ ■

When we wish to find a particular solution of Equation (1) and the function $G(x)$ is the sum of two or more terms, we choose a trial function for each term in $G(x)$ and add them.

EXAMPLE 5 Find the general solution to $y'' - y' = 5e^x - \sin 2x$.

Solution We first check the auxiliary equation

$$r^2 - r = 0.$$

Its roots are $r = 1$ and $r = 0$. Therefore, the complementary solution to the associated homogeneous equation is

$$y_c = c_1e^x + c_2.$$

We now seek a particular solution y_p. That is, we seek a function that will produce $5e^x - \sin 2x$ when substituted into the left-hand side of the given differential equation. One part of y_p is to produce $5e^x$, the other $-\sin 2x$.

Since any function of the form c_1e^x is a solution of the associated homogeneous equation, we choose our trial solution y_p to be the sum

$$y_p = Axe^x + B\cos 2x + C\sin 2x,$$

including xe^x where we might otherwise have included only e^x. When the derivatives of y_p are substituted into the differential equation, the resulting equation is

$$(Axe^x + 2Ae^x - 4B\cos 2x - 4C\sin 2x)$$
$$- (Axe^x + Ae^x - 2B\sin 2x + 2C\cos 2x) = 5e^x - \sin 2x$$

or

$$Ae^x - (4B + 2C)\cos 2x + (2B - 4C)\sin 2x = 5e^x - \sin 2x.$$

This equation will hold if

$$A = 5, \qquad 4B + 2C = 0, \qquad 2B - 4C = -1,$$

or $A = 5$, $B = -1/10$, and $C = 1/5$. Our particular solution is

$$y_p = 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$

The general solution to the differential equation is

$$y = y_c + y_p = c_1e^x + c_2 + 5xe^x - \frac{1}{10}\cos 2x + \frac{1}{5}\sin 2x.$$ ■

You may find the following table helpful in solving the problems at the end of this section.

TABLE 17.1 The method of undetermined coefficients for selected equations of the form

$$ay'' + by' + cy = G(x).$$

If $G(x)$ has a term that is a constant multiple of . . .	And if	Then include this expression in the trial function for y_p.
e^{rx}	r is not a root of the auxiliary equation	Ae^{rx}
	r is a single root of the auxiliary equation	Axe^{rx}
	r is a double root of the auxiliary equation	Ax^2e^{rx}
$\sin kx$, $\cos kx$	ki is not a root of the auxiliary equation	$B\cos kx + C\sin kx$
$px^2 + qx + m$	0 is not a root of the auxiliary equation	$Dx^2 + Ex + F$
	0 is a single root of the auxiliary equation	$Dx^3 + Ex^2 + Fx$
	0 is a double root of the auxiliary equation	$Dx^4 + Ex^3 + Fx^2$

The Method of Variation of Parameters

This is a general method for finding a particular solution of the nonhomogeneous equation (1) once the general solution of the associated homogeneous equation is known. The method consists of replacing the constants c_1 and c_2 in the complementary solution by functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ and requiring (in a way to be explained) that the resulting expression satisfy the nonhomogeneous equation (1). There are two functions to be determined, and requiring that Equation (1) be satisfied is only one condition. As a second condition, we also require that

$$v_1'y_1 + v_2'y_2 = 0. \tag{4}$$

Then we have

$$\begin{aligned} y &= v_1y_1 + v_2y_2, \\ y' &= v_1y_1' + v_2y_2', \\ y'' &= v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'. \end{aligned}$$

If we substitute these expressions into the left-hand side of Equation (1), we obtain

$$v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) + a(v_1'y_1' + v_2'y_2') = G(x).$$

The first two parenthetical terms are zero since y_1 and y_2 are solutions of the associated homogeneous equation (2). So the nonhomogeneous equation (1) is satisfied if, in addition to Equation (4), we require that

$$a(v_1'y_1' + v_2'y_2') = G(x). \tag{5}$$

Equations (4) and (5) can be solved together as a pair

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= \frac{G(x)}{a} \end{aligned}$$

for the unknown functions v_1' and v_2'. The usual procedure for solving this simple system is to use the *method of determinants* (also known as *Cramer's Rule*), which will be demonstrated in the examples to follow. Once the derivative functions v_1' and v_2' are known, the two functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$ can be found by integration. Here is a summary of the method.

Variation of Parameters Procedure

To use the method of variation of parameters to find a particular solution to the nonhomogeneous equation

$$ay'' + by' + cy = G(x),$$

we can work directly with Equations (4) and (5). It is not necessary to rederive them. The steps are as follows.

1. Solve the associated homogeneous equation

$$ay'' + by' + cy = 0$$

to find the functions y_1 and y_2.

2. Solve the equations

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= \frac{G(x)}{a} \end{aligned}$$

simultaneously for the derivative functions v_1' and v_2'.

3. Integrate v_1' and v_2' to find the functions $v_1 = v_1(x)$ and $v_2 = v_2(x)$.
4. Write down the particular solution to nonhomogeneous equation (1) as

$$y_p = v_1y_1 + v_2y_2.$$

EXAMPLE 6 Find the general solution to the equation

$$y'' + y = \tan x.$$

Solution The solution of the homogeneous equation

$$y'' + y = 0$$

is given by

$$y_c = c_1 \cos x + c_2 \sin x.$$

Since $y_1(x) = \cos x$ and $y_2(x) = \sin x$, the conditions to be satisfied in Equations (4) and (5) are

$$\begin{aligned} v_1' \cos x + v_2' \sin x &= 0, \\ -v_1' \sin x + v_2' \cos x &= \tan x. \qquad a = 1 \end{aligned}$$

Solution of this system gives

$$v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\tan x \sin x}{\cos^2 x + \sin^2 x} = \frac{-\sin^2 x}{\cos x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x.$$

After integrating v_1' and v_2', we have

$$\begin{aligned} v_1(x) &= \int \frac{-\sin^2 x}{\cos x}\, dx \\ &= -\int (\sec x - \cos x)\, dx \\ &= -\ln|\sec x + \tan x| + \sin x, \end{aligned}$$

and

$$v_2(x) = \int \sin x\, dx = -\cos x.$$

Note that we have omitted the constants of integration in determining v_1 and v_2. They would merely be absorbed into the arbitrary constants in the complementary solution.

Substituting v_1 and v_2 into the expression for y_p in Step 4 gives

$$\begin{aligned} y_p &= [-\ln|\sec x + \tan x| + \sin x] \cos x + (-\cos x) \sin x \\ &= (-\cos x) \ln|\sec x + \tan x|. \end{aligned}$$

The general solution is

$$y = c_1 \cos x + c_2 \sin x - (\cos x) \ln|\sec x + \tan x|.$$ ■

EXAMPLE 7 Solve the nonhomogeneous equation

$$y'' + y' - 2y = xe^x.$$

Solution The auxiliary equation is

$$r^2 + r - 2 = (r + 2)(r - 1) = 0$$

giving the complementary solution

$$y_c = c_1e^{-2x} + c_2e^x.$$

The conditions to be satisfied in Equations (4) and (5) are

$$\begin{aligned} v_1'e^{-2x} + v_2'e^x &= 0, \\ -2v_1'e^{-2x} + v_2'e^x &= xe^x. \qquad a = 1 \end{aligned}$$

Solving the above system for v_1' and v_2' gives

$$v_1' = \frac{\begin{vmatrix} 0 & e^x \\ xe^x & e^x \end{vmatrix}}{\begin{vmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{vmatrix}} = \frac{-xe^{2x}}{3e^{-x}} = -\frac{1}{3}xe^{3x}.$$

Likewise,

$$v_2' = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & xe^x \end{vmatrix}}{3e^{-x}} = \frac{xe^{-x}}{3e^{-x}} = \frac{x}{3}.$$

Integrating to obtain the parameter functions, we have

$$\begin{aligned} v_1(x) &= \int -\frac{1}{3}xe^{3x}\,dx \\ &= -\frac{1}{3}\left(\frac{xe^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx\right) \\ &= \frac{1}{27}(1 - 3x)e^{3x}, \end{aligned}$$

and

$$v_2(x) = \int \frac{x}{3}\,dx = \frac{x^2}{6}.$$

Therefore,

$$\begin{aligned} y_p &= \left[\frac{(1 - 3x)e^{3x}}{27}\right]e^{-2x} + \left(\frac{x^2}{6}\right)e^x \\ &= \frac{1}{27}e^x - \frac{1}{9}xe^x + \frac{1}{6}x^2e^x. \end{aligned}$$

The general solution to the differential equation is

$$y = c_1e^{-2x} + c_2e^x - \frac{1}{9}xe^x + \frac{1}{6}x^2e^x,$$

where the term $(1/27)e^x$ in y_p has been absorbed into the term c_2e^x in the complementary solution. ■

EXERCISES 17.2

Solve the equations in Exercises 1–16 by the method of undetermined coefficients.

1. $y'' - 3y' - 10y = -3$
2. $y'' - 3y' - 10y = 2x - 3$
3. $y'' - y' = \sin x$
4. $y'' + 2y' + y = x^2$
5. $y'' + y = \cos 3x$
6. $y'' + y = e^{2x}$
7. $y'' - y' - 2y = 20 \cos x$
8. $y'' + y = 2x + 3e^x$
9. $y'' - y = e^x + x^2$
10. $y'' + 2y' + y = 6 \sin 2x$
11. $y'' - y' - 6y = e^{-x} - 7 \cos x$
12. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x$
13. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} = 15x^2$
14. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = -8x + 3$
15. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = e^{3x} - 12x$
16. $\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} = 42x^2 + 5x + 1$

Solve the equations in Exercises 17–28 by variation of parameters.

17. $y'' + y' = x$
18. $y'' + y = \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
19. $y'' + y = \sin x$
20. $y'' + 2y' + y = e^x$
21. $y'' + 2y' + y = e^{-x}$
22. $y'' - y = x$
23. $y'' - y = e^x$
24. $y'' - y = \sin x$
25. $y'' + 4y' + 5y = 10$
26. $y'' - y' = 2^x$
27. $\dfrac{d^2y}{dx^2} + y = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
28. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x \cos x, \quad x > 0$

In each of Exercises 29–32, the given differential equation has a particular solution y_p of the form given. Determine the coefficients in y_p. Then solve the differential equation.

29. $y'' - 5y' = xe^{5x}, \quad y_p = Ax^2e^{5x} + Bxe^{5x}$
30. $y'' - y' = \cos x + \sin x, \quad y_p = A \cos x + B \sin x$
31. $y'' + y = 2 \cos x + \sin x, \quad y_p = Ax \cos x + Bx \sin x$
32. $y'' + y' - 2y = xe^x, \quad y_p = Ax^2e^x + Bxe^x$

In Exercises 33–36, solve the given differential equations **(a)** by variation of parameters and **(b)** by the method of undetermined coefficients.

33. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x + e^{-x}$
34. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 2e^{2x}$
35. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = e^x + 4$
36. $\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} = 9e^{9x}$

Solve the differential equations in Exercises 37–46. Some of the equations can be solved by the method of undetermined coefficients, but others cannot.

37. $y'' + y = \cot x, \quad 0 < x < \pi$
38. $y'' + y = \csc x, \quad 0 < x < \pi$
39. $y'' - 8y' = e^{8x}$
40. $y'' + 4y = \sin x$
41. $y'' - y' = x^3$
42. $y'' + 4y' + 5y = x + 2$
43. $y'' + 2y' = x^2 - e^x$
44. $y'' + 9y = 9x - \cos x$
45. $y'' + y = \sec x \tan x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$
46. $y'' - 3y' + 2y = e^x - e^{2x}$

The method of undetermined coefficients can sometimes be used to solve first-order ordinary differential equations. Use the method to solve the equations in Exercises 47–50.

47. $y' - 3y = e^x$
48. $y' + 4y = x$
49. $y' - 3y = 5e^{3x}$
50. $y' + y = \sin x$

Solve the differential equations in Exercises 51 and 52 subject to the given initial conditions.

51. $\dfrac{d^2y}{dx^2} + y = \sec^2 x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}; \quad y(0) = y'(0) = 1$
52. $\dfrac{d^2y}{dx^2} + y = e^{2x}; \quad y(0) = 0, \ y'(0) = \dfrac{2}{5}$

In Exercises 53–58, verify that the given function is a particular solution to the specified nonhomogeneous equation. Find the general solution and evaluate its arbitrary constants to find the unique solution satisfying the equation and the given initial conditions.

53. $y'' + y' = x, \quad y_p = \dfrac{x^2}{2} - x, \quad y(0) = 0, \ y'(0) = 0$
54. $y'' + y = x, \quad y_p = 2 \sin x + x, \quad y(0) = 0, \ y'(0) = 0$
55. $\dfrac{1}{2}y'' + y' + y = 4e^x(\cos x - \sin x),$
    $y_p = 2e^x \cos x, \quad y(0) = 0, \ y'(0) = 1$
56. $y'' - y' - 2y = 1 - 2x, \quad y_p = x - 1, \quad y(0) = 0, \ y'(0) = 1$
57. $y'' - 2y' + y = 2e^x, \quad y_p = x^2e^x, \quad y(0) = 1, \ y'(0) = 0$
58. $y'' - 2y' + y = x^{-1}e^x, \ x > 0,$
    $y_p = xe^x \ln x, \quad y(1) = e, \ y'(1) = 0$

In Exercises 59 and 60, two linearly independent solutions y_1 and y_2 are given to the associated homogeneous equation of the variable-coefficient nonhomogeneous equation. Use the method of variation of parameters to find a particular solution to the nonhomogeneous equation. Assume $x > 0$ in each exercise.

59. $x^2y'' + 2xy' - 2y = x^2, \quad y_1 = x^{-2}, \ y_2 = x$
60. $x^2y'' + xy' - y = x, \quad y_1 = x^{-1}, \ y_2 = x$

17.3 Applications

In this section we apply second-order differential equations to the study of vibrating springs and electric circuits.

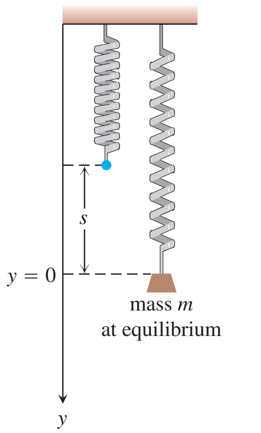


FIGURE 17.2 Mass m stretches a spring by length s to the equilibrium position at $y = 0$.

Vibrations

A spring has its upper end fastened to a rigid support, as shown in Figure 17.2. An object of mass m is suspended from the spring and stretches it a length s when the spring comes to rest in an equilibrium position. According to Hooke's Law (Section 6.5), the tension force in the spring is ks, where k is the spring constant. The force due to gravity pulling down on the spring is mg, and equilibrium requires that

$$ks = mg. \tag{1}$$

Suppose that the object is pulled down an additional amount y_0 beyond the equilibrium position and then released. We want to study the object's motion, that is, the vertical position of its center of mass at any future time.

Let y, with positive direction downward, denote the displacement position of the object away from the equilibrium position $y = 0$ at any time t after the motion has started. Then the forces acting on the object are (see Figure 17.3)

$F_p = mg$, the propulsion force due to gravity,

$F_s = k(s + y)$, the restoring force of the spring's tension,

$F_r = \delta \dfrac{dy}{dt}$, a frictional force assumed proportional to velocity.

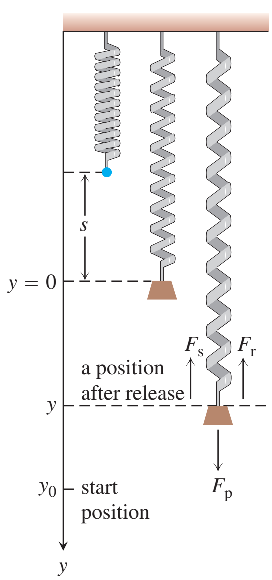


FIGURE 17.3 The propulsion force (weight) F_p pulls the mass downward, but the spring restoring force F_s and frictional force F_r pull the mass upward. The motion starts at $y = y_0$ with the mass vibrating up and down.

The frictional force tends to retard the motion of the object. The resultant of these forces is $F = F_p - F_s - F_r$, and by Newton's second law $F = ma$, we must then have

$$m\frac{d^2y}{dt^2} = mg - ks - ky - \delta\frac{dy}{dt}.$$

By Equation (1), $mg - ks = 0$, so this last equation becomes

$$m\frac{d^2y}{dt^2} + \delta\frac{dy}{dt} + ky = 0, \tag{2}$$

subject to the initial conditions $y(0) = y_0$ and $y'(0) = 0$. (Here we use the prime notation to denote differentiation with respect to time t.)

You might expect that the motion predicted by Equation (2) will be oscillatory about the equilibrium position $y = 0$ and eventually damp to zero because of the retarding frictional force. This is indeed the case, and we will show how the constants m, δ, and k determine the nature of the damping. You will also see that if there is no friction (so $\delta = 0$), then the object will simply oscillate indefinitely.

Simple Harmonic Motion

Suppose first that there is no retarding frictional force. Then $\delta = 0$ and there is no damping. If we substitute $\omega = \sqrt{k/m}$ to simplify our calculations, then the second-order equation (2) becomes

$$y'' + \omega^2 y = 0, \quad \text{with} \quad y(0) = y_0 \quad \text{and} \quad y'(0) = 0.$$

The auxiliary equation is

$$r^2 + \omega^2 = 0,$$

having the imaginary roots $r = \pm\omega i$. The general solution to the differential equation in (2) is

$$y = c_1 \cos \omega t + c_2 \sin \omega t. \tag{3}$$

To fit the initial conditions, we compute

$$y' = -c_1\omega \sin \omega t + c_2\omega \cos \omega t$$

and then substitute the conditions. This yields $c_1 = y_0$ and $c_2 = 0$. The particular solution

$$y = y_0 \cos \omega t \tag{4}$$

describes the motion of the object. Equation (4) represents **simple harmonic motion** of amplitude y_0 and period $T = 2\pi/\omega$.

The general solution given by Equation (3) can be combined into a single term by using the trigonometric identity

$$\sin(\omega t + \phi) = \cos \omega t \sin \phi + \sin \omega t \cos \phi.$$

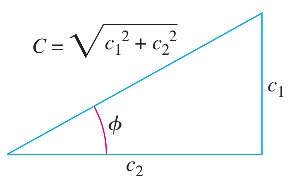


FIGURE 17.4 $c_1 = C \sin \phi$ and $c_2 = C \cos \phi$.

To apply the identity, we take (see Figure 17.4)

$$c_1 = C \sin \phi \qquad \text{and} \qquad c_2 = C \cos \phi,$$

where

$$C = \sqrt{c_1^2 + c_2^2} \qquad \text{and} \qquad \phi = \tan^{-1} \frac{c_1}{c_2}.$$

Then the general solution in Equation (3) can be written in the alternative form

$$y = C \sin(\omega t + \phi). \tag{5}$$

Here C and ϕ may be taken as two new arbitrary constants, replacing the two constants c_1 and c_2. Equation (5) represents simple harmonic motion of amplitude C and period $T = 2\pi/\omega$. The angle $\omega t + \phi$ is called the **phase angle**, and ϕ may be interpreted as its initial value. A graph of the simple harmonic motion represented by Equation (5) is given in Figure 17.5.

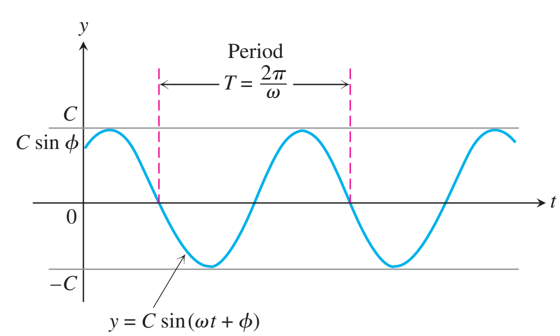


FIGURE 17.5 Simple harmonic motion of amplitude C and period T with initial phase angle ϕ (Equation 5).

Damped Motion

Assume now that there is friction in the spring system, so $\delta \neq 0$. If we substitute $\omega = \sqrt{k/m}$ and $2b = \delta/m$, then the differential equation (2) is

$$y'' + 2by' + \omega^2 y = 0. \tag{6}$$

The auxiliary equation is

$$r^2 + 2br + \omega^2 = 0,$$

with roots $r = -b \pm \sqrt{b^2 - \omega^2}$. Three cases now present themselves, depending upon the relative sizes of b and ω.

Case 1: $b = \omega$. The double root of the auxiliary equation is real and equals $r = \omega$. The general solution to Equation (6) is

$$y = (c_1 + c_2 t)e^{-\omega t}.$$

This situation of motion is called **critical damping** and is not oscillatory. Figure 17.6a shows an example of this kind of damped motion.

Case 2: $b > \omega$. The roots of the auxiliary equation are real and unequal, given by $r_1 = -b + \sqrt{b^2 - \omega^2}$ and $r_2 = -b - \sqrt{b^2 - \omega^2}$. The general solution to Equation (6) is given by

$$y = c_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}.$$

Here again the motion is not oscillatory and both r_1 and r_2 are negative. Thus y approaches zero as time goes on. This motion is referred to as **overdamping** (see Figure 17.6b).

Case 3: $b < \omega$. The roots to the auxiliary equation are complex and given by $r = -b \pm i\sqrt{\omega^2 - b^2}$. The general solution to Equation (6) is given by

$$y = e^{-bt}\left(c_1 \cos\sqrt{\omega^2 - b^2}\,t + c_2 \sin\sqrt{\omega^2 - b^2}\,t\right).$$

This situation, called **underdamping**, represents damped oscillatory motion. It is analogous to simple harmonic motion of period $T = 2\pi/\sqrt{\omega^2 - b^2}$ except that the amplitude is not constant but damped by the factor e^{-bt}. Therefore, the motion tends to zero as t increases, so the vibrations tend to die out as time goes on. Notice that the period $T = 2\pi/\sqrt{\omega^2 - b^2}$ is larger than the period $T_0 = 2\pi/\omega$ in the friction-free system. Moreover, the larger the value of $b = \delta/2m$ in the exponential damping factor, the more quickly the vibrations tend to become unnoticeable. A curve illustrating underdamped motion is shown in Figure 17.6c.

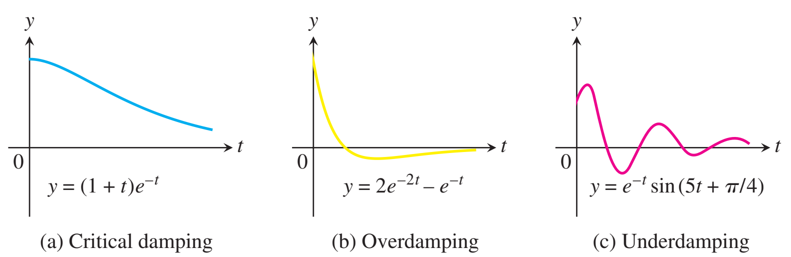


FIGURE 17.6 Three examples of damped vibratory motion for a spring system with friction, so $\delta \neq 0$.

An external force $F(t)$ can also be added to the spring system modeled by Equation (2). The forcing function may represent an external disturbance on the system. For instance, if the equation models an automobile suspension system, the forcing function might represent periodic bumps or potholes in the road affecting the performance of the suspension system; or it might represent the effects of winds when modeling the vertical motion of a suspension bridge. Inclusion of a forcing function results in the second-order nonhomogeneous equation

$$m\frac{d^2y}{dt^2} + \delta\frac{dy}{dt} + ky = F(t). \tag{7}$$

We leave the study of such spring systems to a more advanced course.

Electric Circuits

The basic quantity in electricity is the **charge** q (analogous to the idea of mass). In an electric field we use the flow of charge, or **current** $I = dq/dt$, as we might use velocity in a gravitational field. There are many similarities between motion in a gravitational field and the flow of electrons (the carriers of charge) in an electric field.

Consider the electric circuit shown in Figure 17.7. It consists of four components: voltage source, resistor, inductor, and capacitor. Think of electrical flow as being like a fluid flow, where the voltage source is the pump and the resistor, inductor, and capacitor tend to block the flow. A battery or generator is an example of a source, producing a voltage that causes the current to flow through the circuit when the switch is closed. An electric light bulb or appliance would provide resistance. The inductance is due to a magnetic field that opposes any change in the current as it flows through a coil. The capacitance is normally created by two metal plates that alternate charges and thus reverse the current flow. The following symbols specify the quantities relevant to the circuit:

- q: charge at a cross section of a conductor measured in **coulombs** (abbreviated c);
- I: current or rate of change of charge dq/dt (flow of electrons) at a cross section of a conductor measured in **amperes** (abbreviated A);
- E: electric (potential) source measured in **volts** (abbreviated V);
- V: difference in potential between two points along the conductor measured in **volts** (V).

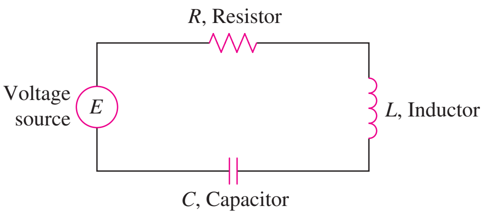


FIGURE 17.7 An electric circuit.

Ohm observed that the current I flowing through a resistor, caused by a potential difference across it, is (approximately) proportional to the potential difference (voltage drop). He named his constant of proportionality $1/R$ and called R the **resistance**. So *Ohm's law* is

$$I = \frac{1}{R}V.$$

Similarly, it is known from physics that the voltage drops across an inductor and a capacitor are

$$L\,\frac{dI}{dt} \qquad \text{and} \qquad \frac{q}{C},$$

where L is the **inductance** and C is the **capacitance** (with q the charge on the capacitor).

The German physicist Gustav R. Kirchhoff (1824–1887) formulated the law that the sum of the voltage drops in a closed circuit is equal to the supplied voltage $E(t)$. Symbolically, this says that

$$RI + L\,\frac{dI}{dt} + \frac{q}{C} = E(t).$$

Since $I = dq/dt$, Kirchhoff's law becomes

$$L\,\frac{d^2q}{dt^2} + R\,\frac{dq}{dt} + \frac{1}{C}\,q = E(t). \tag{8}$$

The second-order differential equation (8), which models an electric circuit, has exactly the same form as Equation (7) modeling vibratory motion. Both models can be solved using the methods developed in Section 17.2.

Summary

The following chart summarizes our analogies for the physics of motion of an object in a spring system versus the flow of charged particles in an electrical circuit.

Linear Second-Order Constant-Coefficient Models

Mechanical System		**Electrical System**	
$my'' + \delta y' + ky = F(t)$		$Lq'' + Rq' + \frac{1}{C}q = E(t)$	
y:	displacement	q:	charge
y':	velocity	q':	current
y'':	acceleration	q'':	change in current
m:	mass	L:	inductance
δ:	damping constant	R:	resistance
k:	spring constant	$1/C$:	where C is the capacitance
$F(t)$:	forcing function	$E(t)$:	voltage source

EXERCISES 17.3

1. A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 1 lb/ft. The resistance in the spring–mass system is numerically equal to the instantaneous velocity. At $t = 0$ the weight is set in motion from a position 2 ft below its equilibrium position by giving it a downward velocity of 2 ft/sec. Write an initial value problem that models the given situation.

2. An 8-lb weight stretches a spring 4 ft. The spring–mass system resides in a medium offering a resistance to the motion that is numerically equal to 1.5 times the instantaneous velocity. If the weight is released at a position 2 ft above its equilibrium position with a downward velocity of 3 ft/sec, write an initial value problem modeling the given situation.

3. A 20-lb weight is hung on an 18-in. spring and stretches it 6 in. The weight is pulled down 5 in. and 5 lb are added to the weight. If the weight is now released with a downward velocity of v_0 in./sec, write an initial value problem modeling the vertical displacement.

4. A 10-lb weight is suspended by a spring that is stretched 2 in. by the weight. Assume a resistance whose magnitude is $20/\sqrt{g}$ lb times the instantaneous velocity v in feet per second. If the weight is pulled down 3 in. below its equilibrium position and released, formulate an initial value problem modeling the behavior of the spring–mass system.

5. An (open) electrical circuit consists of an inductor, a resistor, and a capacitor. There is an initial charge of 2 coulombs on the capacitor. At the instant the circuit is closed, a current of 3 amperes is present and a voltage of $E(t) = 20 \cos t$ is applied. In this circuit the voltage drop across the resistor is 4 times the instantaneous change in the charge, the voltage drop across the capacitor is 10 times the charge, and the voltage drop across the inductor is 2 times the instantaneous change in the current. Write an initial value problem to model the circuit.

6. An inductor of 2 henrys is connected in series with a resistor of 12 ohms, a capacitor of $1/16$ farad, and a 300 volt battery. Initially, the charge on the capacitor is zero and the current is zero. Formulate an initial value problem modeling this electrical circuit.

Mechanical units in the British and metric systems may be helpful in doing the following problems.

Unit	British System	MKS System
Distance	Feet (ft)	Meters (m)
Mass	Slugs	Kilograms (kg)
Time	Seconds (sec)	Seconds (sec)
Force	Pounds (lb)	Newtons (N)
g(earth)	32 ft/sec^2	9.81 m/sec^2

7. A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling and having a spring constant of 1 lb/ft. The resistance in the spring–mass system is numerically equal to the instantaneous velocity. At $t = 0$ the weight is set in motion from a position 2 ft below its equilibrium position by giving it a downward velocity of 2 ft/sec. At the end of π sec, determine whether the mass is above or below the equilibrium position and by what distance.

8. An 8-lb weight stretches a spring 4 ft. The spring–mass system resides in a medium offering a resistance to the motion equal to 1.5 times the instantaneous velocity. If the weight is released at a position 2 ft above its equilibrium position with a downward velocity of 3 ft/sec, find its position relative to the equilibrium position 2 sec later.

9. A 20-lb weight is hung on an 18-in. spring stretching it 6 in. The weight is pulled down 5 in. and 5 lb are added to the weight. If the weight is now released with a downward velocity of v_0 in./sec, find the position of mass relative to the equilibrium in terms of v_0 and valid for any time $t \geq 0$.

10. A mass of 1 slug is attached to a spring whose constant is $25/4$ lb/ft. Initially the mass is released 1 ft above the equilibrium position with a downward velocity of 3 ft/sec, and the subsequent motion takes place in a medium that offers a damping force numerically equal to 3 times the instantaneous velocity. An external force $f(t)$ is driving the system, but assume that initially $f(t) \equiv 0$. Formulate and solve an initial value problem that models the given system. Interpret your results.

11. A 10-lb weight is suspended by a spring that is stretched 2 in. by the weight. Assume a resistance whose magnitude is $40/\sqrt{g}$ lb times the instantaneous velocity in feet per second. If the weight is pulled down 3 in. below its equilibrium position and released, find the time required to reach the equilibrium position for the first time.

12. A weight stretches a spring 6 in. It is set in motion at a point 2 in. below its equilibrium position with a downward velocity of 2 in./sec.
    - **a.** When does the weight return to its starting position?
    - **b.** When does it reach its highest point?
    - **c.** Show that the maximum velocity is $2\sqrt{2g + 1}$ in./sec.

13. A weight of 10 lb stretches a spring 10 in. The weight is drawn down 2 in. below its equilibrium position and given an initial velocity of 4 in./sec. An identical spring has a different weight attached to it. This second weight is drawn down from its equilibrium position a distance equal to the amplitude of the first motion and then given an initial velocity of 2 ft/sec. If the amplitude of the second motion is twice that of the first, what weight is attached to the second spring?

14. A weight stretches one spring 3 in. and a second weight stretches another spring 9 in. If both weights are simultaneously pulled down 1 in. below their respective equilibrium positions and then released, find the first time after $t = 0$ when their velocities are equal.

15. A weight of 16 lb stretches a spring 4 ft. The weight is pulled down 5 ft below the equilibrium position and then released. What initial velocity v_0 given to the weight would have the effect of doubling the amplitude of the vibration?

16. A mass weighing 8 lb stretches a spring 3 in. The spring–mass system resides in a medium with a damping constant of 2 lb-sec/ft. If the mass is released from its equilibrium position with a velocity of 4 in./sec in the downward direction, find the time required for the mass to return to its equilibrium position for the first time.

17. A weight suspended from a spring executes damped vibrations with a period of 2 sec. If the damping factor decreases by 90% in 10 sec, find the acceleration of the weight when it is 3 in. below its equilibrium position and is moving upward with a speed of 2 ft/sec.

18. A 10-lb weight stretches a spring 2 ft. If the weight is pulled down 6 in. below its equilibrium position and released, find the highest point reached by the weight. Assume the spring–mass system resides in a medium offering a resistance of $10/\sqrt{g}$ lb times the instantaneous velocity in feet per second.

19. An *LRC* circuit is set up with an inductance of $1/5$ henry, a resistance of 1 ohm, and a capacitance of $5/6$ farad. Assuming the initial charge is 2 coulombs and the initial current is 4 amperes, find the solution function describing the charge on the capacitor at any time. What is the charge on the capacitor after a long period of time?

20. An (open) electrical circuit consists of an inductor, a resistor, and a capacitor. There is an initial charge of 2 coulombs on the capacitor. At the instant the circuit is closed, a current of 3 amperes is present but no external voltage is being applied. In this circuit the voltage drops at three points are numerically related as follows: across the capacitor, 10 times the charge; across the resistor, 4 times the instantaneous change in the charge; and across the inductor, 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time.

21. A 16-lb weight stretches a spring 4 ft. This spring–mass system is in a medium with a damping constant of 4.5 lb-sec/ft, and an external force given by $f(t) = 4 + e^{-2t}$ (in pounds) is being applied. What is the solution function describing the position of the mass at any time if the mass is released from 2 ft below the equilibrium position with an initial velocity of 4 ft/sec downward?

22. A 10-kg mass is attached to a spring having a spring constant of 140 N/m. The mass is started in motion from the equilibrium position with an initial velocity of 1 m/sec in the upward direction and with an applied external force given by $f(t) = 5 \sin t$ (in newtons). The mass is in a viscous medium with a coefficient of resistance equal to 90 N-sec/m. Formulate an initial value problem that models the given system; solve the model and interpret the results.

23. A 2-kg mass is attached to the lower end of a coil spring suspended from the ceiling. The mass comes to rest in its equilibrium position thereby stretching the spring 1.96 m. The mass is in a viscous medium that offers a resistance in newtons numerically equal to 4 times the instantaneous velocity measured in meters per second. The mass is then pulled down 2 m below its equilibrium position and released with a downward velocity of 3 m/sec. At this same instant an external force given by $f(t) = 20 \cos t$ (in newtons) is applied to the system. At the end of π sec determine if the mass is above or below its equilibrium position and by how much.

24. An 8-lb weight stretches a spring 4 ft. The spring–mass system resides in a medium offering a resistance to the motion equal to 1.5 times the instantaneous velocity, and an external force given by $f(t) = 6 + e^{-t}$ (in pounds) is being applied. If the weight is released at a position 2 ft above its equilibrium position with downward velocity of 3 ft/sec, find its position relative to the equilibrium after 2 sec have elapsed.

25. Suppose $L = 10$ henrys, $R = 10$ ohms, $C = 1/500$ farads, $E = 100$ volts, $q(0) = 10$ coulombs, and $q'(0) = i(0) = 0$. Formulate and solve an initial value problem that models the given *LRC* circuit. Interpret your results.

26. A series circuit consisting of an inductor, a resistor, and a capacitor is open. There is an initial charge of 2 coulombs on the capacitor, and 3 amperes of current is present in the circuit at the instant the circuit is closed. A voltage given by $E(t) = 20 \cos t$ is applied. In this circuit the voltage drops are numerically equal to the following: across the resistor to 4 times the instantaneous change in the charge, across the capacitor to 10 times the charge, and across the inductor to 2 times the instantaneous change in the current. Find the charge on the capacitor as a function of time. Determine the charge on the capacitor and the current at time $t = 10$.

17.4 Euler Equations

In Section 17.1 we introduced the second-order linear homogeneous differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = 0$$

and showed how to solve this equation when the coefficients P, Q, and R are constants. If the coefficients are not constant, we cannot generally solve this differential equation in terms of elementary functions we have studied in calculus. In this section you will learn how to solve the equation when the coefficients have the special forms

$$P(x) = ax^2, \qquad Q(x) = bx, \qquad \text{and} \qquad R(x) = c,$$

where a, b, and c are constants. These special types of equations are called **Euler equations**, in honor of Leonhard Euler who studied them and showed how to solve them. Such equations arise in the study of mechanical vibrations.

The General Solution of Euler Equations

Consider the Euler equation

$$ax^2y'' + bxy' + cy = 0, \quad x > 0. \tag{1}$$

To solve Equation (1), we first make the change of variables

$$z = \ln x \qquad \text{and} \qquad y(x) = Y(z).$$

We next use the chain rule to find the derivatives $y'(x)$ and $y''(x)$:

$$y'(x) = \frac{d}{dx}Y(z) = \frac{d}{dz}Y(z)\frac{dz}{dx} = Y'(z)\frac{1}{x}$$

and

$$y''(x) = \frac{d}{dx}y'(x) = \frac{d}{dx}Y'(z)\frac{1}{x} = -\frac{1}{x^2}Y'(z) + \frac{1}{x}Y''(z)\frac{dz}{dx} = -\frac{1}{x^2}Y'(z) + \frac{1}{x^2}Y''(z).$$

Substituting these two derivatives into the left-hand side of Equation (1), we find

$$ax^2y'' + bxy' + cy = ax^2\left(-\frac{1}{x^2}Y'(z) + \frac{1}{x^2}Y''(z)\right) + bx\left(\frac{1}{x}Y'(z)\right) + cY(z)$$
$$= aY''(z) + (b - a)Y'(z) + cY(z).$$

Therefore, the substitutions give us the second-order linear differential equation with constant coefficients

$$aY''(z) + (b - a)Y'(z) + cY(z) = 0. \tag{2}$$

We can solve Equation (2) using the method of Section 17.1. That is, we find the roots to the associated auxiliary equation

$$ar^2 + (b - a)r + c = 0 \tag{3}$$

to find the general solution for $Y(z)$. After finding $Y(z)$, we can determine $y(x)$ from the substitution $z = \ln x$.

EXAMPLE 1 Find the general solution of the equation $x^2y'' + 2xy' - 2y = 0$.

Solution This is an Euler equation with $a = 1$, $b = 2$, and $c = -2$. The auxiliary equation (3) for $Y(z)$ is

$$r^2 + (2 - 1)r - 2 = (r - 1)(r + 2) = 0,$$

with roots $r = -2$ and $r = 1$. The solution for $Y(z)$ is given by

$$Y(z) = c_1e^{-2z} + c_2e^{z}.$$

Substituting $z = \ln x$ gives the general solution for $y(x)$:

$$y(x) = c_1e^{-2\ln x} + c_2e^{\ln x} = c_1x^{-2} + c_2x$$ ■

EXAMPLE 2 Solve the Euler equation $x^2y'' - 5xy' + 9y = 0$.

Solution Since $a = 1$, $b = -5$, and $c = 9$, the auxiliary equation (3) for $Y(z)$ is

$$r^2 + (-5 - 1)r + 9 = (r - 3)^2 = 0.$$

The auxiliary equation has the double root $r = 3$ giving

$$Y(z) = c_1e^{3z} + c_2ze^{3z}.$$

Substituting $z = \ln x$ into this expression gives the general solution

$$y(x) = c_1e^{3\ln x} + c_2\ln x\, e^{3\ln x} = c_1x^3 + c_2x^3\ln x$$ ■

EXAMPLE 3 Find the particular solution to $x^2y'' - 3xy' + 68y = 0$ that satisfies the initial conditions $y(1) = 0$ and $y'(1) = 1$.

Solution Here $a = 1$, $b = -3$, and $c = 68$ substituted into the auxiliary equation (3) gives

$$r^2 - 4r + 68 = 0.$$

The roots are $r = 2 + 8i$ and $r = 2 - 8i$ giving the solution

$$Y(z) = e^{2z}(c_1 \cos 8z + c_2 \sin 8z).$$

Substituting $z = \ln x$ into this expression gives

$$y(x) = e^{2\ln x}\left(c_1 \cos(8 \ln x) + c_2 \sin(8 \ln x)\right).$$

From the initial condition $y(1) = 0$, we see that $c_1 = 0$ and

$$y(x) = c_2 x^2 \sin(8 \ln x).$$

To fit the second initial condition, we need the derivative

$$y'(x) = c_2\left(8x \cos(8 \ln x) + 2x \sin(8 \ln x)\right).$$

Since $y'(1) = 1$, we immediately obtain $c_2 = 1/8$. Therefore, the particular solution satisfying both initial conditions is

$$y(x) = \frac{1}{8}x^2 \sin(8 \ln x).$$

Since $-1 \le \sin(8 \ln x) \le 1$, the solution satisfies

$$-\frac{x^2}{8} \le y(x) \le \frac{x^2}{8}.$$

A graph of the solution is shown in Figure 17.8. ■

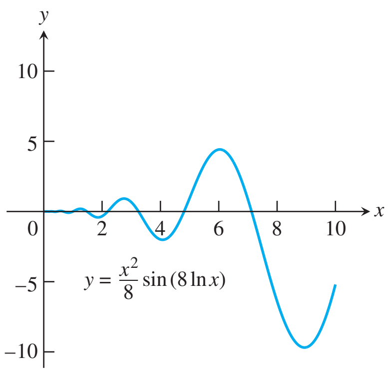


FIGURE 17.8 Graph of the solution to Example 3.

EXERCISES 17.4

In Exercises 1–24, find the general solution to the given Euler equation. Assume $x > 0$ throughout.

1. $x^2y'' + 2xy' - 2y = 0$
2. $x^2y'' + xy' - 4y = 0$
3. $x^2y'' - 6y = 0$
4. $x^2y'' + xy' - y = 0$
5. $x^2y'' - 5xy' + 8y = 0$
6. $2x^2y'' + 7xy' + 2y = 0$
7. $3x^2y'' + 4xy' = 0$
8. $x^2y'' + 6xy' + 4y = 0$
9. $x^2y'' - xy' + y = 0$
10. $x^2y'' - xy' + 2y = 0$
11. $x^2y'' - xy' + 5y = 0$
12. $x^2y'' + 7xy' + 13y = 0$
13. $x^2y'' + 3xy' + 10y = 0$
14. $x^2y'' - 5xy' + 10y = 0$
15. $4x^2y'' + 8xy' + 5y = 0$
16. $4x^2y'' - 4xy' + 5y = 0$
17. $x^2y'' + 3xy' + y = 0$
18. $x^2y'' - 3xy' + 9y = 0$
19. $x^2y'' + xy' = 0$
20. $4x^2y'' + y = 0$
21. $9x^2y'' + 15xy' + y = 0$
22. $16x^2y'' - 8xy' + 9y = 0$
23. $16x^2y'' + 56xy' + 25y = 0$
24. $4x^2y'' - 16xy' + 25y = 0$

In Exercises 25–30, solve the given initial value problem.

25. $x^2y'' + 3xy' - 3y = 0, \quad y(1) = 1,\ y'(1) = -1$
26. $6x^2y'' + 7xy' - 2y = 0, \quad y(1) = 0,\ y'(1) = 1$
27. $x^2y'' - xy' + y = 0, \quad y(1) = 1,\ y'(1) = 1$
28. $x^2y'' + 7xy' + 9y = 0, \quad y(1) = 1,\ y'(1) = 0$
29. $x^2y'' - xy' + 2y = 0, \quad y(1) = -1,\ y'(1) = 1$
30. $x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1,\ y'(1) = 0$

17.5 Power-Series Solutions

In this section we extend our study of second-order linear homogeneous equations with variable coefficients. With the Euler equations in Section 17.4, the power of the variable x in the nonconstant coefficient had to match the order of the derivative with which it was paired: x^2 with y'', x^1 with y', and $x^0 (=1)$ with y. Here we drop that requirement so we can solve more general equations.

Method of Solution

The **power-series method** for solving a second-order homogeneous differential equation consists of finding the coefficients of a power series

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \qquad (1)$$

which solves the equation. To apply the method we substitute the series and its derivatives into the differential equation to determine the coefficients $c_0, c_1, c_2, \ldots$. The technique for finding the coefficients is similar to that used in the method of undetermined coefficients presented in Section 17.2.

In our first example we demonstrate the method in the setting of a simple equation whose general solution we already know. This is to help you become more comfortable with solutions expressed in series form.

EXAMPLE 1 Solve the equation $y'' + y = 0$ by the power-series method.

Solution We assume the series solution takes the form of

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substitution of these forms into the second-order equation gives us

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient Equation		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2} c_0$
x^1	$3(2)c_3 + c_1 = 0$	or	$c_3 = -\frac{1}{3 \cdot 2} c_1$
x^2	$4(3)c_4 + c_2 = 0$	or	$c_4 = -\frac{1}{4 \cdot 3} c_2$
x^3	$5(4)c_5 + c_3 = 0$	or	$c_5 = -\frac{1}{5 \cdot 4} c_3$
x^4	$6(5)c_6 + c_4 = 0$	or	$c_6 = -\frac{1}{6 \cdot 5} c_4$
$\vdots$	$\vdots$		$\vdots$
x^{n-2}	$n(n-1)c_n + c_{n-2} = 0$	or	$c_n = -\frac{1}{n(n-1)} c_{n-2}$

From the table we notice that the coefficients with even indices ($n = 2k, k = 1, 2, 3, \ldots$) are related to each other and the coefficients with odd indices ($n = 2k + 1$) are also interrelated. We treat each group in turn.

Even indices: Here $n = 2k$, so the power is x^{2k-2}. From the last line of the table, we have

$$2k(2k - 1)c_{2k} + c_{2k-2} = 0$$

or

$$c_{2k} = -\frac{1}{2k(2k - 1)} c_{2k-2}.$$

From this recursive relation we find

$$\begin{aligned} c_{2k} &= \left[-\frac{1}{2k(2k - 1)}\right]\left[-\frac{1}{(2k - 2)(2k - 3)}\right]\cdots\left[-\frac{1}{4(3)}\right]\left[-\frac{1}{2}\right]c_0 \\ &= \frac{(-1)^k}{(2k)!} c_0. \end{aligned}$$

Odd indices: Here $n = 2k + 1$, so the power is x^{2k-1}. Substituting this into the last line of the table yields

$$(2k + 1)(2k)c_{2k+1} + c_{2k-1} = 0$$

or

$$c_{2k+1} = -\frac{1}{(2k + 1)(2k)} c_{2k-1}.$$

Thus,

$$\begin{aligned} c_{2k+1} &= \left[-\frac{1}{(2k + 1)(2k)}\right]\left[-\frac{1}{(2k - 1)(2k - 2)}\right]\cdots\left[-\frac{1}{5(4)}\right]\left[-\frac{1}{3(2)}\right]c_1 \\ &= \frac{(-1)^k}{(2k + 1)!} c_1. \end{aligned}$$

Writing the power series by grouping its even and odd powers together and substituting for the coefficients yields

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}. \end{aligned}$$

From Table 9.1 in Section 9.10, we see that the first series on the right-hand side of the last equation represents the cosine function and the second series represents the sine. Thus, the general solution to $y'' + y = 0$ is

$$y = c_0 \cos x + c_1 \sin x.$$ ■

EXAMPLE 2 Find the general solution to $y'' + xy' + y = 0$.

Solution We assume the series solution form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Substitution of these forms into the second-order equation yields

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We equate the coefficients of each power of x to zero as summarized in the following table.

Power of *x*	**Coefficient Equation**		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 + c_1 + c_1 = 0$	or	$c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 + 2c_2 + c_2 = 0$	or	$c_4 = -\frac{1}{4}c_2$
x^3	$5(4)c_5 + 3c_3 + c_3 = 0$	or	$c_5 = -\frac{1}{5}c_3$
x^4	$6(5)c_6 + 4c_4 + c_4 = 0$	or	$c_6 = -\frac{1}{6}c_4$
$\vdots$	$\vdots$		$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0$	or	$c_{n+2} = -\frac{1}{n+2}c_n$

From the table notice that the coefficients with even indices are interrelated and the coefficients with odd indices are also interrelated.

Even indices: Here $n = 2k - 2$, so the power is x^{2k-2}. From the last line in the table, we have

$$c_{2k} = -\frac{1}{2k}c_{2k-2}.$$

From this recurrence relation we obtain

$$c_{2k} = \left(-\frac{1}{2k}\right)\left(-\frac{1}{2k-2}\right)\cdots\left(-\frac{1}{6}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)c_0$$
$$= \frac{(-1)^k}{(2)(4)(6)\cdots(2k)}c_0.$$

Odd indices: Here $n = 2k - 1$, so the power is x^{2k-1}. From the last line in the table, we have

$$c_{2k+1} = -\frac{1}{2k+1}c_{2k-1}.$$

From this recurrence relation we obtain

$$c_{2k+1} = \left(-\frac{1}{2k+1}\right)\left(-\frac{1}{2k-1}\right)\cdots\left(-\frac{1}{5}\right)\left(-\frac{1}{3}\right)c_1$$
$$= \frac{(-1)^k}{(3)(5)\cdots(2k+1)}c_1.$$

Writing the power series by grouping its even and odd powers and substituting for the coefficients yields

$$y = \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1}$$
$$= c_0\sum_{k=0}^{\infty} \frac{(-1)^k}{(2)(4)\cdots(2k)}x^{2k} + c_1\sum_{k=0}^{\infty} \frac{(-1)^k}{(3)(5)\cdots(2k+1)}x^{2k+1}. \quad ■$$

EXAMPLE 3 Find the general solution to

$$(1 - x^2)y'' - 6xy' - 4y = 0, \qquad |x| < 1.$$

Solution Notice that the leading coefficient is zero when $x = \pm 1$. Thus, we assume the solution interval I: $-1 < x < 1$. Substitution of the series form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and its derivatives gives us

$$(1 - x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6\sum_{n=1}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0,$$
$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - 6\sum_{n=1}^{\infty} nc_n x^n - 4\sum_{n=0}^{\infty} c_n x^n = 0.$$

Next, we equate the coefficients of each power of x to zero as summarized in the following table.

Power of x	Coefficient Equation		
x^0	$2(1)c_2 - 4c_0 = 0$	or	$c_2 = \frac{4}{2}c_0$
x^1	$3(2)c_3 - 6(1)c_1 - 4c_1 = 0$	or	$c_3 = \frac{5}{3}c_1$
x^2	$4(3)c_4 - 2(1)c_2 - 6(2)c_2 - 4c_2 = 0$	or	$c_4 = \frac{6}{4}c_2$
x^3	$5(4)c_5 - 3(2)c_3 - 6(3)c_3 - 4c_3 = 0$	or	$c_5 = \frac{7}{5}c_3$
$\vdots$	$\vdots$		$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - [n(n-1) + 6n + 4]c_n = 0$		
	$(n+2)(n+1)c_{n+2} - (n+4)(n+1)c_n = 0$	or	$c_{n+2} = \dfrac{n+4}{n+2}c_n$

Again we notice that the coefficients with even indices are interrelated and those with odd indices are interrelated.

Even indices: Here $n = 2k - 2$, so the power is x^{2k}. From the right-hand column and last line of the table, we get

$$c_{2k} = \frac{2k+2}{2k}c_{2k-2}$$
$$= \left(\frac{2k+2}{2k}\right)\left(\frac{2k}{2k-2}\right)\left(\frac{2k-2}{2k-4}\right)\cdots\frac{6}{4}\left(\frac{4}{2}\right)c_0$$
$$= (k+1)c_0.$$

Odd indices: Here $n = 2k - 1$, so the power is x^{2k+1}. The right-hand column and last line of the table gives us

$$\begin{aligned} c_{2k+1} &= \frac{2k+3}{2k+1} c_{2k-1} \\ &= \left(\frac{2k+3}{2k+1}\right)\left(\frac{2k+1}{2k-1}\right)\left(\frac{2k-1}{2k-3}\right)\cdots\frac{7}{5}\left(\frac{5}{3}\right)c_1 \\ &= \frac{2k+3}{3} c_1. \end{aligned}$$

The general solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1} \\ &= c_0 \sum_{k=0}^{\infty} (k+1) x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{2k+3}{3} x^{2k+1}. \end{aligned}$$ ■

EXAMPLE 4 Find the general solution to $y'' - 2xy' + y = 0$.

Solution Assuming that

$$y = \sum_{n=0}^{\infty} c_n x^n,$$

substitution into the differential equation gives us

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2 \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We next determine the coefficients, listing them in the following table.

Power of x	Coefficient Equation		
x^0	$2(1)c_2 + c_0 = 0$	or	$c_2 = -\frac{1}{2} c_0$
x^1	$3(2)c_3 - 2c_1 + c_1 = 0$	or	$c_3 = \frac{1}{3 \cdot 2} c_1$
x^2	$4(3)c_4 - 4c_2 + c_2 = 0$	or	$c_4 = \frac{3}{4 \cdot 3} c_2$
x^3	$5(4)c_5 - 6c_3 + c_3 = 0$	or	$c_5 = \frac{5}{5 \cdot 4} c_3$
x^4	$6(5)c_6 - 8c_4 + c_4 = 0$	or	$c_6 = \frac{7}{6 \cdot 5} c_4$
$\vdots$	$\vdots$		$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - (2n-1)c_n = 0$	or	$c_{n+2} = \frac{2n-1}{(n+2)(n+1)} c_n$

From the recursive relation

$$c_{n+2} = \frac{2n - 1}{(n + 2)(n + 1)} c_n,$$

we write out the first few terms of each series for the general solution:

$$y = c_0\left(1 - \frac{1}{2}x^2 - \frac{3}{4!}x^4 - \frac{21}{6!}x^6 - \cdots\right)$$
$$+ c_1\left(x + \frac{1}{3!}x^3 + \frac{5}{5!}x^5 + \frac{45}{7!}x^7 + \cdots\right).$$

EXERCISES 17.5

In Exercises 1–18, use power series to find the general solution of the differential equation.

1. $y'' + 2y' = 0$
2. $y'' + 2y' + y = 0$
3. $y'' + 4y = 0$
4. $y'' - 3y' + 2y = 0$
5. $x^2y'' - 2xy' + 2y = 0$
6. $y'' - xy' + y = 0$
7. $(1 + x)y'' - y = 0$
8. $(1 - x^2)y'' - 4xy' + 6y = 0$
9. $(x^2 - 1)y'' + 2xy' - 2y = 0$
10. $y'' + y' - x^2y = 0$
11. $(x^2 - 1)y'' - 6y = 0$
12. $xy'' - (x + 2)y' + 2y = 0$
13. $(x^2 - 1)y'' + 4xy' + 2y = 0$
14. $y'' - 2xy' + 4y = 0$
15. $y'' - 2xy' + 3y = 0$
16. $(1 - x^2)y'' - xy' + 4y = 0$
17. $y'' - xy' + 3y = 0$
18. $x^2y'' - 4xy' + 6y = 0$

1 The Laplace Transform

Definition 1. *Let f be an arbitrary (complex valued or real valued) function, defined on the semi-infinite interval $[0,\infty)$; then the integral*

$$f^L(\lambda) = (\mathcal{L}f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)\, dt \tag{1}$$

is said to be the Laplace transform *of f, if the integral* (1) *converges for some value $\lambda = \lambda_0$. Therefore the Laplace transform of a function (if it exists) depends on a parameter λ which could be either a real number or a complex number.*

Saying that a function $f(t)$ has a Laplace transform $f^L(\lambda)$ means, that for some $\lambda = \lambda_0$, the limit

$$\lim_{N\to\infty} \int_0^N f(t)\, e^{-\lambda_0 t}\, dt = f^L(\lambda)$$

exists. The integral in the right-hand side of Eq.(1) is an integral over an unbounded interval. Such integrals are called *improper integrals* and they are defined as a limit of integrals over finite intervals. If such a limit does not exist the improper integral is said to diverge.

From the definition of the integral, it follows that if the Laplace transform exists for a particular function then it does not depend on the values of a function at a finite number of points. Namely, we can change the values of a function at a finite number of points and its Laplace transform will still be the same.

The parameter λ in the definition of the Laplace transform is not necessarily a positive or real number, but a complex number. Thus, $\lambda = \alpha + i\beta$ where α is the real part of λ, denoted by $\alpha = \Re\lambda$ and β is an imaginary part of a complex number λ, $\beta = \Im\lambda$. The set of all complex numbers is denoted as $\mathbb{C}$ whereas the set of all real numbers is denoted as $\mathbb{R}$.

Theorem 1. *The Laplace transform is a linear operator, that is*

$$(\mathcal{L}Cf)(\lambda) = C\,(\mathcal{L}f)(\lambda), \ \ \text{and} \ \ (\mathcal{L}(f+g))(\lambda) = (\mathcal{L}f)(\lambda) + (\mathcal{L}g)(\lambda),$$

where C is a constant and f and g are arbitrary functions for which the Laplace transforms exist.

Theorem 2. *If a function f is absolutely integrable over finite intervals and the integral* (1) *converges for some complex number $\lambda = \mu$, then it converges in the half-space $\Re\lambda > \Re\mu$, i.e. in $\{\lambda \in \mathbb{C} : \Re\lambda > \Re\mu\}$.*

There is some real value σ_c called the **abscissa of convergence** of the function f such that the integral (1) is convergent in the half-plane $\Re\lambda > \sigma_c$ and divergent in the half-plane $\Re\lambda < \sigma_c$. We don't know precisely whether or not there are points of convergence on the line $\Re\lambda = \sigma_c$ itself.

Example 1. *Let p be any positive number (not necessarily an integer). Then the Laplace transform of the function $f(t) = t^p$, $t > 0$ is*

$$\begin{aligned}(\mathcal{L}t^p)(\lambda) &= \int_0^\infty e^{-\lambda t}\, t^p\, dt = \int_0^\infty e^{-\lambda t}\, (\lambda t)^p \lambda^{-p}\, dt \\ &= \lambda^{-p-1} \int_0^\infty e^{-\tau}\, \tau^{-p}\, d\tau = \frac{\Gamma(p+1)}{\lambda^{p+1}},\end{aligned}$$

where

$$\Gamma(\nu) = \int_0^\infty e^{-\tau}\, \tau^{\nu-1}\, d\tau \tag{2}$$

is the **Gamma function** *of Euler. This improper integral converges for $\nu > 0$ and by integrating by parts we obtain*

$$\Gamma(\nu+1) = \nu\,\Gamma(\nu). \tag{3}$$

Indeed, for $\nu > 0$ we have

$$\begin{aligned}\Gamma(\nu+1) &= \int_0^\infty e^{-\tau}\, \tau^\nu\, d\tau = -\int_0^\infty \tau^\nu\, de^{-\tau} \\ &= -\tau^\nu\, e^{-\tau}\Big|_{\tau=0}^{\tau=\infty} + \nu \int_0^\infty e^{-\tau}\, \tau^{\nu-1}\, d\tau = \nu\,\Gamma(\nu).\end{aligned}$$

The most remarkable property of the Γ*-function is obtained when we set* $\nu = n$*, an integer. The comparison with the result of the previous example yields*

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \ldots.$$

Definition 2. *A function* f *is said to be* **piecewise continuous** *on a finite interval* $[a, b]$ *if this interval can be subdivided into finitely many intervals so that* $f(t)$ *is continuous on each subinterval and approaches a finite limit at the end points of each subinterval from the interior. That is, there are finite number of points* $\{\alpha_j\}$, $j = 1, 2, \ldots, N$ *where a function* f *has a jump discontinuity when both*

$$\lim_{\substack{h\to 0\\ h\geq 0}} f(\alpha_j + h) = f(\alpha_j + 0) \quad \text{and} \quad \lim_{\substack{h\to 0\\ h\geq 0}} f(\alpha_j - h) = f(\alpha_j - 0)$$

exist but are different.

Note that an infinite number of discontinuities is allowed, as long as just a finite number occur on a finite interval. However, all these jumps must be finite.

Remember that for a continuous function f we have $f(t) = f(t+0) = f(t-0)$. If, at some point $t = t_0$ this is not valid, then a function is discontinuous at $t = t_0$. In other words, the finite discontinuity occurs if the left hand side and the right hand side limits are finite and are not equal.

Definition 3. *The* Heaviside function $H(t)$ *is the unit step function, equal to zero for* t *negative and unity for* t *positive, with* $H(0) = 1/2$*, i.e.*

$$H(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases} \tag{4}$$

Remark. Of course, we can define the value of the Heaviside function at the point $t = 0$ whatever we want. As we will see in the next example, it will not effect the value of its Laplace transformation. That is why we can change the value at a finite number of points of any function and it will not change the value of the corresponding Laplace transform. But if we wish to restore a function from it's Laplace transform value we will get a function that possesses the same property as the Heaviside function. Namely, a value of the function at any point will be equal to the mean of it's right hand side and left hand side limit values:

$$f(t) = \frac{1}{2}[f(t+0) + f(t-0)]. \quad \square$$

Definition 4. *A function* $f(t)$, $t \in [0, \infty)$ *is said to be a* **function-original** *if it has on every finite interval only a finite number of points of discontinuity and*

$$|f(t)| < M\, e^{ct} \quad (t > T) \tag{5}$$

for some values of c, M *and* T*, which may be very large. Moreover we assume that at points of discontinuity the value of a function-original is equal to the corresponding* mean value*, thus*

$$f(t_0) = \frac{1}{2}[f(t_0+0) + f(t_0-0)] = \lim_{\substack{\varepsilon\to 0\\ \varepsilon\geq 0}} \frac{f(t_0+\varepsilon) + f(t_0-\varepsilon)}{2}. \tag{6}$$

The Laplace transform of such a function is called the **image***.*

Definition 5. *We say that a function* f *is of* exponential order *if for some values* c, M*, and* T (5) *holds. We abbreviate this as* $f = O\left(e^{ct}\right)$*. A function* f *is said to be of* exponential order α*, or* $eo(\alpha)$ *for abbreviation, if* $f = O\left(e^{ct}\right)$ *for any real number* $c > \alpha$*, but not when* $c < \alpha$*.*

Definition 6. *The integral* (1) *is said to be* absolutely convergent*, if the integral*

$$\int_0^\infty e^{-\Re\lambda t}\, |f(t)|\, dt \tag{7}$$

converges. The greatest lower bound σ_a *of such numbers* $\Re\lambda$ *for which the integral* (7) *converges is called the* abscissa of absolute convergence.

Theorem 3. *If* $|f(t)| \leq C$ *for* $t \geq T$*, then the Laplace transform* (1) *converges absolutely for any* λ_0 *with* $\Re\lambda_0 > 0$*. In particular, the Laplace transform exists for any positive (real)* λ*.*

Theorem 4. *The integral* (1) *converges for any function-original. Moreover, if a function* f *is of exponential order* α*, then the integral* (1) *absolutely converges for* $\Re\lambda > \alpha$*. Furthermore, if* f *and* g *are piecewise continuous functions whose Laplace transforms exist and satisfy* $(\mathcal{L}f) = (\mathcal{L}g)$*, then* $f = g$ *at their points of continuity. Thus, if* $F(\lambda)$ *has a continuous inverse* f*, then* f *is unique.*

#	Function-original	Its Laplace Transform
1.	$H(t)$	$\frac{1}{\lambda}$
2.	$H(t-a)$	$\frac{1}{\lambda}\, e^{-a\lambda}$
3.	t	$\frac{1}{\lambda^2}$
4.	$t^n,\ n=1,2,\dots$	$\frac{n!}{\lambda^{n+1}}$
5.	t^p	$\frac{\Gamma(p+1)}{\lambda^{p+1}}$
6.	$e^{\alpha t}$	$\frac{1}{\lambda-\alpha},\ \Re\lambda > \Re\alpha$
7.	$t^n\, e^{\alpha t},\ n=1,2,\dots$	$\frac{n!}{(\lambda-\alpha)^{n+1}},\ \Re\lambda > \Re\alpha$
8.	$\sin\alpha t$	$\frac{\alpha}{\lambda^2+\alpha^2},\ \Re\lambda > 0$
9.	$\cos\alpha t$	$\frac{\lambda}{\lambda^2+\alpha^2},\ \Re\lambda > 0$
10.	$e^{\alpha t}\ \sin\beta t$	$\frac{\beta}{(\lambda-\alpha)^2+\beta^2},\ \Re\lambda > \Re\alpha$
11.	$e^{\alpha t}\ \cos\beta t$	$\frac{\lambda-\alpha}{(\lambda-\alpha)^2+\beta^2},\ \Re\lambda > \Re\alpha$
12.	$\sinh\beta t$	$\frac{\beta}{\lambda^2-\beta^2},\ \Re\lambda > \Re\beta$
13.	$\cosh\beta t$	$\frac{\lambda}{\lambda^2-\beta^2},\ \Re\lambda > \Re\beta$
14.	$t\sin\beta t$	$\frac{2\beta\lambda}{(\lambda^2+\beta^2)^2},\ \Re\lambda > 0$
15.	$t\cos\beta t$	$\frac{\lambda^2-\beta^2}{(\lambda^2+\beta^2)^2},\ \Re\lambda > 0$
16.	$e^{\alpha t}-e^{\beta t}$	$\frac{\alpha-\beta}{(\lambda-\alpha)(\lambda-\beta)}, \Re\lambda > \Re\alpha, \Re\beta$
17.	$e^{\alpha t}\left[\cos\beta t + \frac{\alpha}{\beta}\sin\beta t\right]$	$\frac{\lambda}{(\lambda-\alpha)^2+\beta^2}\ \Re\lambda > \Re\alpha$
18.	$\frac{\sin\beta t}{2\beta^3} - \frac{t\cos\beta t}{2\beta^2}$	$\frac{1}{(\lambda^2+\beta^2)^2}\ \Re\lambda > 0$
19.	$\frac{t\sin\beta t}{2\beta}$	$\frac{\lambda}{(\lambda^2+\beta^2)^2}\ \Re\lambda > 0$
20.	$e^{\alpha t}\ \sinh\beta t$	$\frac{\beta}{(\lambda-\alpha)^2-\beta^2},\ \Re\lambda > \Re(\alpha\pm\beta)$
21.	$e^{\alpha t}\ \cosh\beta t$	$\frac{\lambda-\alpha}{(\lambda-\alpha)^2-\beta^2},\ \Re\lambda > \Re(\alpha\pm\beta)$

Table 1: A Table of Elementary Laplace Transforms. Note: Each function in the left column is zero for negative t; that is, they must be multiplied by the Heaviside function $H(t)$.

2 Properties of the Laplace Transform

The success of transformation techniques in solving initial value problems and other applications hinges on their *operational properties*. Rules that govern how operations in the time domain translate to operations in the image domain are called *operational laws* or *rules*. In this section we present the basic 6 rules that are useful in applications of the Laplace Transformation to differential equations. The justification of these laws involve technical details that are beyond the scope of the text and therefore, is omitted. So we simply point to [1], [2]. We start with the following

Definition 7. *The* **convolution** *of two functions f and g, defined on the half-line $[0, \infty)$, is the integral*

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = (g * f)(t).$$

It is easy to veryfy that the convolution of two contsants is

$$1 * 1 = \int_0^t d\tau = t.$$

Many examples of convolutions the reader will find later in the following sections. Now we list the properties of the Laplace transform.

1° **The differential rule**

$$\mathcal{L}\left[f^{(n)}(t)\right](\lambda) = \lambda^n \mathcal{L}f(\lambda) - \sum_{k=1}^{n} \lambda^{n-k} f^{(k-1)}(+0). \tag{8}$$

Integration by parts gives us the equality (8). In particular,

$$\mathcal{L}\left[f'(t)\right](\lambda) = \lambda\, f^L(\lambda) - f(0). \tag{9}$$

$$\mathcal{L}\left[f''(t)\right](\lambda) = \lambda^2 f^L(\lambda) - \lambda f(0) - f'(0). \tag{10}$$

2° **The convolution rule**

The Laplace transform of the convolution of two functions is equal to the product of its images:

$$\mathcal{L}(f * g)(\lambda) = f^L(\lambda) g^L(\lambda). \tag{11}$$

3° **The similarity rule**

$$\mathcal{L}[f(at)](\lambda) = \frac{1}{a} f^L\left(\frac{\lambda}{a}\right), \quad \Re\lambda > a\sigma_c, \tag{12}$$

if a is a positive number.

4° **The shift rule**

If we know $G(\lambda)$, which is the Laplace transform of $g(t)$, then the retarded function $f(t) = g(t-a)H(t-a)$ has the Laplace transform $G(\lambda)e^{-\lambda a}$, namely,

$$\mathcal{L}\left[H(t-a)g(t-a)\right](\lambda) = e^{-a\lambda} g^L(\lambda), \quad a > 0, \tag{13}$$

Similarly

$$\mathcal{L}\left[f(t+a)\right](\lambda) = e^{a\lambda}\left\{f^L(\lambda) - \int_0^a e^{-\lambda t} f(t)\,dt\right\}, \quad a > 0. \tag{14}$$

where H is the Heaviside function.

5° **The attenuation rule**

$$\mathcal{L}\left[e^{-at} f(t)\right](\lambda) = f^L(\lambda + a). \tag{15}$$

6° **The integration rule**

$$\mathcal{L}[t^{n-1} * f(t)](\lambda) = \mathcal{L} \int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau = \frac{(n-1)!}{\lambda^n} f^L(\lambda), \quad n = 1, 2, \ldots. \tag{16}$$

If $n = 1$, then

$$\frac{1}{\lambda} f^L(\lambda) = \mathcal{L} \int_0^t f(\tau)\, d\tau. \tag{17}$$

Remark. We can unite (12) and (15) so that

$$\mathcal{L}\left[\frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right)\right](\lambda) = f^L(a\lambda + b). \tag{18}$$

3 The Inverse Laplace Transform

We employ the symbol $\mathcal{L}^{-1}[F(\lambda)]$, corresponding to the direct Laplace transform defined by Eq.(1.1), to denote a function $f(t)$ whose Laplace transform is $F(\lambda)$. Thus, we have the Laplace pair

$$F(\lambda) = (\mathcal{L} f(t))(\lambda), \quad f(t) = \mathcal{L}^{-1}[F(\lambda)](t).$$

It has already been demonstrated that the Laplace transform $f^L(\lambda)$ of a given function $f(t)$ can be calculated by direct integration. The inverse Laplace transform is more complicated. However, it is very important because the solution of practical problems usually provides a known $F(\lambda)$ from which the function $f(t)$ must be found such that $f^L(\lambda) = F(\lambda)$. Thomas John l'Anson Bromwich (1875 – 1929) answered the question of how to find this function, $f(t)$, which is the inverse Laplace transform of a given function $F(\lambda)$, in 1916. He expressed the inverse Laplace transform as the contour integral

$$\frac{1}{2}[f(t+0) + f(-0)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^L(\lambda)\, e^{\lambda t}\, d\lambda, \tag{19}$$

where c is any number greater than the abscissa of convergence for $f^L(\lambda)$ and the integration is defined in the sense of the Cauchy principle value.

Remark. From this formula (19), it follows that the inverse Laplace transform restores a function-original from its image in such a way that the value of a function-original at any point is equal to a mean of its right-hand side limit value and its left-hand side limit value. If a function is continuous at a point then its value at this point coincides with its mean value. □

In this section we will not use Eq. (19) as it is very complicated. Instead, we consider three methods to find the inverse Laplace transform: *Partial Fraction Decomposition*, the *Convolution Theorem*, and the *Cauchy Residue Theorem*. We will restrict ourselves to finding the inverse Laplace transform of rational functions or their products on exponentials, that is,

$$F(\lambda) = \frac{P(\lambda)}{Q(\lambda)} \quad \text{or} \quad F_a(\lambda) = \frac{P(\lambda)}{Q(\lambda)}\, e^{-\alpha\lambda},$$

where $P(\lambda)$ and $Q(\lambda)$ are polynomials[1]. This case is one of the most important in applications of the Laplace transform to differential equations with constant coefficients. In this section we only consider cases in which the degree of the denominator is larger than the degree of the numerator.

The case of the product of a rational function and an exponential can be easily reduced to the case without the exponential multiplier by the shift rule (13). In fact, suppose we know

$$f(t) = \mathcal{L}^{-1}[F(\lambda)](t) = \mathcal{L}^{-1}\left[\frac{P(\lambda)}{Q(\lambda)}\right](t),$$

the original of a rational function $F(\lambda)$. Then according to (13) we have

$$H(t-a) f(t-a) = \mathcal{L}^{-1}[F(\lambda)\, e^{-a\lambda}].$$

[1]The result is valid for the case when Q is an entire function, that is, $Q(\lambda)$ is represented by a series which converges everywhere except infinity.

3.1 Partial Fraction Decomposition

The fraction

$$F(\lambda) = \frac{P(\lambda)}{Q(\lambda)}$$

can be easily expanded into partial fractions, that is, P/Q can be represented as a linear combination of simple rational functions of the form $1/(\lambda-\alpha)$, $1/(\lambda-\alpha)^2$, and so forth. To do this, it is first necessary to find all nulls of the denominator $Q(\lambda)$ or, equivalently, to find all roots of the equation

$$Q(\lambda) = 0. \tag{20}$$

Then $Q(\lambda)$ can be factored as

$$Q(\lambda) = c_0(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k},$$

where λ_1, λ_2, ..., λ_k are the distinct roots of Eq. (20) and m_1, m_2, ..., m_k are their respective **multiplicities**. A root of the Eq. (20) is called **simple** if its multiplicity equals 1. A root which appears twice is often called a **double root**. Recall that a polynomial of degree n has n roots, counting miltiplicities, so $m_1+m_2+\cdots+m_k = n$. Thus if the equation (20) has a simple real root $\lambda = \lambda_0$ then the polynomial $Q(\lambda)$ has a factor $\lambda - \lambda_0$. To this factor in $F = P/Q$ corresponds the partial fraction decomposition of the form

$$\frac{A}{\lambda - \lambda_0},$$

where A is a constant to be found. The inverse Laplace transform of this fraction is (see Table 1, formula 6)

$$\mathcal{L}^{-1}\left[\frac{A}{\lambda-\lambda_0}\right] = A\,e^{\lambda_0 t}\,H(t),$$

where H is the Heaviside function (4).

The attenuation rule (15) gives us the clue about how get rid of λ_0 in the denominator. Thus, using formula 1 from Table 1, yields

$$\mathcal{L}^{-1}\left[\frac{A}{\lambda}\right] = AH(t).$$

Therefore, $\mathcal{L}\left[e^{\lambda_0 t}AH(t)\right] = (\lambda-\lambda_0)^{-1}$.

If a polynomial $Q(\lambda)$ has a repeated factor $(\lambda-\lambda_0)^m$, that is , if Eq. (20) has a root λ_0 with multiplicity m, then the partial fraction decomposition of $F = P/Q$ contains a sum of m fractions

$$\frac{A_m}{(\lambda-\lambda_0)^m} + \frac{A_{m-1}}{(\lambda-\lambda_0)^{m-1}} + \cdots + \frac{A_1}{\lambda-\lambda_0}.$$

The inverse Laplace transform of each term is (see Table 1, formula 7):

$$\mathcal{L}^{-1}\left[\frac{A_m}{(\lambda-\lambda_0)^m}\right] = A_m\,\frac{t^{m-1}}{(m-1)!}\,e^{\lambda_0 t}\,H(t).$$

Suppose a polynomial $Q(\lambda)$ has an unrepeated complex factor $(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)$, where $\lambda_0 = \alpha + i\beta$, and $\overline{\lambda}_0 = \alpha - i\beta$ is the complex conjugate of λ_0. The coefficients of $Q(\lambda)$ are real, complex roots occur in conjugate pairs. The pair of conjugate roots of Eq. (20) corresponding to this factor gives rise to the term

$$\frac{A\lambda + B}{(\lambda-\alpha)^2+\beta^2}$$

in the partial fraction decomposition, since

$$(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0) = (\lambda-\alpha)^2+\beta^2.$$

The expansion of the fraction $F = P/Q$ can be rewritten as

$$\frac{A(\lambda-\alpha)+\alpha A + B}{(\lambda-\alpha)^2+\beta^2}.$$

From Table 1 formulas 9 and 10 and the shift rule (13) we obtain the inverse transform

$$\mathcal{L}^{-1}\left\{\frac{A\lambda+B}{(\lambda-\alpha)^2+\beta^2}\right\}=e^{\alpha t}\left[A\cos\beta t+\frac{\alpha A+B}{\beta}\sin\beta t\right]H(t). \tag{21}$$

If the polynomial $Q(\lambda)$ has the repeated complex factor $[(\lambda-\lambda_0)(\lambda-\overline{\lambda}_0)]^2$ then the sum of the form

$$\frac{A\lambda+B}{[(\lambda-\alpha)^2+\beta^2]^2}+\frac{C\lambda+D}{(\lambda-\alpha)^2+\beta^2}.$$

corresponds to this factor in partial fraction decomposition of $F=P/Q$. The last fraction is as in Eq. (21). To find the inverse Laplace transform of the first factor we can use formulas 17 and 18 from Table 1 and the shift rule (13). This leads us to

$$\mathcal{L}^{-1}\left\{\frac{A\lambda+B}{[(\lambda-\alpha)^2+\beta^2]^2}\right\}=e^{\alpha t}\left[\frac{A}{2\beta}t\,\sin\beta t++\frac{\alpha A+B}{2\beta^2}(\sin\beta t-\beta t\cos\beta t)\right]H(t).$$

3.2 Convolution Theorem

Let the given function $F(\lambda)$ be represented as a product of two other functions: $F(\lambda)=F_1(\lambda)\cdot F_2(\lambda)$. Assume that we know the inverse Laplace transforms $f_1(t)$ and $f_2(t)$ of these functions F_1 and F_2. Then the inverse Laplace transform can be defined according to the convolution rule (11) to obtain

$$\mathcal{L}^{-1}\left\{F(\lambda)\right\}(t)=(f_1*f_2)(t)\equiv\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau=\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau.$$

It turns out that one can calculate the inverse of such a product in terms of the known inverses, with the intervention of an integral.

3.3 Residue Method

Suppose a function $F(\lambda)=P(\lambda)/Q(\lambda)$ is a fraction of two polynomials (or entire functions). We denote by $\lambda_j,\ j=1,2,\ldots,N$ all nulls[2] of the denominator $Q(\lambda)$. Then the inverse Laplace transform of a function F can be found as

$$f(t)=\mathcal{L}^{-1}\left\{F(\lambda)\right\}=\sum_{j=1}^{N}\operatorname*{Res}_{\lambda_j}F(\lambda)e^{\lambda t}, \tag{22}$$

where the sum covers all zeros of the equation (20) and residues $\operatorname{Res}_{\lambda_j} F(\lambda)e^{\lambda t}$ are evaluated as follows.

If λ_j is a simple root of Eq. (20) then

$$\operatorname*{Res}_{\lambda_j}F(\lambda)e^{\lambda t}=\frac{P(\lambda_j)}{Q'(\lambda_j)}\,e^{\lambda_j t}. \tag{23}$$

If λ_j is a double root of Eq. (20) then

$$\operatorname*{Res}_{\lambda_j}F(\lambda)e^{\lambda t}=\lim_{\lambda\to\lambda_j}\frac{d}{d\lambda}\left\{(\lambda-\lambda_j)^2F(\lambda)\,e^{\lambda t}\right\}. \tag{24}$$

In general, when λ_j is a n-fold root of Eq. (20) then

$$\operatorname*{Res}_{\lambda_j}F(\lambda)e^{\lambda t}=\lim_{\lambda\to\lambda_j}\frac{1}{(n-1)!}\frac{d^{n-1}}{d\lambda^{n-1}}\left\{(\lambda-\lambda_j)^nF(\lambda)\,e^{\lambda t}\right\}.\ \square \tag{25}$$

[2] $N=\infty$ if $Q(\lambda)$ is an entire function.

CHAPTER 16

THE FOURIER SERIES

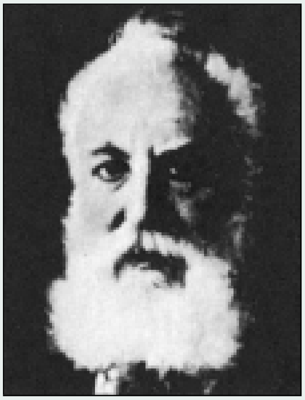
Do not worry about your difficulties in mathematics, I assure you that mine are greater.

—Albert Einstein

Historical Profiles

Jean Baptiste Joseph Fourier (1768–1830), a French mathematician, first presented the series and transform that bear his name. Fourier's results were not enthusiastically received by the scientific world. He could not even get his work published as a paper.

Born in Auxerre, France, Fourier was orphaned at age 8. He attended a local military college run by Benedictine monks, where he demonstrated great proficiency in mathematics. Like most of his contemporaries, Fourier was swept into the politics of the French Revolution. He played an important role in Napoleon's expeditions to Egypt in the later 1790s. Due to his political involvement, he narrowly escaped death twice.



Alexander Graham Bell (1847–1922) inventor of the telephone, was a Scottish-American scientist.

Bell was born in Edinburgh, Scotland, a son of Alexander Melville Bell, a well-known speech teacher. Alexander the younger also became a speech teacher after graduating from the University of Edinburgh and the University of London. In 1866 he became interested in transmitting speech electrically. After his older brother died of tuberculosis, his father decided to move to Canada. Alexander was asked to come to Boston to work at the School for the Deaf. There he met Thomas A. Watson, who became his assistant in his electromagnetic transmitter experiment. On March 10, 1876, Alexander sent the famous first telephone message: "Watson, come here I want you." The bel, the logarithmic unit introduced in Chapter 14, is named in his honor.

16.1 INTRODUCTION

We have spent a considerable amount of time on the analysis of circuits with sinusoidal sources. This chapter is concerned with a means of analyzing circuits with periodic, nonsinusoidal excitations. The notion of periodic functions was introduced in Chapter 9; it was mentioned there that the sinusoid is the most simple and useful periodic function. This chapter introduces the Fourier series, a technique for expressing a periodic function in terms of sinusoids. Once the source function is expressed in terms of sinusoids, we can apply the phasor method to analyze circuits.

The Fourier series is named after Jean Baptiste Joseph Fourier (1768–1830). In 1822, Fourier's genius came up with the insight that any practical periodic function can be represented as a sum of sinusoids. Such a representation, along with the superposition theorem, allows us to find the response of circuits to arbitrary periodic inputs using phasor techniques.

We begin with the trigonometric Fourier series. Later we consider the exponential Fourier series. We then apply Fourier series in circuit analysis. Finally, practical applications of Fourier series in spectrum analyzers and filters are demonstrated.

16.2 TRIGONOMETRIC FOURIER SERIES

While studying heat flow, Fourier discovered that a nonsinusoidal periodic function can be expressed as an infinite sum of sinusoidal functions. Recall that a periodic function is one that repeats every T seconds. In other words, a periodic function $f(t)$ satisfies

$$f(t) = f(t + nT) \tag{16.1}$$

where n is an integer and T is the period of the function.

According to the *Fourier theorem*, any practical periodic function of frequency ω_0 can be expressed as an infinite sum of sine or cosine functions that are integral multiples of ω_0. Thus, $f(t)$ can be expressed as

$$\begin{aligned} f(t) = a_0 &+ a_1 \cos \omega_0 t + b_1 \sin \omega_0 t + a_2 \cos 2\omega_0 t \\ &+ b_2 \sin 2\omega_0 t + a_3 \cos 3\omega_0 t + b_3 \sin 3\omega_0 t + \cdots \end{aligned} \tag{16.2}$$

or

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}} \tag{16.3}$$

where $\omega_0 = 2\pi/T$ is called the *fundamental frequency* in radians per second. The sinusoid $\sin n\omega_0 t$ or $\cos n\omega_0 t$ is called the nth harmonic of $f(t)$; it is an odd harmonic if n is odd and an even harmonic if n is even. Equation 16.3 is called the *trigonometric Fourier series* of $f(t)$. The constants a_n and b_n are the *Fourier coefficients*. The coefficient a_0 is the dc component or the average value of $f(t)$. (Recall that sinusoids

The harmonic frequency ω_n is an integral multiple of the fundamental frequency ω_0, i.e., $\omega_n = n\omega_0$.

have zero average values.) The coefficients a_n and b_n (for $n \neq 0$) are the amplitudes of the sinusoids in the ac component. Thus,

The **Fourier series** of a periodic function $f(t)$ is a representation that resolves $f(t)$ into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

A function that can be represented by a Fourier series as in Eq. (16.3) must meet certain requirements, because the infinite series in Eq. (16.3) may or may not converge. These conditions on $f(t)$ to yield a convergent Fourier series are as follows:

1. $f(t)$ is single-valued everywhere.
2. $f(t)$ has a finite number of finite discontinuities in any one period.
3. $f(t)$ has a finite number of maxima and minima in any one period.
4. The integral $\int_{t_0}^{t_0+T} |f(t)|\,dt < \infty$ for any t_0.

These conditions are called *Dirichlet conditions*. Although they are not necessary conditions, they are sufficient conditions for a Fourier series to exist.

Historical note: Although Fourier published his theorem in 1822, it was P. G. L. Dirichlet (1805–1859) who later supplied an acceptable proof of the theorem.

A major task in Fourier series is the determination of the Fourier coefficients a_0, a_n, and b_n. The process of determining the coefficients is called *Fourier analysis*. The following trigonometric integrals are very helpful in Fourier analysis. For any integers m and n,

A software package like Mathcad or Maple can be used to evaluate the Fourier coefficients.

$$\int_0^T \sin n\omega_0 t\,dt = 0 \tag{16.4a}$$

$$\int_0^T \cos n\omega_0 t\,dt = 0 \tag{16.4b}$$

$$\int_0^T \sin n\omega_0 t \cos m\omega_0 t\,dt = 0 \tag{16.4c}$$

$$\int_0^T \sin n\omega_0 t \sin m\omega_0 t\,dt = 0, \qquad (m \neq n) \tag{16.4d}$$

$$\int_0^T \cos n\omega_0 t \cos m\omega_0 t\,dt = 0, \qquad (m \neq n) \tag{16.4e}$$

$$\int_0^T \sin^2 n\omega_0 t\,dt = \frac{T}{2} \tag{16.4f}$$

$$\int_0^T \cos^2 n\omega_0 t\,dt = \frac{T}{2} \tag{16.4g}$$

Let us use these identities to evaluate the Fourier coefficients.

We begin by finding a_0. We integrate both sides of Eq. (16.3) over one period and obtain

$$\begin{aligned}\int_0^T f(t)\,dt &= \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right] dt \\ &= \int_0^T a_0\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t\,dt \right. \\ &\qquad \left. + \int_0^T b_n \sin n\omega_0 t\,dt\right] dt\end{aligned} \tag{16.5}$$

Invoking the identities of Eqs. (16.4a) and (16.4b), the two integrals involving the ac terms vanish. Hence,

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt = a_0 T$$

or

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt \tag{16.6}$$

showing that a_0 is the average value of $f(t)$.

To evaluate a_n, we multiply both sides of Eq. (16.3) by $\cos m\omega_0 t$ and integrate over one period:

$$\begin{aligned}&\int_0^T f(t)\cos m\omega_0 t\,dt \\ &\quad = \int_0^T \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)\right]\cos m\omega_0 t\,dt \\ &\quad = \int_0^T a_0 \cos m\omega_0 t\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos n\omega_0 t \cos m\omega_0 t\,dt \right. \\ &\qquad \left. + \int_0^T b_n \sin n\omega_0 t \cos m\omega_0 t\,dt\right] dt\end{aligned} \tag{16.7}$$

The integral containing a_0 is zero in view of Eq. (16.4b), while the integral containing b_n vanishes according to Eq. (16.4c). The integral containing a_n will be zero except when $m = n$, in which case it is $T/2$, according to Eqs. (16.4e) and (16.4g). Thus,

$$\int_0^T f(t)\cos m\omega_0 t\,dt = a_n \frac{T}{2}, \qquad \text{for } m = n$$

or

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \tag{16.8}$$

In a similar vein, we obtain b_n by multiplying both sides of Eq. (16.3) by $\sin m\omega_0 t$ and integrating over the period. The result is

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \tag{16.9}$$

Be aware that since $f(t)$ is periodic, it may be more convenient to carry the integrations above from $-T/2$ to $T/2$ or generally from t_0 to $t_0 + T$ instead of 0 to T. The result will be the same.

An alternative form of Eq. (16.3) is the *amplitude-phase* form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \tag{16.10}$$

We can use Eqs. (9.11) and (9.12) to relate Eq. (16.3) to Eq. (16.10), or we can apply the trigonometric identity

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \tag{16.11}$$

to the ac terms in Eq. (16.10) so that

$$\begin{aligned} a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) = a_0 + \sum_{n=1}^{\infty} (A_n \cos\phi_n) \cos n\omega_0 t \\ - (A_n \sin\phi_n) \sin n\omega_0 t \end{aligned} \tag{16.12}$$

Equating the coefficients of the series expansions in Eqs. (16.3) and (16.12) shows that

$$a_n = A_n \cos\phi_n, \qquad b_n = -A_n \sin\phi_n \tag{16.13a}$$

or

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n} \tag{16.13b}$$

To avoid any confusion in determining ϕ_n, it may be better to relate the terms in complex form as

$$A_n \angle \phi_n = a_n - jb_n \tag{16.14}$$

The convenience of this relationship will become evident in Section 16.6. The plot of the amplitude A_n of the harmonics versus $n\omega_0$ is called the *amplitude spectrum* of $f(t)$; the plot of the phase ϕ_n versus $n\omega_0$ is the *phase spectrum* of $f(t)$. Both the amplitude and phase spectra form the *frequency spectrum* of $f(t)$.

The frequency spectrum is also known as the *line spectrum* in view of the discrete frequency components.

> The frequency spectrum of a signal consists of the plots of the amplitudes and phases of the harmonics versus frequency.

Thus, the Fourier analysis is also a mathematical tool for finding the spectrum of a periodic signal. Section 16.6 will elaborate more on the spectrum of a signal.

To evaluate the Fourier coefficients a_0, a_n, and b_n, we often need to apply the following integrals:

$$\int \cos at\, dt = \frac{1}{a}\sin at \tag{16.15a}$$

$$\int \sin at\, dt = -\frac{1}{a}\cos at \tag{16.15b}$$

$$\int t \cos at\, dt = \frac{1}{a^2}\cos at + \frac{1}{a} t \sin at \tag{16.15c}$$

$$\int t \sin at\, dt = \frac{1}{a^2}\sin at - \frac{1}{a} t \cos at \tag{16.15d}$$

It is also useful to know the values of the cosine, sine, and exponential functions for integral multiples of π. These are given in Table 16.1, where n is an integer.

TABLE 16.1 Values of cosine, sine, and exponential functions for integral multiples of π.

Function	Value
$\cos 2n\pi$	1
$\sin 2n\pi$	0
$\cos n\pi$	$(-1)^n$
$\sin n\pi$	0
$\cos \frac{n\pi}{2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ 0, & n = \text{odd} \end{cases}$
$\sin \frac{n\pi}{2}$	$\begin{cases} (-1)^{(n-1)/2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$
$e^{j2n\pi}$	1
$e^{jn\pi}$	$(-1)^n$
$e^{jn\pi/2}$	$\begin{cases} (-1)^{n/2}, & n = \text{even} \\ j(-1)^{(n-1)/2}, & n = \text{odd} \end{cases}$

EXAMPLE 16.1

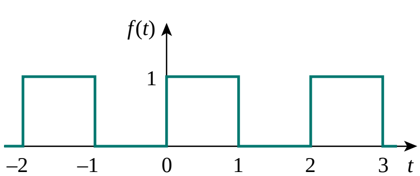


Figure 16.1 For Example 16.1; a square wave.

Determine the Fourier series of the waveform shown in Fig. 16.1. Obtain the amplitude and phase spectra.

Solution:

The Fourier series is given by Eq. (16.3), namely,

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{16.1.1}$$

Our goal is to obtain the Fourier coefficients a_0, a_n, and b_n using Eqs. (16.6), (16.8), and (16.9). First, we describe the waveform as

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases} \tag{16.1.2}$$

and $f(t) = f(t + T)$. Since $T = 2$, $\omega_0 = 2\pi/T = \pi$. Thus,

$$a_0 = \frac{1}{T}\int_0^T f(t)\, dt = \frac{1}{2}\left[\int_0^1 1\, dt + \int_1^2 0\, dt\right] = \frac{1}{2}t\Big|_0^1 = \frac{1}{2} \tag{16.1.3}$$

Using Eq. (16.8) along with Eq. (16.15a),

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt \\ &= \frac{2}{2}\left[\int_0^1 1\cos n\pi t\, dt + \int_1^2 0\cos n\pi t\, dt\right] \\ &= \frac{1}{n\pi}\sin n\pi t\Big|_0^1 = \frac{1}{n\pi}\sin n\pi = 0 \end{aligned} \tag{16.1.4}$$

From Eq. (16.9) with the aid of Eq. (16.15b),

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\, dt \\ &= \frac{2}{2}\left[\int_0^1 1\sin n\pi t\, dt + \int_1^2 0\sin n\pi t\, dt\right] \\ &= -\frac{1}{n\pi}\cos n\pi t\Big|_0^1 \\ &= -\frac{1}{n\pi}(\cos n\pi - 1), \qquad \cos n\pi = (-1)^n \\ &= \frac{1}{n\pi}[1-(-1)^n] = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \end{aligned} \tag{16.1.5}$$

Substituting the Fourier coefficients in Eqs. (16.1.3) to (16.1.5) into Eq. (16.1.1) gives the Fourier series as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin \pi t + \frac{2}{3\pi}\sin 3\pi t + \frac{2}{5\pi}\sin 5\pi t + \cdots \tag{16.1.6}$$

Since $f(t)$ contains only the dc component and the sine terms with the fundamental component and odd harmonics, it may be written as

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k-1 \tag{16.1.7}$$

By summing the terms one by one as demonstrated in Fig. 16.2, we notice how superposition of the terms can evolve into the original square. As more and more Fourier components are added, the sum gets closer and closer to the square wave. However, it is not possible in practice to sum the series in Eq. (16.1.6) or (16.1.7) to infinity. Only a partial sum ($n = 1, 2, 3, \ldots, N$, where N is finite) is possible. If we plot the partial sum (or truncated series) over one period for a large N as in Fig. 16.3, we notice that the partial sum oscillates above and below the actual value of $f(t)$. At the neighborhood of the points of discontinuity ($x = 0, 1, 2, \ldots$), there is overshoot and damped oscillation. In fact, an overshoot of about 9 percent of the peak value is always present, regardless of the number of terms used to approximate $f(t)$. This is called the *Gibbs phenomenon*.

Summing the Fourier terms by hand calculation may be tedious. A computer is helpful to compute the terms and plot the sum like those shown in Fig. 16.2.

Historical note: Named after the mathematical physicist Josiah Willard Gibbs, who first observed it in 1899.

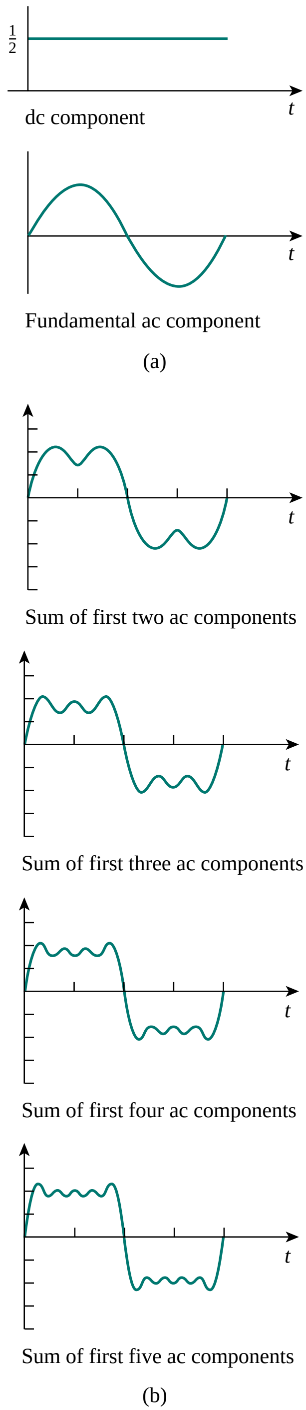


Figure 16.2 Evolution of a square wave from its Fourier components.

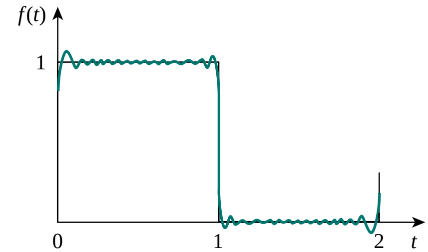


Figure 16.3 Truncating the Fourier series at $N = 11$; Gibbs phenomenon.

Finally, let us obtain the amplitude and phase spectra for the signal in Fig. 16.1. Since $a_n = 0$,

$$A_n = \sqrt{a_n^2 + b_n^2} = |b_n| = \begin{cases} \dfrac{2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{16.1.8}$$

and

$$\phi_n = -\tan^{-1}\frac{b_n}{a_n} = \begin{cases} -90^\circ, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{16.1.9}$$

The plots of A_n and ϕ_n for different values of $n\omega_0 = n\pi$ provide the amplitude and phase spectra in Fig. 16.4. Notice that the amplitudes of the harmonics decay very fast with frequency.

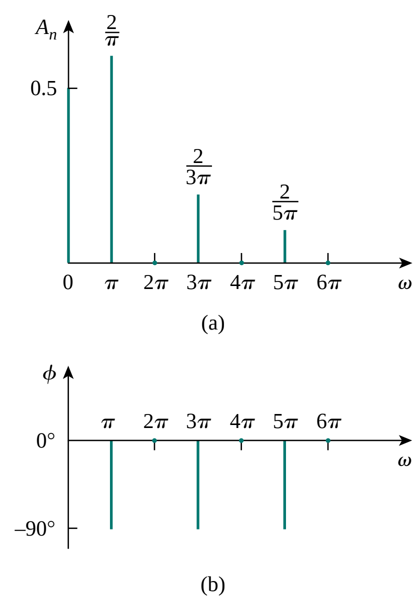


Figure 16.4 For Example 16.1: (a) amplitude and (b) phase spectrum of the function shown in Fig. 16.1.

PRACTICE PROBLEM 16.1

Find the Fourier series of the square wave in Fig. 16.5. Plot the amplitude and phase spectra.

Answer: $f(t) = \dfrac{4}{\pi}\sum_{k=1}^{\infty}\dfrac{1}{n}\sin n\pi t,\ n = 2k-1$. See Fig. 16.6 for the spectra.

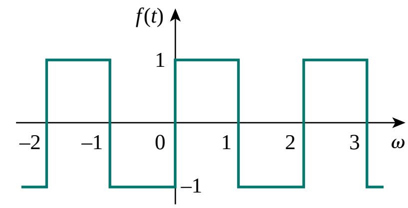


Figure 16.5 For Practice Prob. 16.1.

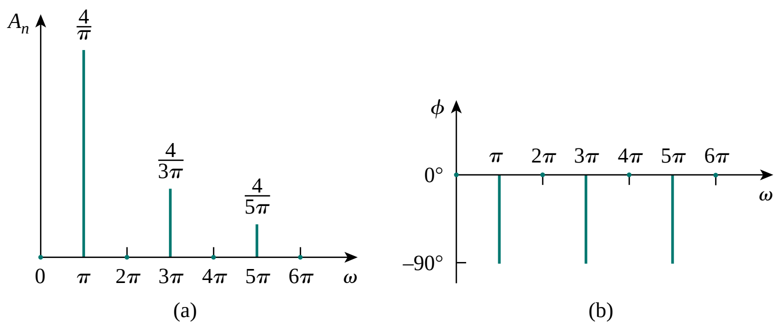


Figure 16.6 For Practice Prob. 16.1: amplitude and phase spectra for the function shown in Fig. 16.5.

EXAMPLE 16.2

Obtain the Fourier series for the periodic function in Fig. 16.7 and plot the amplitude and phase spectra.

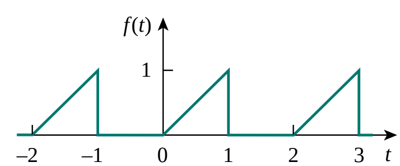


Figure 16.7 For Example 16.2.

Solution:

The function is described as

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

Since $T = 2$, $\omega_0 = 2\pi/T = \pi$. Then

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 0\,dt\right] = \frac{1}{2}\frac{t^2}{2}\bigg|_0^1 = \frac{1}{4} \tag{16.2.1}$$

To evaluate a_n and b_n, we need the integrals in Eq. (16.15):

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt \\ &= \frac{2}{2}\left[\int_0^1 t\cos n\pi t\,dt + \int_1^2 0\cos n\pi t\,dt\right] \\ &= \left[\frac{1}{n^2\pi^2}\cos n\pi t + \frac{t}{n\pi}\sin n\pi t\right]\bigg|_0^1 \\ &= \frac{1}{n^2\pi^2}(\cos n\pi - 1) + 0 = \frac{(-1)^n - 1}{n^2\pi^2} \end{aligned} \tag{16.2.2}$$

since $\cos n\pi = (-1)^n$; and

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt \\ &= \frac{2}{2}\left[\int_0^1 t\sin n\pi t\,dt + \int_1^2 0\sin n\pi t\,dt\right] \\ &= \left[\frac{1}{n^2\pi^2}\sin n\pi t - \frac{t}{n\pi}\cos n\pi t\right]\Bigg|_0^1 \\ &= 0 - \frac{\cos n\pi}{n\pi} = \frac{(-1)^{n+1}}{n\pi} \end{aligned} \tag{16.2.3}$$

Substituting the Fourier coefficients just found into Eq. (16.3) yields

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{[(-1)^n - 1]}{(n\pi)^2}\cos n\pi t + \frac{(-1)^{n+1}}{n\pi}\sin n\pi t\right]$$

To obtain the amplitude and phase spectra, we notice that, for even harmonics, $a_n = 0$, $b_n = -1/n\pi$, so that

$$A_n\underline{/\phi_n} = a_n - jb_n = 0 + j\frac{1}{n\pi} \tag{16.2.4}$$

Hence,

$$\begin{aligned} A_n &= |b_n| = \frac{1}{n\pi}, \qquad n = 2, 4, \ldots \\ \phi_n &= 90^\circ, \qquad n = 2, 4, \ldots \end{aligned} \tag{16.2.5}$$

For odd harmonics, $a_n = -2/(n^2\pi^2)$, $b_n = 1/(n\pi)$ so that

$$A_n\underline{/\phi_n} = a_n - jb_n = -\frac{2}{n^2\pi^2} - j\frac{1}{n\pi} \tag{16.2.6}$$

That is,

$$\begin{aligned} A_n &= \sqrt{a_n^2 + b_n^2} = \sqrt{\frac{4}{n^4\pi^4} + \frac{1}{n^2\pi^2}} \\ &= \frac{1}{n^2\pi^2}\sqrt{4 + n^2\pi^2}, \qquad n = 1, 3, \ldots \end{aligned} \tag{16.2.7}$$

From Eq. (16.2.6), we observe that ϕ lies in the third quadrant, so that

$$\phi_n = 180^\circ + \tan^{-1}\frac{n\pi}{2}, \qquad n = 1, 3, \ldots \tag{16.2.8}$$

From Eqs. (16.2.5), (16.2.7), and (16.2.8), we plot A_n and ϕ_n for different values of $n\omega_0 = n\pi$ to obtain the amplitude spectrum and phase spectrum as shown in Fig. 16.8.

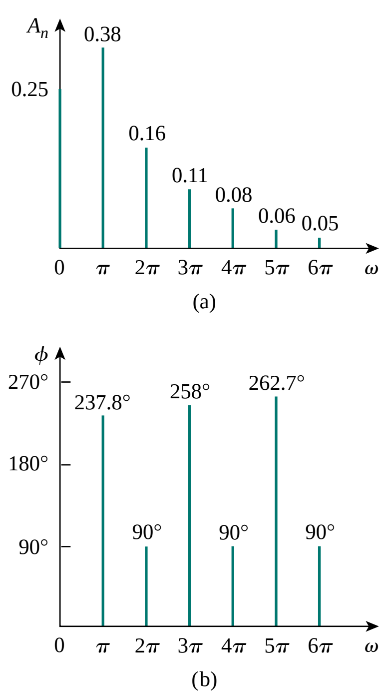


Figure 16.8 For Example 16.2: (a) amplitude spectrum, (b) phase spectrum.

PRACTICE PROBLEM 16.2

Determine the Fourier series of the sawtooth waveform in Fig. 16.9.

Answer: $f(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin 2\pi n t.$

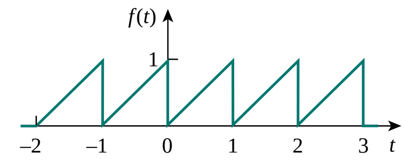


Figure 16.9 For Practice Prob. 16.2.

16.3 SYMMETRY CONSIDERATIONS

We noticed that the Fourier series of Example 16.1 consisted only of the sine terms. One may wonder if a method exists whereby one can know in advance that some Fourier coefficients would be zero and avoid the unnecessary work involved in the tedious process of calculating them. Such a method does exist; it is based on recognizing the existence of symmetry. Here we discuss three types of symmetry: (1) even symmetry, (2) odd symmetry, (3) half-wave symmetry.

16.3.1 Even Symmetry

A function $f(t)$ is *even* if its plot is symmetrical about the vertical axis; that is,

$$f(t) = f(-t) \tag{16.16}$$

Examples of even functions are t^2, t^4, and $\cos t$. Figure 16.10 shows more examples of periodic even functions. Note that each of these examples satisfies Eq. (16.16). A main property of an even function $f_e(t)$ is that:

$$\int_{-T/2}^{T/2} f_e(t)\,dt = 2\int_{0}^{T/2} f_e(t)\,dt \tag{16.17}$$

because integrating from $-T/2$ to 0 is the same as integrating from 0 to $T/2$. Utilizing this property, the Fourier coefficients for an even function become

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^{T/2} f(t)\,dt \\ a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt \\ b_n &= 0 \end{aligned} \tag{16.18}$$

Since $b_n = 0$, Eq. (16.3) becomes a *Fourier cosine series*. This makes sense because the cosine function is itself even. It also makes intuitive sense that an even function contains no sine terms since the sine function is odd.

To confirm Eq. (16.18) quantitatively, we apply the property of an even function in Eq. (16.17) in evaluating the Fourier coefficients in Eqs. (16.6), (16.8), and (16.9). It is convenient in each case to integrate over the interval $-T/2 < t < T/2$, which is symmetrical about the origin. Thus,

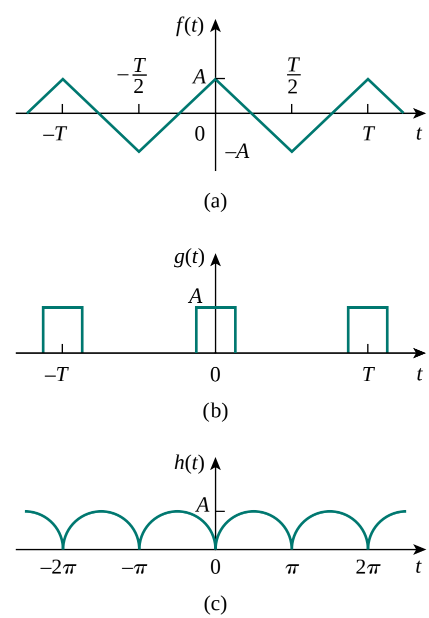


Figure 16.10 Typical examples of even periodic functions.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)\,dt + \int_{0}^{T/2} f(t)\,dt\right] \quad (16.19)$$

We change variables for the integral over the interval $-T/2 < t < 0$ by letting $t = -x$, so that $dt = -dx$, $f(t) = f(-t) = f(x)$, since $f(t)$ is an even function, and when $t = -T/2$, $x = T/2$. Then,

$$\begin{aligned} a_0 &= \frac{1}{T}\left[\int_{T/2}^{0} f(x)(-dx) + \int_{0}^{T/2} f(t)\,dt\right] \\ &= \frac{1}{T}\left[\int_{0}^{T/2} f(x)\,dx + \int_{0}^{T/2} f(t)\,dt\right] \end{aligned} \quad (16.20)$$

showing that the two integrals are identical. Hence,

$$a_0 = \frac{2}{T}\int_{0}^{T/2} f(t)\,dt \quad (16.21)$$

as expected. Similarly, from Eq. (16.8),

$$a_n = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos n\omega_0 t\,dt + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \quad (16.22)$$

We make the same change of variables that led to Eq. (16.20) and note that both $f(t)$ and $\cos n\omega_0 t$ are even functions, implying that $f(-t) = f(t)$ and $\cos(-n\omega_0 t) = \cos n\omega_0 t$. Equation (16.22) becomes

$$\begin{aligned} a_n &= \frac{2}{T}\left[\int_{T/2}^{0} f(-x)\cos(-n\omega_0 x)(-dx) + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_{T/2}^{0} f(x)\cos(n\omega_0 x)(-dx) + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_{0}^{T/2} f(x)\cos(n\omega_0 x)\,dx + \int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt\right] \end{aligned} \quad (16.23a)$$

or

$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos n\omega_0 t\,dt \quad (16.23b)$$

as expected. For b_n, we apply Eq. (16.9),

$$b_n = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\sin n\omega_0 t\,dt + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \quad (16.24)$$

We make the same change of variables but keep in mind that $f(-t) = f(t)$ but $\sin(-n\omega_0 t) = -\sin n\omega_0 t$. Equation (16.24) yields

$$\begin{aligned} b_n &= \frac{2}{T}\left[\int_{T/2}^{0} f(-x)\sin(-n\omega_0 x)(-dx) + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[\int_{T/2}^{0} f(x)\sin n\omega_0 x\,dx + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= \frac{2}{T}\left[-\int_{0}^{T/2} f(x)\sin(n\omega_0 x)\,dx + \int_{0}^{T/2} f(t)\sin n\omega_0 t\,dt\right] \\ &= 0 \end{aligned} \quad (16.25)$$

confirming Eq. (16.18).

16.3.2 Odd Symmetry

A function $f(t)$ is said to be *odd* if its plot is antisymmetrical about the vertical axis:

$$f(-t) = -f(t) \tag{16.26}$$

Examples of odd functions are t, t^3, and $\sin t$. Figure 16.11 shows more examples of periodic odd functions. All these examples satisfy Eq. (16.26). An odd function $f_o(t)$ has this major characteristic:

$$\int_{-T/2}^{T/2} f_o(t)\, dt = 0 \tag{16.27}$$

because integration from $-T/2$ to 0 is the negative of that from 0 to $T/2$. With this property, the Fourier coefficients for an odd function become

$$a_0 = 0, \qquad a_n = 0$$
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt \tag{16.28}$$

which give us a *Fourier sine series*. Again, this makes sense because the sine function is itself an odd function. Also, note that there is no dc term for the Fourier series expansion of an odd function.

The quantitative proof of Eq. (16.28) follows the same procedure taken to prove Eq. (16.18) except that $f(t)$ is now odd, so that $f(t) = -f(t)$. With this fundamental but simple difference, it is easy to see that $a_0 = 0$ in Eq. (16.20), $a_n = 0$ in Eq. (16.23a), and b_n in Eq. (16.24) becomes

$$b_n = \frac{2}{T}\left[\int_{T/2}^{0} f(-x)\sin(-n\omega_0 x)(-dx) + \int_0^{T/2} f(t)\sin n\omega_0 t\, dt\right]$$
$$= \frac{2}{T}\left[-\int_{T/2}^{0} f(x)\sin n\omega_0 x\, dx + \int_0^{T/2} f(t)\sin n\omega_0 t\, dt\right]$$
$$= \frac{2}{T}\left[\int_0^{T/2} f(x)\sin(n\omega_0 x)\, dx + \int_0^{T/2} f(t)\sin n\omega_0 t\, dt\right]$$
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt \tag{16.29}$$

as expected.

It is interesting to note that any periodic function $f(t)$ with neither even nor odd symmetry may be decomposed into even and odd parts. Using the properties of even and odd functions from Eqs. (16.16) and (16.26), we can write

$$f(t) = \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{\text{even}} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{\text{odd}} = f_e(t) + f_o(t) \tag{16.30}$$

Notice that $f_e(t) = \frac{1}{2}[f(t) + f(-t)]$ satisfies the property of an even function in Eq. (16.16), while $f_o(t) = \frac{1}{2}[f(t) - f(-t)]$ satisfies the property of an odd function in Eq. (16.26). The fact that $f_e(t)$ contains

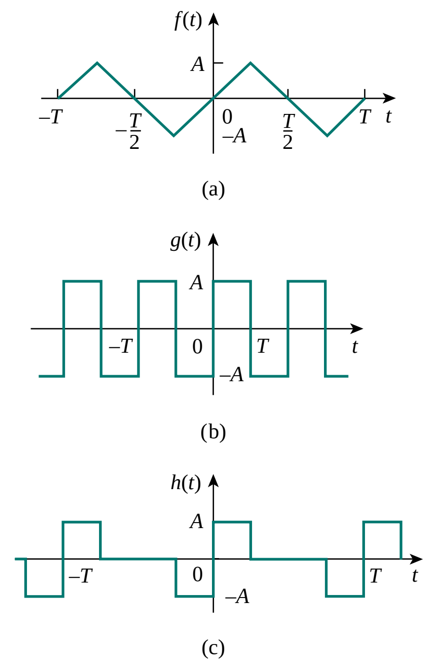


Figure 16.11 Typical examples of odd periodic functions.

only the dc term and the cosine terms, while $f_o(t)$ has only the sine terms, can be exploited in grouping the Fourier series expansion of $f(t)$ as

$$f(t) = \underbrace{a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t}_{\text{even}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin n\omega_0 t}_{\text{odd}} = f_e(t) + f_o(t) \quad (16.31)$$

It follows readily from Eq. (16.31) that when $f(t)$ is even, $b_n = 0$, and when $f(t)$ is odd, $a_0 = 0 = a_n$.

Also, note the following properties of odd and even functions:

1. The product of two even functions is also an even function.
2. The product of two odd functions is an even function.
3. The product of an even function and an odd function is an odd function.
4. The sum (or difference) of two even functions is also an even function.
5. The sum (or difference) of two odd functions is an odd function.
6. The sum (or difference) of an even function and an odd function is neither even nor odd.

Each of these properties can be proved using Eqs. (16.16) and (16.26).

16.3.3 Half-Wave Symmetry

A function is half-wave (odd) symmetric if

$$f\left(t - \frac{T}{2}\right) = -f(t) \quad (16.32)$$

which means that each half-cycle is the mirror image of the next half-cycle. Notice that functions $\cos n\omega_0 t$ and $\sin n\omega_0 t$ satisfy Eq. (16.32) for odd values of n and therefore possess half-wave symmetry when n is odd. Figure 16.12 shows other examples of half-wave symmetric functions. The functions in Figs. 16.11(a) and 16.11(b) are also half-wave symmetric. Notice that for each function, one half-cycle is the inverted

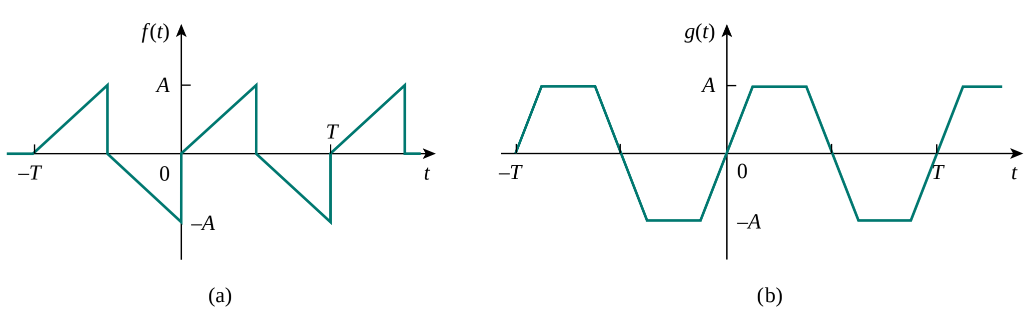


Figure 16.12 Typical examples of half-wave odd symmetric functions.

version of the adjacent half-cycle. The Fourier coefficients become

$$
\begin{aligned}
a_0 &= 0\\
a_n &= \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\, dt, & \text{for } n \text{ odd}\\ 0, & \text{for } n \text{ even}\end{cases}\\
b_n &= \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\sin n\omega_0 t\, dt, & \text{for } n \text{ odd}\\ 0, & \text{for } n \text{ even}\end{cases}
\end{aligned}
\tag{16.33}
$$

showing that the Fourier series of a half-wave symmetric function contains only odd harmonics.

To derive Eq. (16.33), we apply the property of half-wave symmetric functions in Eq. (16.32) in evaluating the Fourier coefficients in Eqs. (16.6), (16.8), and (16.9). Thus,

$$
a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{T}\left[\int_{-T/2}^{0} f(t)\,dt + \int_0^{T/2} f(t)\,dt\right] \tag{16.34}
$$

We change variables for the integral over the interval $-T/2 < t < 0$ by letting $x = t + T/2$, so that $dx = dt$; when $t = -T/2, x = 0$; and when $t = 0, x = T/2$. Also, we keep Eq. (16.32) in mind; that is, $f(x - T/2) = -f(x)$. Then,

$$
\begin{aligned}
a_0 &= \frac{1}{T}\left[\int_0^{T/2} f\left(x - \frac{T}{2}\right)dx + \int_0^{T/2} f(t)\,dt\right]\\
&= \frac{1}{T}\left[-\int_0^{T/2} f(x)\,dx + \int_0^{T/2} f(t)\,dt\right] = 0
\end{aligned}
\tag{16.35}
$$

confirming the expression for a_0 in Eq. (16.33). Similarly,

$$
a_n = \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos n\omega_0 t\,dt + \int_0^{T/2} f(t)\cos n\omega_0 t\,dt\right] \tag{16.36}
$$

We make the same change of variables that led to Eq. (16.35) so that Eq. (16.36) becomes

$$
\begin{aligned}
a_n = \frac{2}{T}\Bigg[&\int_0^{T/2} f\left(x - \frac{T}{2}\right)\cos n\omega_0\left(x - \frac{T}{2}\right)dx\\
&+ \int_0^{T/2} f(t)\cos n\omega_0 t\,dt\Bigg]
\end{aligned}
\tag{16.37}
$$

Since $f(x - T/2) = -f(x)$ and

$$
\begin{aligned}
\cos n\omega_0\left(x - \frac{T}{2}\right) &= \cos(n\omega_0 t - n\pi)\\
&= \cos n\omega_0 t\cos n\pi + \sin n\omega_0 t\sin n\pi\\
&= (-1)^n\cos n\omega_0 t
\end{aligned}
\tag{16.38}
$$

substituting these in Eq. (16.37) leads to

$$a_n = \frac{2}{T}[1-(-1)^n]\int_0^{T/2} f(t)\cos n\omega_0 t\, dt$$

$$= \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{T/2} f(t)\cos n\omega_0 t\, dt, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even} \end{cases} \qquad (16.39)$$

confirming Eq. (16.33). By following a similar procedure, we can derive b_n as in Eq. (16.33).

Table 16.2 summarizes the effects of these symmetries on the Fourier coefficients. Table 16.3 provides the Fourier series of some common periodic functions.

TABLE 16.2 Effects of symmetry on Fourier coefficients.

Symmetry	a_0	a_n	b_n	Remarks
Even	$a_0 \neq 0$	$a_n \neq 0$	$b_n = 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Odd	$a_0 = 0$	$a_n = 0$	$b_n \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.
Half-wave	$a_0 = 0$	$a_{2n} = 0$ $a_{2n+1} \neq 0$	$b_{2n} = 0$ $b_{2n+1} \neq 0$	Integrate over $T/2$ and multiply by 2 to get the coefficients.

TABLE 16.3 The Fourier series of common functions.

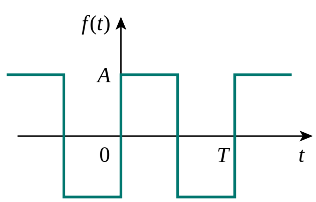
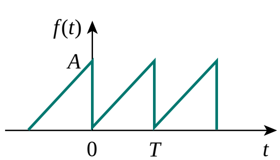
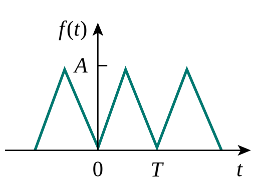
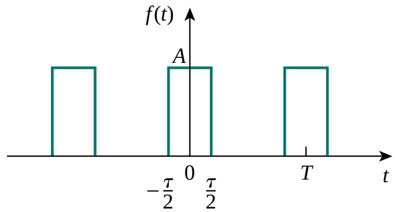
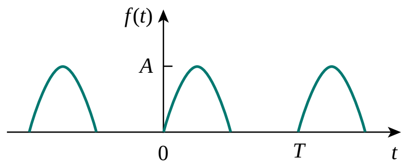
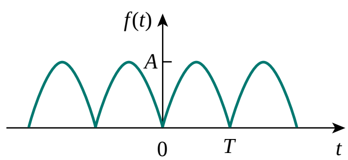
Function	Fourier series
1. Square wave 	$f(t) = \dfrac{4A}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\omega_0 t$
2. Sawtooth wave 	$f(t) = \dfrac{A}{2} - \dfrac{A}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{\sin n\omega_0 t}{n}$
3. Triangular wave 	$f(t) = \dfrac{A}{2} - \dfrac{4A}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos(2n-1)\omega_0 t$

TABLE 16.3 *(continued)*

Function	Fourier series
4. Rectangular pulse train 	$f(t) = \frac{A\tau}{T} + \frac{2A}{T}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi\tau}{T}\cos n\omega_0 t$
5. Half-wave rectified sine 	$f(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega_0 t - \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2n\omega_0 t$
6. Full-wave rectified sine 	$f(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos n\omega_0 t$

EXAMPLE 16.3

Find the Fourier series expansion of $f(t)$ given in Fig. 16.13.

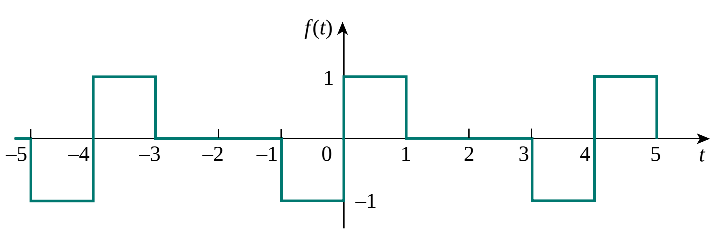


Figure 16.13 For Example 16.3.

Solution:

The function $f(t)$ is an odd function. Hence $a_0 = 0 = a_n$. The period is $T = 4$, and $\omega_0 = 2\pi/T = \pi/2$, so that

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\,dt$$

$$= \frac{4}{4}\left[\int_0^1 1\sin\frac{n\pi}{2}t\,dt + \int_1^2 0\sin\frac{n\pi}{2}t\,dt\right]$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^1 = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

Hence,

$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}t$$

which is a Fourier sine series.

PRACTICE PROBLEM 16.3

Find the Fourier series of the function $f(t)$ in Fig. 16.14.

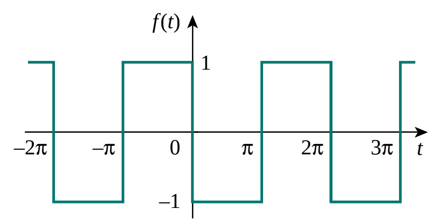


Figure 16.14 For Practice Prob. 16.3.

Answer: $f(t) = -\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin nt,\ n = 2k-1.$

EXAMPLE 16.4

Determine the Fourier series for the half-wave rectified cosine function shown in Fig. 16.15.

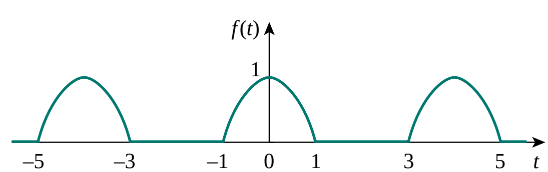


Figure 16.15 A half-wave rectified cosine function; for Example 16.4.

Solution:

This is an even function so that $b_n = 0$. Also, $T = 4$, $\omega_0 = 2\pi/T = \pi/2$. Over a period,

$$f(t) = \begin{cases} 0, & -2 < t < -1 \\ \cos\dfrac{\pi}{2}t, & -1 < t < 1 \\ 0, & 1 < t < 2 \end{cases}$$

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{4}\left[\int_0^1 \cos\frac{\pi}{2}t\,dt + \int_1^2 0\,dt\right]$$

$$= \frac{1}{2}\frac{2}{\pi}\sin\frac{\pi}{2}t\bigg|_0^1 = \frac{1}{\pi}$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega_0 t\,dt = \frac{4}{4}\left[\int_0^1 \cos\frac{\pi}{2}t\cos\frac{n\pi t}{2}\,dt + 0\right]$$

But $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$. Then

$$a_n = \frac{1}{2}\int_0^1 \left[\cos\frac{\pi}{2}(n+1)t + \cos\frac{\pi}{2}(n-1)t\right] dt$$

For $n = 1$,

$$a_1 = \frac{1}{2}\int_0^1 [\cos\pi t + 1]\,dt = \frac{1}{2}\left[\frac{\sin\pi t}{\pi} + t\right]\bigg|_0^1 = \frac{1}{2}$$

For $n > 1$,

$$a_n = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1)$$

For $n =$ odd ($n = 1, 3, 5, \ldots$), $(n+1)$ and $(n-1)$ are both even, so

$$\sin\frac{\pi}{2}(n+1) = 0 = \sin\frac{\pi}{2}(n-1), \qquad n = \text{odd}$$

For $n =$ even ($n = 2, 4, 6, \ldots$), $(n+1)$ and $(n-1)$ are both odd. Also,

$$\sin\frac{\pi}{2}(n+1) = -\sin\frac{\pi}{2}(n-1) = \cos\frac{n\pi}{2} = (-1)^{n/2}, \qquad n = \text{even}$$

Hence,

$$a_n = \frac{(-1)^{n/2}}{\pi(n+1)} + \frac{-(-1)^{n/2}}{\pi(n-1)} = \frac{-2(-1)^{n/2}}{\pi(n^2-1)}, \qquad n = \text{even}$$

Thus,

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{n=\text{even}}^{\infty}\frac{(-1)^{n/2}}{(n^2-1)}\cos\frac{n\pi}{2}t$$

To avoid using $n = 2, 4, 6, \ldots$ and also to ease computation, we can replace n by $2k$, where $k = 1, 2, 3, \ldots$ and obtain

$$f(t) = \frac{1}{\pi} + \frac{1}{2}\cos\frac{\pi}{2}t - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(4k^2-1)}\cos k\pi t$$

which is a Fourier cosine series.

PRACTICE PROBLEM 16.4

Find the Fourier series expansion of the function in Fig. 16.16.

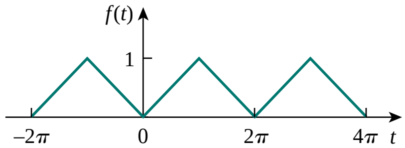


Figure 16.16 For Practice Prob. 16.4.

Answer: $f(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{n^2}\cos nt, n = 2k-1.$

EXAMPLE 16.5

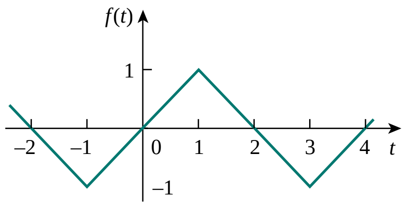


Figure 16.17 For Example 16.5.

Calculate the Fourier series for the function in Fig. 16.17.

Solution:

The function in Fig. 16.17 is half-wave odd symmetric, so that $a_0 = 0 = a_n$. It is described over half the period as

$$f(t) = t, \qquad -1 < t < 1$$

$T = 4$, $\omega_0 = 2\pi/T = \pi/2$. Hence,

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega_0 t\, dt$$

Instead of integrating $f(t)$ from 0 to 2, it is more convenient to integrate from -1 to 1. Applying Eq. (16.15d),

$$\begin{aligned} b_n &= \frac{4}{4}\int_{-1}^{1} t\sin\frac{n\pi t}{2}\,dt = \left[\frac{\sin n\pi t/2}{n^2\pi^2/4} - \frac{t\cos n\pi t/2}{n\pi/2}\right]\Bigg|_{-1}^{1} \\ &= \frac{4}{n^2\pi^2}\left[\sin\frac{n\pi}{2} - \sin\left(-\frac{n\pi}{2}\right)\right] - \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} + \cos\left(-\frac{n\pi}{2}\right)\right] \\ &= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{4}{n\pi}\cos\frac{n\pi}{2} \end{aligned}$$

since $\sin(-x) = -\sin x$ as an odd function, while $\cos(-x) = \cos x$ as an even function. Using the identities for $\sin n\pi/2$ and $\cos n\pi/2$ in Table 16.1,

$$b_n = \begin{cases} \dfrac{8}{n^2\pi^2}(-1)^{(n-1)/2}, & n = \text{odd} = 1, 3, 5, \ldots \\ \dfrac{4}{n\pi}(-1)^{(n+2)/2}, & n = \text{even} = 2, 4, 6, \ldots \end{cases}$$

Thus,

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{2}t$$

where b_n is given above.

PRACTICE PROBLEM 16.5

Determine the Fourier series of the function in Fig. 16.12(a). Take $A = 1$ and $T = 2\pi$.

Answer: $f(t) = \frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{-2}{n^2\pi}\cos nt + \frac{1}{n}\sin nt\right), n = 2k-1.$

16.4 CIRCUIT APPLICATIONS

We find that in practice, many circuits are driven by nonsinusoidal periodic functions. To find the steady-state response of a circuit to a nonsinusoidal periodic excitation requires the application of a Fourier series, ac phasor analysis, and the superposition principle. The procedure usually involves three steps.

Steps for Applying Fourier Series:

1. Express the excitation as a Fourier series.
2. Find the response of each term in the Fourier series.
3. Add the individual responses using the superposition principle.

The first step is to determine the Fourier series expansion of the excitation. For the periodic voltage source shown in Fig. 16.18(a), for example, the Fourier series is expressed as

$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \theta_n) \tag{16.40}$$

(The same could be done for a periodic current source.) Equation (16.40) shows that $v(t)$ consists of two parts: the dc component V_0 and the ac component $\mathbf{V}_n = V_n \angle \theta_n$ with several harmonics. This Fourier series representation may be regarded as a set of series-connected sinusoidal sources, with each source having its own amplitude and frequency, as shown in Fig. 16.18(b).

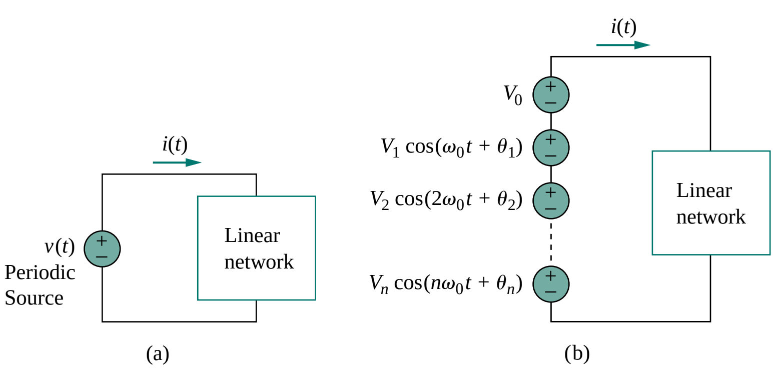


Figure 16.18 (a) Linear network excited by a periodic voltage source, (b) Fourier series representation (time-domain).

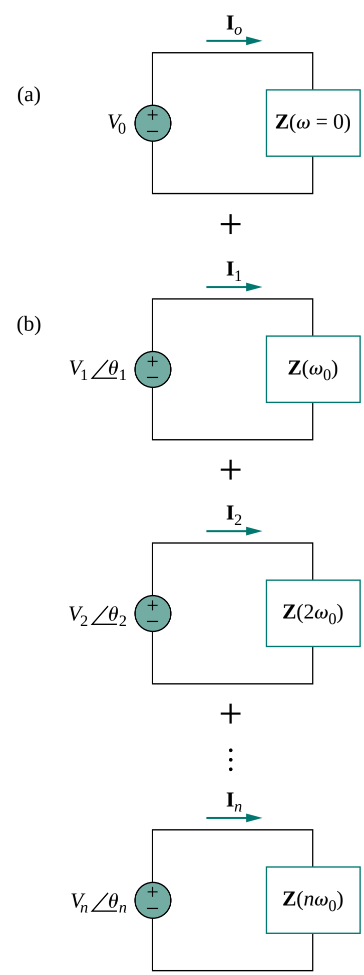


Figure 16.19 Steady-state responses: (a) dc component, (b) ac component (frequency domain).

The second step is finding the response to each term in the Fourier series. The response to the dc component can be determined in the frequency domain by setting $n = 0$ or $\omega = 0$ as in Fig. 16.19(a), or in the time domain by replacing all inductors with short circuits and all capacitors with open circuits. The response to the ac component is obtained by the phasor techniques covered in Chapter 9, as shown in Fig. 16.19(b). The network is represented by its impedance $\mathbf{Z}(n\omega_0)$ or admittance $\mathbf{Y}(n\omega_0)$. $\mathbf{Z}(n\omega_0)$ is the input impedance at the source when ω is everywhere replaced by $n\omega_0$, and $\mathbf{Y}(n\omega_0)$ is the reciprocal of $\mathbf{Z}(n\omega_0)$.

Finally, following the principle of superposition, we add all the individual responses. For the case shown in Fig. 16.19,

$$\begin{aligned} i(t) &= i_0(t) + i_1(t) + i_2(t) + \cdots \\ &= \mathbf{I}_0 + \sum_{n=1}^{\infty} |\mathbf{I}_n| \cos(n\omega_0 t + \psi_n) \end{aligned} \tag{16.41}$$

where each component $\mathbf{I}_n$ with frequency $n\omega_0$ has been transformed to the time domain to get $i_n(t)$, and ψ_n is the argument of $\mathbf{I}_n$.

EXAMPLE 16.6

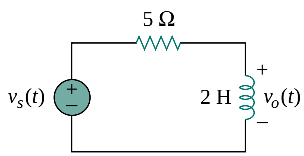


Figure 16.20 For Example 16.6.

Let the function $f(t)$ in Example 16.1 be the voltage source $v_s(t)$ in the circuit of Fig. 16.20. Find the response $v_o(t)$ of the circuit.

Solution:

From Example 16.1,

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin n\pi t, \qquad n = 2k-1$$

where $\omega_n = n\omega_0 = n\pi$ rad/s. Using phasors, we obtain the response $\mathbf{V}_o$ in the circuit of Fig. 16.20 by voltage division:

$$\mathbf{V}_o = \frac{j\omega_n L}{R + j\omega_n L}\mathbf{V}_s = \frac{j2n\pi}{5 + j2n\pi}\mathbf{V}_s$$

For the dc component ($\omega_n = 0$ or $n = 0$)

$$\mathbf{V}_s = \frac{1}{2} \qquad \Longrightarrow \qquad \mathbf{V}_o = 0$$

This is expected, since the inductor is a short circuit to dc. For the nth harmonic,

$$\mathbf{V}_s = \frac{2}{n\pi}\angle -90^\circ \tag{16.6.1}$$

and the corresponding response is

$$\begin{aligned} \mathbf{V}_o &= \frac{2n\pi\angle 90^\circ}{\sqrt{25 + 4n^2\pi^2}\angle \tan^{-1} 2n\pi/5}\,\frac{2}{n\pi}\angle -90^\circ \\ &= \frac{4\angle -\tan^{-1}\ 2n\pi/5}{\sqrt{25+4n^2\pi^2}} \end{aligned} \tag{16.6.2}$$

In the time domain,

$$v_o(t) = \sum_{k=1}^{\infty}\frac{4}{\sqrt{25+4n^2\pi^2}}\cos\left(n\pi t - \tan^{-1}\frac{2n\pi}{5}\right), \qquad n = 2k-1$$

The first three terms ($k = 1, 2, 3$ or $n = 1, 3, 5$) of the odd harmonics in the summation give us

$$\begin{aligned} v_o(t) = 0.4981\cos(\pi t - 51.49^\circ) + 0.2051\cos(3\pi t - 75.14^\circ) \\ + 0.1257\cos(5\pi t - 80.96^\circ) + \cdots \text{ V} \end{aligned}$$

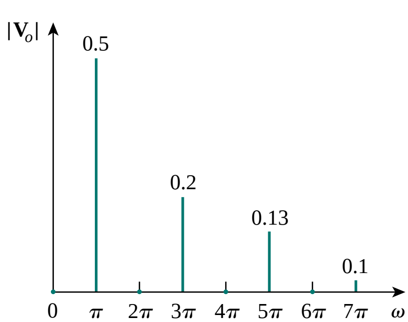


Figure 16.21 For Example 16.6: Amplitude spectrum of the output voltage.

Figure 16.21 shows the amplitude spectrum for output voltage $v_o(t)$, while that of the input voltage $v_s(t)$ is in Fig. 16.4(a). Notice that the two spectra are close. Why? We observe that the circuit in Fig. 16.20 is a highpass filter with the corner frequency $\omega_c = R/L = 2.5$ rad/s, which is less than the fundamental frequency $\omega_0 = \pi$ rad/s. The dc component is not passed and the first harmonic is slightly attenuated, but higher harmonics are passed. In fact, from Eqs. (16.6.1) and (16.6.2), $\mathbf{V}_o$ is identical to $\mathbf{V}_s$ for large n, which is characteristic of a highpass filter.

PRACTICE PROBLEM 16.6

If the sawtooth waveform in Fig. 16.9 (see Practice Prob. 16.2) is the voltage source $v_s(t)$ in the circuit of Fig. 16.22, find the response $v_o(t)$.

Answer: $$v_o(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi n t - \tan^{-1} 4n\pi)}{n\sqrt{1+16n^2\pi^2}} \text{ V.}$$

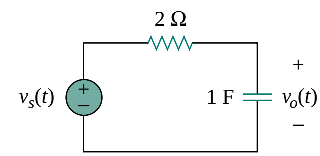


Figure 16.22 For Practice Prob. 16.6.

EXAMPLE 16.7

Find the response $i_o(t)$ in the circuit in Fig. 16.23 if the input voltage $v(t)$ has the Fourier series expansion

$$v(t) = 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{1+n^2}(\cos nt - n \sin nt)$$

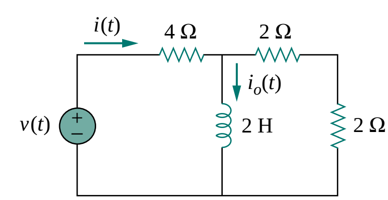


Figure 16.23 For Example 16.7.

Solution:

Using Eq. (16.13), we can express the input voltage as

$$\begin{aligned} v(t) &= 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n}{\sqrt{1+n^2}} \cos(nt + \tan^{-1} n) \\ &= 1 - 1.414\cos(t + 45^\circ) + 0.8944\cos(2t + 63.45^\circ) \\ &\quad - 0.6345\cos(3t + 71.56^\circ) - 0.4851\cos(4t + 78.7^\circ) + \cdots \end{aligned}$$

We notice that $\omega_0 = 1$, $\omega_n = n$ rad/s. The impedance at the source is

$$\mathbf{Z} = 4 + j\omega_n 2 \parallel 4 = 4 + \frac{j\omega_n 8}{4 + j\omega_n 2} = \frac{8 + j\omega_n 8}{2 + j\omega_n}$$

The input current is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{2 + j\omega_n}{8 + j\omega_n 8}\mathbf{V}$$

where $\mathbf{V}$ is the phasor form of the source voltage $v(t)$. By current division,

$$\mathbf{I}_o = \frac{4}{4 + j\omega_n 2}\mathbf{I} = \frac{\mathbf{V}}{4 + j\omega_n 4}$$

Since $\omega_n = n$, $\mathbf{I}_o$ can be expressed as

$$\mathbf{I}_o = \frac{\mathbf{V}}{4\sqrt{1+n^2}\,\underline{/\tan^{-1} n}}$$

For the dc component ($\omega_n = 0$ or $n = 0$)

$$\mathbf{V} = 1 \qquad \Longrightarrow \qquad \mathbf{I}_o = \frac{\mathbf{V}}{4} = \frac{1}{4}$$

For the nth harmonic,

$$\mathbf{V} = \frac{2(-1)^n}{\sqrt{1+n^2}} \angle \tan^{-1} n$$

so that

$$\mathbf{I}_o = \frac{1}{4\sqrt{1+n^2}\angle \tan^{-1} n} \frac{2(-1)^n}{\sqrt{1+n^2}} \angle \tan^{-1} n = \frac{(-1)^n}{2(1+n^2)}$$

In the time domain,

$$i_o(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2(1+n^2)} \cos nt \text{ A}$$

PRACTICE PROBLEM 16.7

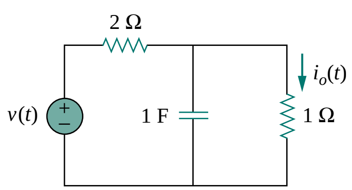


Figure 16.24 For Practice Prob. 16.7.

If the input voltage in the circuit of Fig. 16.24 is

$$v(t) = \frac{1}{3} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \cos nt - \frac{\pi}{n} \sin nt \right) \text{ V}$$

determine the response $i_o(t)$.

Answer: $\frac{1}{9} + \sum_{n=1}^{\infty} \frac{\sqrt{1+n^2\pi^2}}{n^2\pi^2\sqrt{9+4n^2}} \cos\left(nt - \tan^{-1}\frac{2n}{3} + \tan^{-1} n\pi \right)$ A.

16.5 AVERAGE POWER AND RMS VALUES

Recall the concepts of average power and rms value of a periodic signal that we discussed in Chapter 11. To find the average power absorbed by a circuit due to a periodic excitation, we write the voltage and current in amplitude-phase form [see Eq. (16.10)] as

$$v(t) = V_{\text{dc}} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_n) \tag{16.42}$$

$$i(t) = I_{\text{dc}} + \sum_{m=1}^{\infty} I_m \cos(m\omega_0 t - \phi_m) \tag{16.43}$$

Following the passive sign convention (Fig. 16.25), the average power is

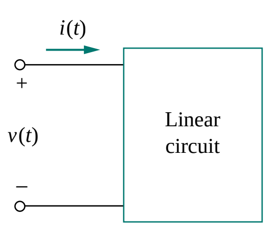


Figure 16.25 The voltage polarity reference and current reference direction.

$$P = \frac{1}{T} \int_0^T vi\, dt \tag{16.44}$$

Substituting Eqs. (16.42) and (16.43) into Eq. (16.44) gives

$$\begin{aligned} P = {} & \frac{1}{T}\int_0^T V_{\text{dc}} I_{\text{dc}}\, dt + \sum_{m=1}^{\infty} \frac{I_m V_{\text{dc}}}{T} \int_0^T \cos(m\omega_0 t - \phi_m)\, dt \\ & + \sum_{n=1}^{\infty} \frac{V_n I_{\text{dc}}}{T} \int_0^T \cos(n\omega_0 t - \theta_n)\, dt \\ & + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{V_n I_m}{T} \int_0^T \cos(n\omega_0 t - \theta_n)\cos(m\omega_0 t - \phi_m)\, dt \end{aligned} \tag{16.45}$$

The second and third integrals vanish, since we are integrating the cosine over its period. According to Eq. (16.4e), all terms in the fourth integral are zero when $m \neq n$. By evaluating the first integral and applying Eq. (16.4g) to the fourth integral for the case $m = n$, we obtain

$$P = V_{dc} I_{dc} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n) \tag{16.46}$$

This shows that in average-power calculation involving periodic voltage and current, the total average power is the sum of the average powers in each harmonically related voltage and current.

Given a periodic function $f(t)$, its rms value (or the effective value) is given by

$$F_{rms} = \sqrt{\frac{1}{T} \int_0^T f^2(t)\, dt} \tag{16.47}$$

Substituting $f(t)$ in Eq. (16.10) into Eq. (16.47) and noting that $(a+b)^2 = a^2 + 2ab + b^2$, we obtain

$$\begin{aligned} F_{rms}^2 = \frac{1}{T} \int_0^T \Bigg[& a_0^2 + 2 \sum_{n=1}^{\infty} a_0 A_n \cos(n\omega_0 t + \phi_n) \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \cos(n\omega_0 t + \phi_n) \cos(m\omega_0 t + \phi_m) \Bigg] dt \\ = \frac{1}{T} \int_0^T a_0^2\, dt & + 2 \sum_{n=1}^{\infty} a_0 A_n \frac{1}{T} \int_0^T \cos(n\omega_0 t + \phi_n)\, dt \\ & + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n A_m \frac{1}{T} \int_0^T \cos(n\omega_0 t + \phi_n) \cos(m\omega_0 t + \phi_m)\, dt \end{aligned} \tag{16.48}$$

Distinct integers n and m have been introduced to handle the product of the two series summations. Using the same reasoning as above, we get

$$F_{rms}^2 = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2$$

or

$$F_{rms} = \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} A_n^2} \tag{16.49}$$

In terms of Fourier coefficients a_n and b_n, Eq. (16.49) may be written as

$$F_{rms} = \sqrt{a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)} \tag{16.50}$$

If $f(t)$ is the current through a resistor R, then the power dissipated in the resistor is

$$P = R F_{rms}^2 \tag{16.51}$$

Or if $f(t)$ is the voltage across a resistor R, the power dissipated in the resistor is

$$P = \frac{F_{\mathrm{rms}}^2}{R} \tag{16.52}$$

One can avoid specifying the nature of the signal by choosing a 1-Ω resistance. The power dissipated by the 1-Ω resistance is

$$P_{1\Omega} = F_{\mathrm{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{16.53}$$

Historical note: Named after the French mathematician Marc-Antoine Parseval Deschemes (1755–1836).

This result is known as *Parseval's theorem*. Notice that a_0^2 is the power in the dc component, while $1/2(a_n^2 + b_n^2)$ is the ac power in the nth harmonic. Thus, Parseval's theorem states that the average power in a periodic signal is the sum of the average power in its dc component and the average powers in its harmonics.

EXAMPLE 16.8

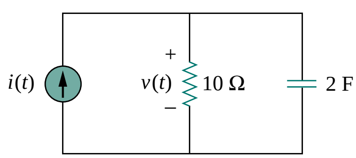


Figure 16.26 For Example 16.8.

Determine the average power supplied to the circuit in Fig. 16.26 if $i(t) = 2 + 10\cos(t + 10^\circ) + 6\cos(3t + 35^\circ)$ A.

Solution:

The input impedance of the network is

$$\mathbf{Z} = 10 \left\| \frac{1}{j2\omega} = \frac{10(1/j2\omega)}{10 + 1/j2\omega} = \frac{10}{1 + j20\omega}\right.$$

Hence,

$$\mathbf{V} = \mathbf{IZ} = \frac{10\mathbf{I}}{\sqrt{1 + 400\omega^2}\,\underline{/\tan^{-1} 20\omega}}$$

For the dc component, $\omega = 0$,

$$\mathbf{I} = 2\ \mathrm{A} \qquad \Longrightarrow \qquad \mathbf{V} = 10(2) = 20\ \mathrm{V}$$

This is expected, because the capacitor is an open circuit to dc and the entire 2-A current flows through the resistor. For $\omega = 1$ rad/s,

$$\mathbf{I} = 10\underline{/10^\circ} \qquad \Longrightarrow \qquad \mathbf{V} = \frac{10(10\underline{/10^\circ})}{\sqrt{1 + 400}\,\underline{/\tan^{-1} 20}}$$

$$= 5\underline{/-77.14^\circ}$$

For $\omega = 3$ rad/s,

$$\mathbf{I} = 6\underline{/45^\circ} \qquad \Longrightarrow \qquad \mathbf{V} = \frac{10(6\underline{/45^\circ})}{\sqrt{1 + 3600}\,\underline{/\tan^{-1} 60}}$$

$$= 1\underline{/-44.05^\circ}$$

Thus, in the time domain,

$$v(t) = 20 + 5\cos(t - 77.14^\circ) + 1\cos(3t - 44.05^\circ)\ \mathrm{V}$$

We obtain the average power supplied to the circuit by applying Eq. (16.46), as

$$P = V_{dc} I_{dc} + \frac{1}{2} \sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

To get the proper signs of θ_n and ϕ_n, we have to compare v and i in this example with Eqs. (16.42) and (16.43). Thus,

$$\begin{aligned} P &= 20(2) + \frac{1}{2}(5)(10)\cos[77.14^\circ - (-10^\circ)] \\ &\quad + \frac{1}{2}(1)(6)\cos[44.05^\circ - (-35^\circ)] \\ &= 40 + 1.247 + 0.05 = 41.5 \ \text{W} \end{aligned}$$

Alternatively, we can find the average power absorbed by the resistor as

$$\begin{aligned} P &= \frac{V_{dc}^2}{R} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{|V_n|}{R} = \frac{20^2}{10} + \frac{1}{2} \cdot \frac{5^2}{10} + \frac{1}{2} \cdot \frac{1^2}{10} \\ &= 40 + 1.25 + 0.05 = 41.5 \text{ W} \end{aligned}$$

which is the same as the power supplied, since the capacitor absorbs no average power.

PRACTICE PROBLEM 16.8

The voltage and current at the terminals of a circuit are

$$v(t) = 80 + 120\cos 120\pi t + 60\cos(360\pi t - 30^\circ)$$
$$i(t) = 5\cos(120\pi t - 10^\circ) + 2\cos(360\pi t - 60^\circ)$$

Find the average power absorbed by the circuit.

Answer: 347.4 W.

EXAMPLE 16.9

Find an estimate for the rms value of the voltage in Example 16.7.

Solution:

From Example 16.7, $v(t)$ is expressed as

$$\begin{aligned} v(t) = 1 &- 1.414\cos(t + 45^\circ) + 0.8944\cos(2t + 63.45^\circ) \\ &- 0.6345\cos(3t + 71.56^\circ) \\ &- 0.4851\cos(4t + 78.7^\circ) + \cdots \text{ V} \end{aligned}$$

Using Eq. (16.49),

$$\begin{aligned} V_{rms} &= \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2} \\ &= \sqrt{1^2 + \frac{1}{2}\left[(-1.414)^2 + (0.8944)^2 + (-0.6345)^2 + (-0.4851)^2 + \cdots\right]} \\ &= \sqrt{2.7186} = 1.649 \text{ V} \end{aligned}$$

This is only an estimate, as we have not taken enough terms of the series. The actual function represented by the Fourier series is

$$v(t) = \frac{\pi e^t}{\sinh \pi}, \qquad -\pi < t < \pi$$

with $v(t) = v(t + T)$. The exact rms value of this is 1.776 V.

PRACTICE PROBLEM 16.9

Find the rms value of the periodic current

$$i(t) = 8 + 30\cos 2t - 20\sin 2t + 15\cos 4t - 10\sin 4t \text{ A}$$

Answer: 29.61 A.

16.6 EXPONENTIAL FOURIER SERIES

A compact way of expressing the Fourier series in Eq. (16.3) is to put it in exponential form. This requires that we represent the sine and cosine functions in the exponential form using Euler's identity:

$$\cos n\omega_0 t = \frac{1}{2}[e^{jn\omega_0 t} + e^{-jn\omega_0 t}] \tag{16.54a}$$

$$\sin n\omega_0 t = \frac{1}{2j}[e^{jn\omega_0 t} - e^{-jn\omega_0 t}] \tag{16.54b}$$

Substituting Eq. (16.54) into Eq. (16.3) and collecting terms, we obtain

$$f(t) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty}[(a_n - jb_n)e^{jn\omega_0 t} + (a_n + jb_n)e^{-jn\omega_0 t}] \tag{16.55}$$

If we define a new coefficient c_n so that

$$c_0 = a_0, \qquad c_n = \frac{(a_n - jb_n)}{2}, \qquad c_{-n} = c_n^* = \frac{(a_n + jb_n)}{2} \tag{16.56}$$

then $f(t)$ becomes

$$f(t) = c_0 + \sum_{n=1}^{\infty}(c_n e^{jn\omega_0 t} + c_{-n}e^{-jn\omega_0 t}) \tag{16.57}$$

or

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{16.58}$$

This is the *complex* or *exponential Fourier series* representation of $f(t)$. Note that this exponential form is more compact than the sine-cosine form in Eq. (16.3). Although the exponential Fourier series coefficients c_n can also be obtained from a_n and b_n using Eq. (16.56), they can also be obtained directly from $f(t)$ as

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\, dt \tag{16.59}$$

where $\omega_0 = 2\pi/T$, as usual. The plots of the magnitude and phase of c_n versus $n\omega_0$ are called the *complex amplitude spectrum* and *complex phase spectrum* of $f(t)$, respectively. The two spectra form the complex frequency spectrum of $f(t)$.

> The exponential Fourier series of a periodic function $f(t)$ describes the spectrum of $f(t)$ in terms of the amplitude and phase angle of ac components at positive and negative harmonic frequencies.

The coefficients of the three forms of Fourier series (sine-cosine form, amplitude-phase form, and exponential form) are related by

$$\boxed{A_n \underline{/\phi_n} = a_n - jb_n = 2c_n} \tag{16.60}$$

or

$$c_n = |c_n| \underline{/\theta_n} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \underline{/-\tan^{-1} b_n/a_n} \tag{16.61}$$

if only $a_n > 0$. Note that the phase θ_n of c_n is equal to ϕ_n.

In terms of the Fourier complex coefficients c_n, the rms value of a periodic signal $f(t)$ can be found as

$$\begin{aligned} F_{\text{rms}}^2 &= \frac{1}{T}\int_0^T f^2(t)\,dt = \frac{1}{T}\int_0^T f(t)\left[\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}\right] dt \\ &= \sum_{n=-\infty}^{\infty} c_n \left[\frac{1}{T}\int_0^T f(t) e^{jn\omega_0 t}\right] \\ &= \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned} \tag{16.62}$$

or

$$F_{\text{rms}} = \sqrt{\sum_{n=-\infty}^{\infty} |c_n|^2} \tag{16.63}$$

Equation (16.62) can be written as

$$F_{\text{rms}}^2 = |c_0|^2 + 2\sum_{n=1}^{\infty} |c_n|^2 \tag{16.64}$$

Again, the power dissipated by a 1-Ω resistance is

$$P_{1\Omega} = F_{\text{rms}}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{16.65}$$

which is a restatement of Parseval's theorem. The *power spectrum* of the signal $f(t)$ is the plot of $|c_n|^2$ versus $n\omega_0$. If $f(t)$ is the voltage across a resistor R, the average power absorbed by the resistor is F_{rms}^2/R; if $f(t)$ is the current through R, the power is $F_{\text{rms}}^2 R$.

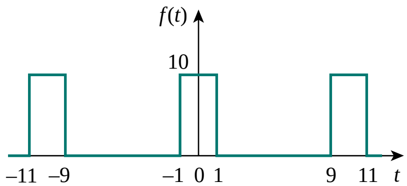


Figure 16.27 The periodic pulse train.

As an illustration, consider the periodic pulse train of Fig. 16.27. Our goal is to obtain its amplitude and phase spectra. The period of the pulse train is $T = 10$, so that $\omega_0 = 2\pi/T = \pi/5$. Using Eq. (16.59),

$$\begin{aligned} c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{10}\int_{-1}^{1} 10e^{-jn\omega_0 t}\,dt \\ &= \frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{-1}^{1} = \frac{1}{-jn\omega_0}(e^{-jn\omega_0} - e^{jn\omega_0}) \\ &= \frac{2}{n\omega_0}\,\frac{e^{jn\omega_0} - e^{-jn\omega_0}}{2j} = 2\frac{\sin n\omega_0}{n\omega_0}, \qquad \omega_0 = \frac{\pi}{5} \\ &= 2\frac{\sin n\pi/5}{n\pi/5} \end{aligned} \tag{16.66}$$

and

$$f(t) = 2\sum_{n=-\infty}^{\infty} \frac{\sin n\pi/5}{n\pi/5} e^{jn\pi t/5} \tag{16.67}$$

Notice from Eq. (16.66) that c_n is the product of 2 and a function of the form $\sin x/x$. This function is known as the *sinc function*; we write it as

The sinc function is called the *sampling function* in communication theory, where it is very useful.

$$\text{sinc}(x) = \frac{\sin x}{x} \tag{16.68}$$

Some properties of the sinc function are important here. For zero argument, the value of the sinc function is unity,

$$\text{sinc}(0) = 1 \tag{16.69}$$

This is obtained applying L'Hopital's rule to Eq. (16.68). For an integral multiple of π, the value of the sinc function is zero,

$$\text{sinc}(n\pi) = 0, \qquad n = 1, 2, 3, \dots \tag{16.70}$$

Also, the sinc function shows even symmetry. With all this in mind, we can obtain the amplitude and phase spectra of $f(t)$. From Eq. (16.66), the magnitude is

$$|c_n| = 2\left|\frac{\sin n\pi/5}{n\pi/5}\right| \tag{16.71}$$

while the phase is

$$\theta_n = \begin{cases} 0^\circ, & \sin\dfrac{n\pi}{5} > 0 \\ 180^\circ, & \sin\dfrac{n\pi}{5} < 0 \end{cases} \tag{16.72}$$

Figure 16.28 shows the plot of $|c_n|$ versus n for n varying from -10 to 10, where $n = \omega/\omega_0$ is the normalized frequency. Figure 16.29 shows the plot of θ_n versus n. Both the amplitude spectrum and phase spectrum are called *line spectra*, because the value of $|c_n|$ and θ_n occur only at discrete values of frequencies. The spacing between the lines is ω_0. The power spectrum, which is the plot of $|c_n|^2$ versus $n\omega_0$, can also be plotted. Notice that the sinc function forms the envelope of the amplitude spectrum.

Examining the input and output spectra allows visualization of the effect of a circuit on a periodic signal.

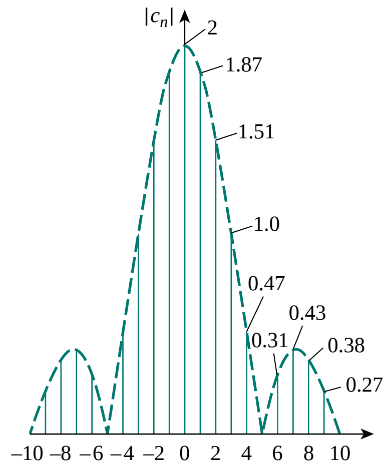


Figure 16.28 The amplitude of a periodic pulse train.

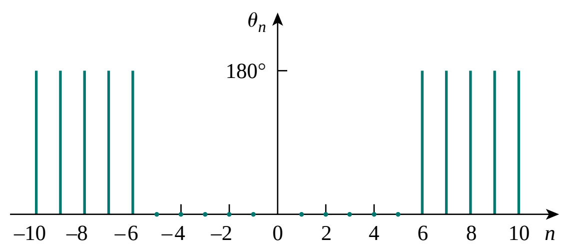


Figure 16.29 The phase spectrum of a periodic pulse train.

EXAMPLE 16.10

Find the exponential Fourier series expansion of the periodic function $f(t) = e^t, 0 < t < 2\pi$ with $f(t + 2\pi) = f(t)$.

Solution:

Since $T = 2\pi$, $\omega_0 = 2\pi/T = 1$. Hence,

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^t e^{-jnt}\,dt$$

$$= \frac{1}{2\pi}\frac{1}{1-jn}e^{(1-jn)t}\Big|_0^{2\pi} = \frac{1}{2\pi(1-jn)}[e^{2\pi}e^{-j2\pi n} - 1]$$

But by Euler's identity,

$$e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1 - j0 = 1$$

Thus,

$$c_n = \frac{1}{2\pi(1-jn)}[e^{2\pi} - 1] = \frac{85}{1-jn}$$

The complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{85}{1-jn}e^{jnt}$$

We may want to plot the complex frequency spectrum of $f(t)$. If we let $c_n = |c_n|\angle\theta_n$, then

$$|c_n| = \frac{85}{\sqrt{1+n^2}}, \qquad \theta_n = \tan^{-1} n$$

By inserting in negative and positive values of n, we obtain the amplitude and the phase plots of c_n versus $n\omega_0 = n$, as in Fig. 16.30.

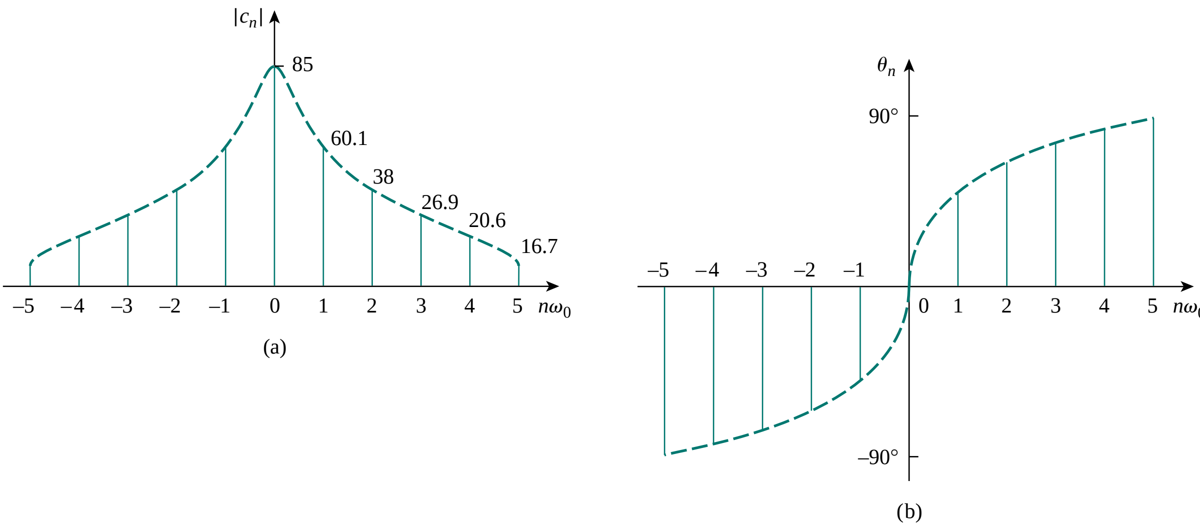


Figure 16.30 The complex frequency spectrum of the function in Example 16.10: (a) amplitude spectrum, (b) phase spectrum.

PRACTICE PROBLEM 16.10

Obtain the complex Fourier series of the function in Fig. 16.1.

Answer: $f(t) = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n\neq 0 \\ n=\text{odd}}}^{\infty} \frac{j}{n\pi} e^{jn\pi t}$.

EXAMPLE 16.11

Find the complex Fourier series of the sawtooth wave in Fig. 16.9. Plot the amplitude and the phase spectra.

Solution:

From Fig. 16.9, $f(t) = t$, $0 < t < 1$, $T = 1$ so that $\omega_0 = 2\pi/T = 2\pi$. Hence,

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt = \frac{1}{1}\int_0^1 te^{-j2n\pi t}\,dt \qquad (16.11.1)$$

But

$$\int te^{at}\,dt = \frac{e^{at}}{a^2}(ax-1)+C$$

Applying this to Eq. (16.11.1) gives

$$\begin{aligned} c_n &= \frac{e^{-j2n\pi t}}{(-j2n\pi)^2}(-j2n\pi t-1)\Big|_0^1 \\ &= \frac{e^{-j2n\pi}(-j2n\pi-1)+1}{-4n^2\pi^2} \end{aligned} \qquad (16.11.2)$$

Again,

$$e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n = 1 - j0 = 1$$

so that Eq. (16.11.2) becomes

$$c_n = \frac{-j2n\pi}{-4n^2\pi^2} = \frac{j}{2n\pi} \qquad (16.11.3)$$

This does not include the case when $n = 0$. When $n = 0$,

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{1}\int_0^1 t\,dt = \frac{t^2}{2}\Big|_1^0 = 0.5 \qquad (16.11.4)$$

Hence,

$$f(t) = 0.5 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j}{2n\pi}e^{j2n\pi t} \qquad (16.11.5)$$

and

$$|c_n| = \begin{cases} \dfrac{1}{2|n|\pi}, & n \neq 0 \\ 0.5, & n = 0 \end{cases}, \qquad \theta_n = 90^\circ, \quad n \neq 0 \qquad (16.11.6)$$

By plotting $|c_n|$ and θ_n for different n, we obtain the amplitude spectrum and the phase spectrum shown in Fig. 16.31.

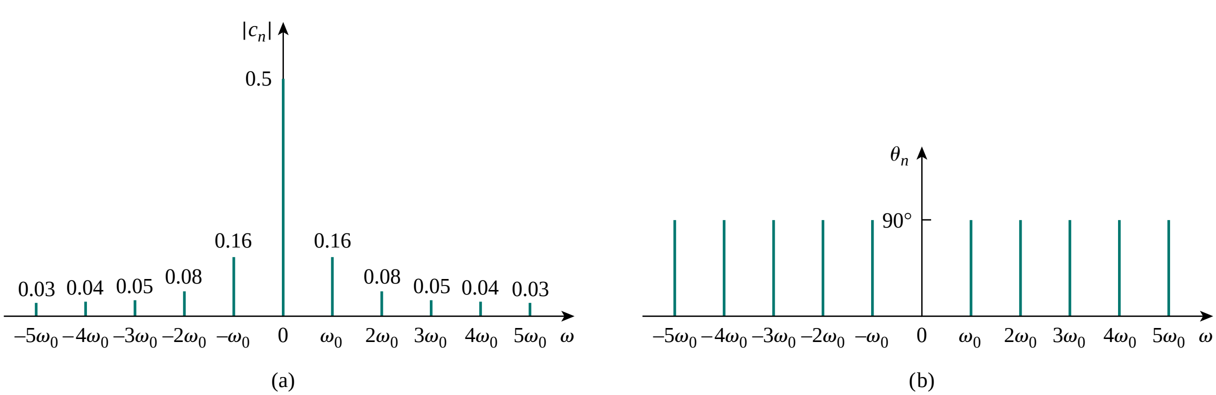


Figure 16.31 For Example 16.11: (a) amplitude spectrum, (b) phase spectrum.

PRACTICE PROBLEM 16.11

Obtain the complex Fourier series expansion of $f(t)$ in Fig. 16.17. Show the amplitude and phase spectra.

Answer: $f(t) = -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j(-1)^n}{n\pi} e^{jn\pi t}$. See Fig. 16.32 for the spectra.

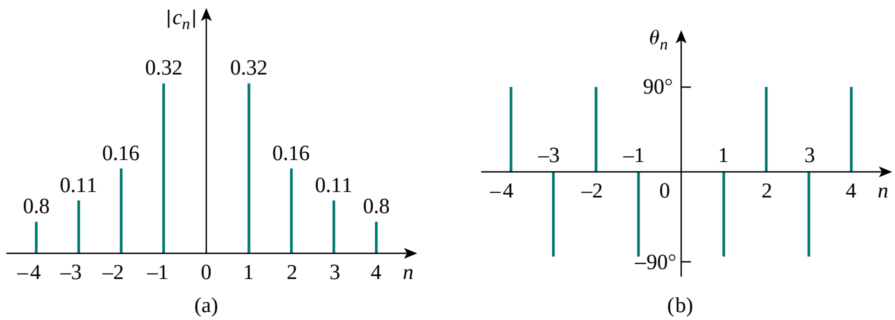


Figure 16.32 For Practice Prob. 16.11: (a) amplitude spectrum, (b) phase spectrum.

16.7 FOURIER ANALYSIS WITH *PSPICE*

Fourier analysis is usually performed with *PSpice* in conjunction with transient analysis. Therefore, we must do a transient analysis in order to perform a Fourier analysis.

To perform the Fourier analysis of a waveform, we need a circuit whose input is the waveform and whose output is the Fourier decomposition. A suitable circuit is a current (or voltage) source in series with a 1-Ω resistor as shown in Fig. 16.33. The waveform is inputted as $v_s(t)$ using VPULSE for a pulse or VSIN for a sinusoid, and the attributes of the waveform are set over its period T. The output V(1) from node 1 is the dc level (a_0) and the first nine harmonics (A_n) with their corresponding phases ψ_n; that is,

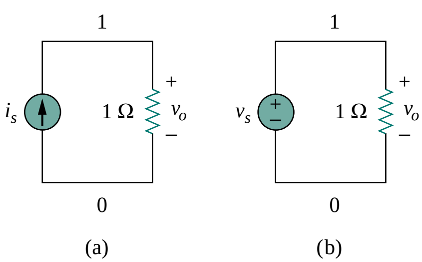


Figure 16.33 Fourier analysis with *PSpice* using: (a) a current source, (b) a voltage source.

$$v_o(t) = a_0 + \sum_{n=1}^{9} A_n \sin(n\omega_0 t + \psi_n) \tag{16.73}$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \psi_n = \phi_n - \frac{\pi}{2}, \qquad \phi_n = \tan^{-1}\frac{b_n}{a_n} \tag{16.74}$$

Notice in Eq. (16.74) that the *PSpice* output is in the sine and angle form rather than the cosine and angle form in Eq. (16.10). The *PSpice* output also includes the normalized Fourier coefficients. Each coefficient a_n is normalized by dividing it by the magnitude of the fundamental a_1 so that the normalized component is a_n/a_1. The corresponding phase ψ_n is normalized by subtracting from it the phase ψ_1 of the fundamental, so that the normalized phase is $\psi_n - \psi_1$.

There are two types of Fourier analyses offered by *PSpice for Windows: Discrete Fourier Transform* (DFT) performed by the *PSpice* program and *Fast Fourier Transform* (FFT) performed by the *Probe* program. While DFT is an approximation of the exponential Fourier series, FTT is an algorithm for rapid efficient numerical computation of DFT. A full discussion of DFT and FTT is beyond the scope of this book.

16.7.1 Discrete Fourier Transform

A discrete Fourier transform (DFT) is performed by the *PSpice* program, which tabulates the harmonics in an output file. To enable a Fourier analysis, we select **Analysis/Setup/Transient** and bring up the Transient dialog box, shown in Fig. 16.34. The *Print Step* should be a small fraction of the period T, while the *Final Time* could be $6T$. The *Center Frequency* is the fundamental frequency $f_0 = 1/T$. The particular variable whose DFT is desired, V(1) in Fig. 16.34, is entered in the **Output Vars** command box. In addition to filling in the Transient dialog box, **DCLICK** *Enable Fourier*. With the Fourier analysis enabled and the schematic saved, run *PSpice* by selecting **Analysis/Simulate** as usual. The program executes a harmonic decomposition into Fourier components of the result of the transient analysis. The results are sent to an output file which you can retrieve by selecting **Analysis/Examine Output**. The output file includes the dc value and the first nine harmonics by default, although you can specify more in the *Number of harmonics* box (see Fig. 16.34).

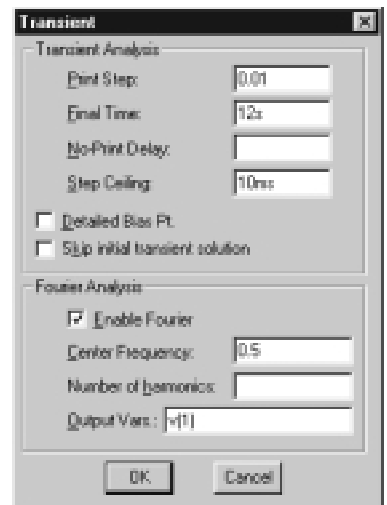


Figure 16.34 Transient dialog box.

16.7.2 Fast Fourier Transform

A fast Fourier transform (FFT) is performed by the *Probe* program and displays as a *Probe* plot the complete spectrum of a transient expression. As explained above, we first construct the schematic in Fig. 16.33(b) and enter the attributes of the waveform. We also need to enter the *Print Step* and the *Final Time* in the Transient dialog box. Once this is done, we can obtain the FFT of the waveform in two ways.

One way is to insert a voltage marker at node 1 in the schematic of the circuit in Fig. 16.33(b). After saving the schematic and selecting **Analysis/Simulate**, the waveform V(1) will be displayed in the Probe window. Double clicking the FFT icon in the Probe menu will automatically replace the waveform with its FFT. From the FFT-generated graph, we can obtain the harmonics. In case the FFT-generated graph is crowded, we can use the *User Defined* data range (see Fig. 16.35) to specify a smaller range.

Another way of obtaining the FFT of V(1) is to not insert a voltage marker at node 1 in the schematic. After selecting **Analysis/Simulate**, the Probe window will come up with no graph on it. We select **Trace/Add** and type V(1) in the **Trace Command** box and **DCLICKL OK**. We now select **Plot/X-Axis Settings** to bring up the *X Axis Setting* dialog box shown in Fig. 16.35 and then select **Fourier/OK**. This will cause the FFT of the selected trace (or traces) to be displayed. This second approach is useful for obtaining the FFT of any trace associated with the circuit.

A major advantage of the FFT method is that it provides graphical output. But its major disadvantage is that some of the harmonics may be too small to see.

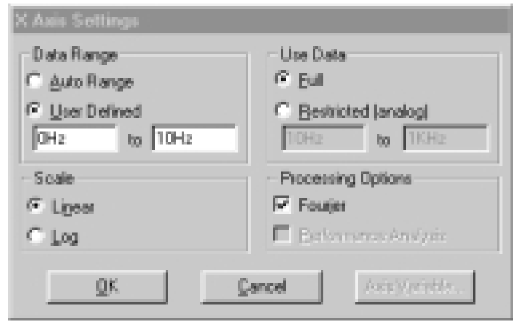


Figure 16.35 *X* axis settings dialog box.

In both DFT and FFT, we should let the simulation run for a large number of cycles and use a small value of *Step Ceiling* (in the Transient dialog box) to ensure accurate results. The *Final Time* in the Transient dialog box should be at least five times the period of the signal to allow the simulation to reach steady state.

EXAMPLE 16.12

Use *PSpice* to determine the Fourier coefficients of the signal in Fig. 16.1.

Solution:

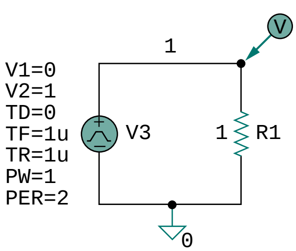


Figure 16.36 Schematic for Example 16.12.

Figure 16.36 shows the schematic for obtaining the Fourier coefficients. With the signal in Fig. 16.1 in mind, we enter the attributes of the voltage source VPULSE as shown in Fig. 16.36. We will solve this example using both the DFT and FFT approaches.

METHOD 1 **DFT Approach:** (The voltage marker in Fig. 16.36 is not needed for this method.) From Fig. 16.1, it is evident that $T = 2$ s,

$$f_0 = \frac{1}{T} = \frac{1}{2} = 0.5 \text{ Hz}$$

So, in the transient dialog box, we select the *Final Time* as 6T = 12 s, the *Print Step* as 0.01 s, the *Step Ceiling* as 10 ms, the *Center Frequency* as 0.5 Hz, and the output variable as V(1). (In fact, Fig. 16.34 is for this particular example.) When *PSpice* is run, the output file contains the following result.

```
FOURIER COEFFICIENTS OF TRANSIENT RESPONSE V(1)

DC COMPONENT = 4.989950E-01

HARMONIC  FREQUENCY   FOURIER    NORMALIZED     PHASE     NORMALIZED
  NO        (HZ)     COMPONENT   COMPONENT      (DEG)     PHASE (DEG)

  1       5.000E-01  6.366E-01  1.000E+00   -1.809E-01   0.000E+00
  2       1.000E+00  2.012E-03  3.160E-03   -9.226E+01  -9.208E+01
  3       1.500E+00  2.122E-01  3.333E-01   -5.427E-01  -3.619E-01
```

(continued)

(continued)

4	2.000E+00	2.016E-03	3.167E-03	-9.451E+01	-9.433E+01
5	2.500E+00	1.273E-01	1.999E-01	-9.048E-01	-7.239E-01
6	3.000E+00	2.024E-03	3.180E-03	-9.676E+01	-9.658E+01
7	3.500E+00	9.088E-02	1.427E-01	-1.267E+00	-1.086E+00
8	4.000E+00	2.035E-03	3.197E-03	-9.898E+01	-9.880E+01
9	4.500E+00	7.065E-02	1.110E-01	-1.630E+00	-1.449E+00

Comparing the result with that in Eq. (16.1.7) (see Example 16.1) or with the spectra in Fig. 16.4 shows a close agreement. From Eq. (16.1.7), the dc component is 0.5 while *PSpice* gives 0.498995. Also, the signal has only odd harmonics with phase $\psi_n = -90^\circ$, whereas *PSpice* seems to indicate that the signal has even harmonics although the magnitudes of the even harmonics are small.

METHOD 2 **FFT Approach:** With voltage marker in Fig. 16.36 in place, we run *PSpice* and obtain the waveform V(1) shown in Fig. 16.37(a) on the Probe window. By double clicking the FFT icon in the Probe menu and changing the X-axis setting to 0 to 10 Hz, we obtain the FFT of V(1) as shown in Fig. 16.37(b). The FFT-generated graph contains the dc and harmonic components within the selected frequency range. Notice that the magnitudes and frequencies of the harmonics agree with the DFT-generated tabulated values.

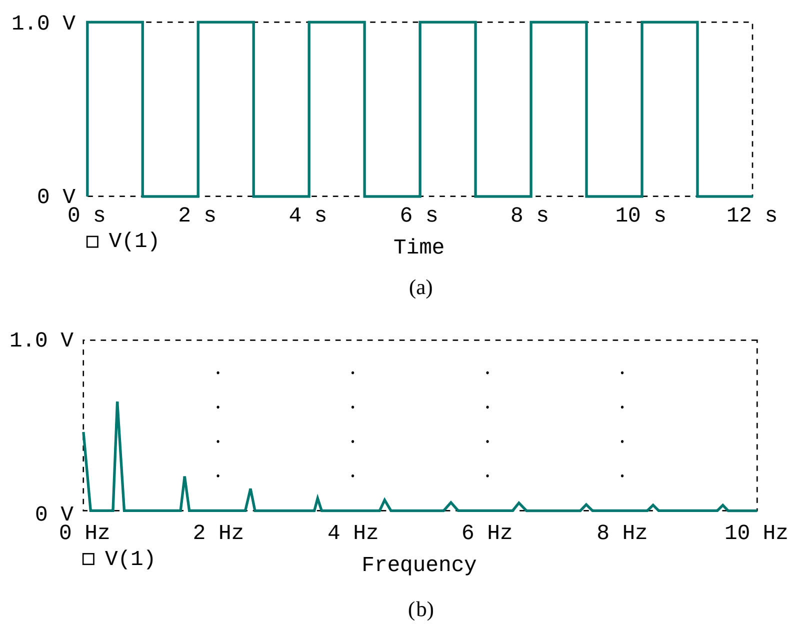


Figure 16.37 (a) Original waveform of Fig. 16.1, (b) FFT of the waveform.

PRACTICE PROBLEM 16.12

Obtain the Fourier coefficients of the function in Fig. 16.7 using *PSpice*.

Answer:

FOURIER COEFFICIENTS OF TRANSIENT RESPONSE V(1)

DC COMPONENT = 4.950000E-01

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.000E+00	3.184E-01	1.000E+00	-1.782E+02	0.000E+00
2	2.000E+00	1.593E-01	5.002E-01	-1.764E+02	1.800E+00
3	3.000E+00	1.063E-01	3.338E-01	-1.746E+02	3.600E+00
4	4.000E+00	7.979E-02	2.506E-03	-1.728E+02	5.400E+00
5	5.000E+00	6.392E-01	2.008E-01	-1.710E+02	7.200E+00
6	6.000E+00	5.337E-02	1.676E-03	-1.692E+02	9.000E+00
7	7.000E+00	4.584E-02	1.440E-01	-1.674E+02	1.080E+01
8	8.000E+00	4.021E-02	1.263E-01	-1.656E+02	1.260E+01
9	9.000E+00	3.584E-02	1.126E-01	-1.638E+02	1.440E+01

EXAMPLE 16.13

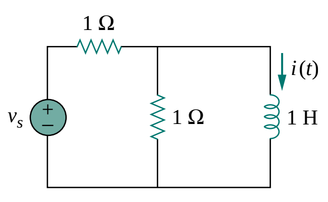


Figure 16.38 For Example 16.13.

If v_s in the circuit of Fig. 16.38 is a sinusoidal voltage source of amplitude 12 V and frequency 100 Hz, find current $i(t)$.

Solution:

The schematic is shown in Fig. 16.39. We may use the DFT approach to obtain the Fourier coefficents of $i(t)$. Since the period of the input waveform is $T = 1/100 = 10$ ms, in the Transient dialog box we select *Print Step:* 0.1 ms, *Final Time:* 100 ms, *Center Frequency:* 100 Hz, *Number of harmonics:* 4, and *Output Vars:* I(L1). When the circuit is simulated, the output file includes the following.

FOURIER COEFFICIENTS OF TRANSIENT RESPONSE I(VD)

DC COMPONENT = 8.583269E-03

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.000E+02	8.730E-03	1.000E+00	-8.984E+01	0.000E+00
2	2.000E+02	1.017E-04	1.165E-02	-8.306E+01	6.783E+00
3	3.000E+02	6.811E-05	7.802E-03	-8.235E+01	7.490E+00
4	4.000E+02	4.403E-05	5.044E-03	-8.943E+01	4.054E+00

With the Fourier coefficients, the Fourier series describing the current $i(t)$ can be obtained using Eq. (16.73); that is,

$$\begin{aligned} i(t) &= 8.5833 + 8.73\sin(2\pi \cdot 100t - 89.84^\circ) \\ &\quad + 0.1017\sin(2\pi \cdot 200t - 83.06^\circ) \\ &\quad + 0.068\sin(2\pi \cdot 300t - 82.35^\circ) + \cdots \text{ mA} \end{aligned}$$

We can also use the FFT approach to cross-check our result. The current marker is inserted at pin 1 of the inductor as shown in Fig. 16.39. Running *PSpice* will automatically produce the plot of I(L1) in the Probe window, as shown in Fig. 16.40(a). By double clicking the FFT icon and setting the range of the X-axis from 0 to 200 Hz, we generate the FFT of I(L1) shown in Fig. 16.40(b). It is clear from the FFT-generated plot that only the dc component and the first harmonic are visible. Higher harmonics are negligibly small.

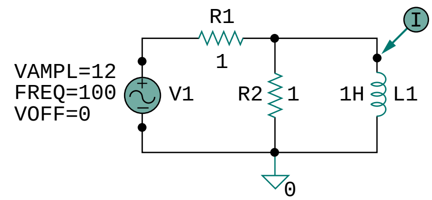


Figure 16.39 Schematic of the circuit in Fig. 16.38.

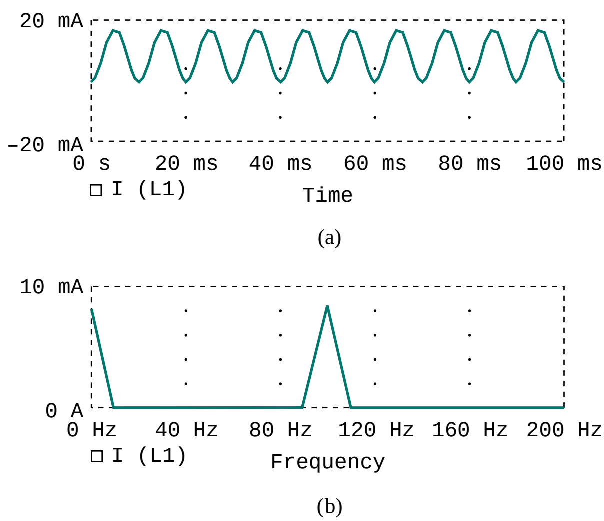


Figure 16.40 For Example 16.13: (a) plot of $i(t)$, (b) the FFT of $i(t)$.

PRACTICE PROBLEM 16.13

A sinusoidal current source of amplitude 4 A and frequency 2 kHz is applied to the circuit in Fig. 16.41. Use *PSpice* to find $v(t)$.

Answer: $v(t) = -150.72 + 145.5\sin(4\pi \cdot 10^3 t + 90^\circ) + \cdots$ μV. The Fourier components are shown below.

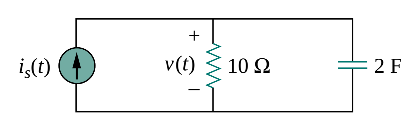


Figure 16.41 For Practice Prob. 16.14.

FOURIER COEFFICIENTS OF TRANSIENT RESPONSE V(R1:1)

DC COMPONENT = -1.507169E-04

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	2.000E+03	1.455E-04	1.000E+00	9.006E+01	0.000E+00
2	4.000E+03	1.851E-06	1.273E-02	9.597E+01	5.910E+00
3	6.000E+03	1.406E-06	9.662E-03	9.323E+01	3.167E+00
4	8.000E+03	1.010E-06	6.946E-02	8.077E+01	-9.292E+00

†16.8 APPLICATIONS

We demonstrated in Section 16.4 that the Fourier series expansion permits the application of the phasor techniques used in ac analysis to circuits containing nonsinusoidal periodic excitations. The Fourier series has many other practical applications, particularly in communications and signal processing. Typical applications include spectrum analysis, filtering, rectification, and harmonic distortion. We will consider two of these: spectrum analyzers and filters.

TABLE 16.4 Frequency ranges of typical signals.

Signal	Frequency Range
Audible sounds	20 Hz to 15 kHz
AM radio	540–1600 kHz
Short-wave radio	3–36 MHz
Video signals (U.S. standards)	dc to 4.2 MHz
VHF television, FM radio	54–216 MHz
UHF television	470–806 MHz
Cellular telephone	824–891.5 MHz
Microwaves	2.4–300 GHz
Visible light	10^5–10^6 GHz
X-rays	10^8–10^9 GHz

16.8.1 Spectrum Analyzers

The Fourier series provides the spectrum of a signal. As we have seen, the spectrum consists of the amplitudes and phases of the harmonics versus frequency. By providing the spectrum of a signal $f(t)$, the Fourier series helps us identify the pertinent features of the signal. It demonstrates which frequencies are playing an important role in the shape of the output and which ones are not. For example, audible sounds have significant components in the frequency range of 20 Hz to 15 kHz, while visible light signals range from 10^5 GHz to 10^6 GHz. Table 16.4 presents some other signals and the frequency ranges of their components. A periodic function is said to be *band-limited* if its amplitude spectrum contains only a finite number of coefficients A_n or c_n. In this case, the Fourier series becomes

$$f(t) = \sum_{n=-N}^{N} c_n e^{jn\omega_0 t} = a_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t + \phi_n) \qquad (16.75)$$

This shows that we need only $2N+1$ terms (namely, $a_0, A_1, A_2, \dots, A_N, \phi_1, \phi_2, \dots, \phi_N$) to completely specify $f(t)$ if ω_0 is known. This leads to the *sampling theorem:* a band-limited periodic function whose Fourier series contains N harmonics is uniquely specified by its values at $2N+1$ instants in one period.

A *spectrum analyzer* is an instrument that displays the amplitude of the components of a signal versus frequency. In other words, it shows the various frequency components (spectral lines) that indicate the amount of energy at each frequency. It is unlike an oscilloscope, which displays the entire signal (all components) versus time. An oscilloscope shows the signal in the time domain, while the spectrum analyzer shows the signal in the frequency domain. There is perhaps no instrument more useful to a circuit analyst than the spectrum analyzer. An analyzer can conduct noise and spurious signal analysis, phase checks, electromagnetic interference and filter examinations, vibration measurements, radar measurements, and more. Spectrum analyzers are commercially available in various sizes and shapes. Figure 16.42 displays a typical one.

16.8.2 Filters

Filters are an important component of electronics and communications systems. Chapter 14 presented a full discussion on passive and active filters. Here, we investigate how to design filters to select the fundamental component (or any desired harmonic) of the input signal and reject other harmonics. This filtering process cannot be accomplished without the

Figure 16.42 A typical spectrum analyzer.
(Courtesy of Hewlett-Packer.)

Fourier series expansion of the input signal. For the purpose of illustration, we will consider two cases, a lowpass filter and a bandpass filter. In Example 16.6, we already looked at a highpass RL filter.

The output of a lowpass filter depends on the input signal, the transfer function $H(\omega)$ of the filter, and the corner or half-power frequency ω_c. We recall that $\omega_c = 1/RC$ for an RC passive filter. As shown in Fig. 16.43(a), the lowpass filter passes the dc and low-frequency components, while blocking the high-frequency components. By making ω_c sufficiently large ($\omega_c \gg \omega_0$, e.g., making C small), a large number of the

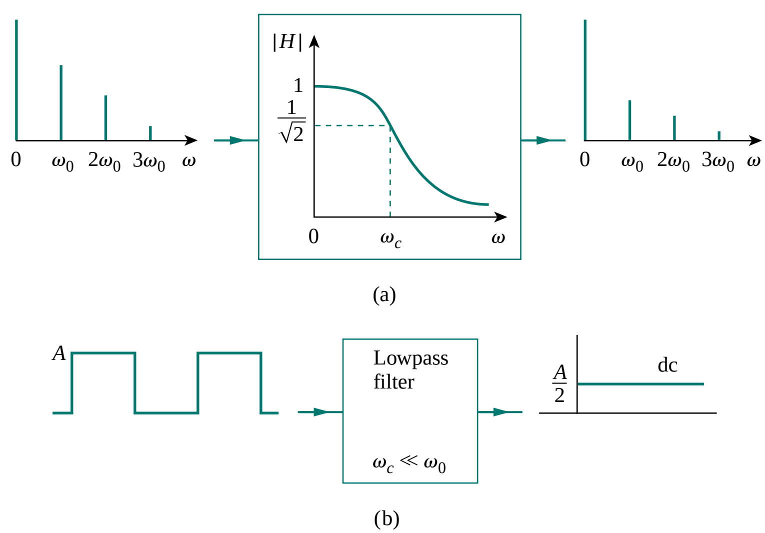


Figure 16.43 (a) Input and output spectra of a lowpass filter, (b) the lowpass filter passes only the dc component when $\omega_c \ll \omega_0$.

harmonics can be passed. On the other hand, by making ω_c sufficiently small ($\omega_c \ll \omega_0$), we can block out all the ac components and pass only dc, as shown typically in Fig. 16.43(b). (See Fig. 16.2(a) for the Fourier series expansion of the square wave.)

Similarly, the output of a bandpass filter depends on the input signal, the transfer function of the filter $H(\omega)$, its bandwidth B, and its center frequency ω_c. As illustrated in Fig. 16.44(a), the filter passes all the harmonics of the input signal within a band of frequencies ($\omega_1 < \omega < \omega_2$) centered around ω_c. We have assumed that ω_0, $2\omega_0$, and $3\omega_0$ are within that band. If the filter is made highly selective ($B \ll \omega_0$) and $\omega_c = \omega_0$, where ω_0 is the fundamental frequency of the input signal, the filter passes only the fundamental component ($n = 1$) of the input and blocks out all higher harmonics. As shown in Fig. 16.44(b), with a square wave as input, we obtain a sine wave of the same frequency as the output. (Again, refer to Fig. 16.2(a).)

In this section, we have used ω_c for the center frequency of the bandpass filter instead of ω_0 as in Chapter 14, to avoid confusing ω_0 with the fundamental frequency of the input signal.

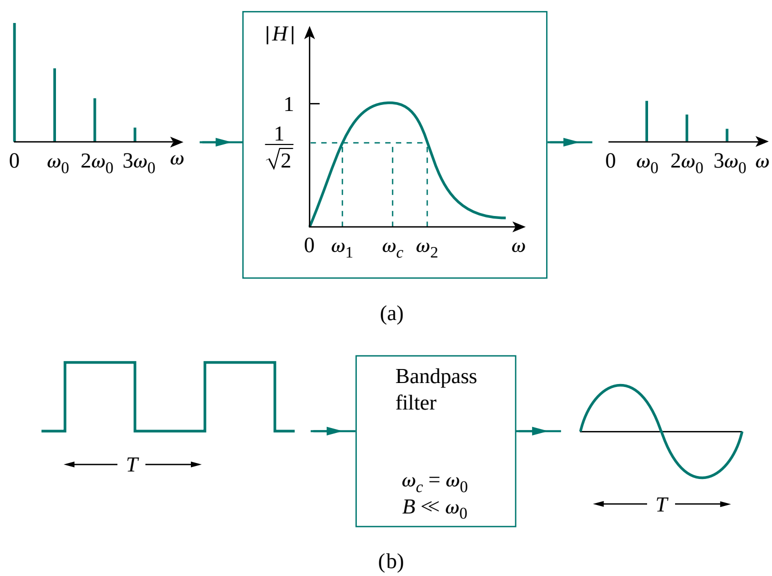


Figure 16.44 (a) Input and output spectra of a bandpass filter, (b) the bandpass filter passes only the fundamental component when $B \ll \omega_0$.

EXAMPLE 16.14

If the sawtooth waveform in Fig. 16.45(a) is applied to an ideal lowpass filter with the transfer function shown in Fig. 16.45(b), determine the output.

Solution:

The input signal in Fig. 16.45(a) is the same as the signal in Fig. 16.9. From Practice Prob. 16.2, we know that the Fourier series expansion is

$$x(t) = \frac{1}{2} - \frac{1}{\pi}\sin\omega_0 t - \frac{1}{2\pi}\sin 2\omega_0 t - \frac{1}{3\pi}\sin 3\omega_0 t - \cdots$$

where the period is $T = 1$ s and the fundamental frequency is $\omega_0 = 2\pi$ rad/s. Since the corner frequency of the filter is $\omega_c = 10$ rad/s, only the dc component and harmonics with $n\omega_0 < 10$ will be passed. For $n = 2$, $n\omega_0 = 4\pi = 12.566$ rad/s, which is higher than 10 rad/s, meaning that second and higher harmonics will be rejected. Thus, only the dc and fundamental components will be passed. Hence the output of the filter is

$$y(t) = \frac{1}{2} - \frac{1}{\pi}\sin 2\pi t$$

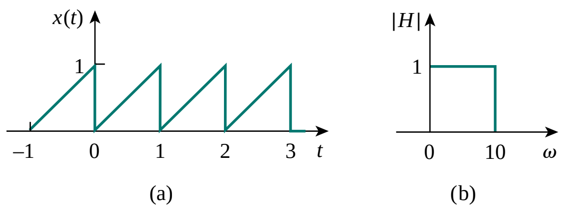


Figure 16.45 For Example 16.14.

PRACTICE PROBLEM 16.14

Rework Example 16.14 if the lowpass filter is replaced by the ideal bandpass filter shown in Fig. 16.46.

Answer: $y(t) = -\frac{1}{3\pi}\sin 3\omega_0 t - \frac{1}{4\pi}\sin 4\omega_0 t - \frac{1}{5\pi}\sin 5\omega_0 t.$

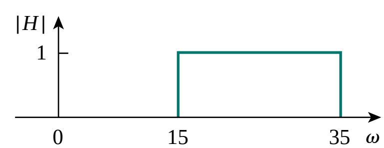


Figure 16.46 For Practice Prob. 16.14.

16.9 SUMMARY

1. A periodic function is one that repeats itself every T seconds; that is, $f(t \pm nT) = f(t), n = 1, 2, 3, \ldots$.
2. Any nonsinusoidal periodic function $f(t)$ that we encounter in electrical engineering can be expressed in terms of sinusoids using Fourier series:

$$f(t) = \underbrace{a_0}_{\text{dc}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{\text{ac}}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency. The Fourier series resolves the function into the dc component a_0 and an ac component containing infinitely many harmonically related sinusoids. The

Fourier coefficients are determined as

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt, \qquad a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt$$

If $f(t)$ is an even function, $b_n = 0$, and when $f(t)$ is odd, $a_0 = 0$ and $a_n = 0$. If $f(t)$ is half-wave symmetric, $a_0 = a_n = b_n = 0$ for even values of n.

3. An alternative to the trigonometric (or sine-cosine) Fourier series is the amplitude-phase form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\frac{b_n}{a_n}$$

4. Fourier series representation allows us to apply the phasor method in analyzing circuits when the source function is a nonsinusoidal periodic function. We use phasor technique to determine the response of each harmonic in the series, transform the responses to the time domain, and add them up.
5. The average-power of periodic voltage and current is

$$P = V_{dc}I_{dc} + \frac{1}{2}\sum_{n=1}^{\infty} V_n I_n \cos(\theta_n - \phi_n)$$

In other words, the total average power is the sum of the average powers in each harmonically related voltage and current.

6. A periodic function can also be represented in terms of an exponential (or complex) Fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}\,dt$$

and $\omega_0 = 2\pi/T$. The exponential form describes the spectrum of $f(t)$ in terms of the amplitude and phase of ac components at positive and negative harmonic frequencies. Thus, there are three basic forms of Fourier series representation: the trigonometric form, the amplitude-phase form, and the exponential form.

7. The frequency (or line) spectrum is the plot of A_n and ϕ_n or $|c_n|$ and θ_n versus frequency.
8. The rms value of a periodic function is given by

$$F_{rms} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

The power dissipated by a 1-Ω resistance is

$$P_{1\Omega} = F_{\text{rms}}^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

This relationship is known as *Parseval's theorem*.

9. Using *PSpice*, a Fourier analysis of a circuit can be performed in conjunction with the transient analysis.
10. Fourier series find application in spectrum analyzers and filters. The spectrum analyzer is an instrument that displays the discrete Fourier spectra of an input signal, so that an analyst can determine the frequencies and relative energies of the signal's components. Because the Fourier spectra are discrete spectra, filters can be designed for great effectiveness in blocking frequency components of a signal that are outside a desired range.

REVIEW QUESTIONS

16.1 Which of the following cannot be a Fourier series?

(a) $t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5}$
(b) $5\sin t + 3\sin 2t - 2\sin 3t + \sin 4t$
(c) $\sin t - 2\cos 3t + 4\sin 4t + \cos 4t$
(d) $\sin t + 3\sin 2.7t - \cos\pi t + 2\tan\pi t$
(e) $1 + e^{-j\pi t} + \frac{e^{-j2\pi t}}{2} + \frac{e^{-j3\pi t}}{3}$

16.2 If $f(t) = t$, $0 < t < \pi$, $f(t + n\pi) = f(t)$, the value of ω_0 is

(a) 1 (b) 2 (c) π (d) 2π

16.3 Which of the following are even functions?

(a) $t + t^2$ (b) $t^2\cos t$ (c) e^{t^2}
(d) $t^2 + t^4$ (e) $\sinh t$

16.4 Which of the following are odd functions?

(a) $\sin t + \cos t$ (b) $t\sin t$
(c) $t\ln t$ (d) $t^3\cos t$
(e) $\sinh t$

16.5 If $f(t) = 10 + 8\cos t + 4\cos 3t + 2\cos 5t + \cdots$, the magnitude of the dc component is:

(a) 10 (b) 8 (c) 4
(d) 2 (e) 0

16.6 If $f(t) = 10 + 8\cos t + 4\cos 3t + 2\cos 5t + \cdots$, the angular frequency of the 6th harmonic is

(a) 12 (b) 11 (c) 9
(d) 6 (e) 1

16.7 The function in Fig. 16.14 is half-wave symmetric.

(a) True (b) False

16.8 The plot of $|c_n|$ versus $n\omega_0$ is called:

(a) complex frequency spectrum
(b) complex amplitude spectrum
(c) complex phase spectrum

16.9 When the periodic voltage $2 + 6\sin\omega_0 t$ is applied to a 1-Ω resistor, the integer closest to the power (in watts) dissipated in the resistor is:

(a) 5 (b) 8 (c) 20
(d) 22 (e) 40

16.10 The instrument for displaying the spectrum of a signal is known as:

(a) oscilloscope (b) spectrogram
(c) spectrum analyzer (d) Fourier spectrometer

Answers: 16.1a,d, 16.2b, 16.3b,c,d, 16.4d,e, 16.5a, 16.6d, 16.7a, 16.8b, 16.9d ,16.10c.

PROBLEMS

Section 16.2 Trigonometric Fourier Series

16.1 Evaluate each of the following functions and see if it is periodic. If periodic, find its period.

(a) $f(t) = \cos\pi t + 2\cos 3\pi t + 3\cos 5\pi t$
(b) $y(t) = \sin t + 4\cos 2\pi t$
(c) $g(t) = \sin 3t\cos 4t$
(d) $h(t) = \cos^2 t$
(e) $z(t) = 4.2\sin(0.4\pi t + 10^\circ) + 0.8\sin(0.6\pi t + 50^\circ)$
(f) $p(t) = 10$
(g) $q(t) = e^{-\pi t}$

16.2 Determine the period of these periodic functions:
(a) $f_1(t) = 4\sin 5t + 3\sin 6t$
(b) $f_2(t) = 12 + 5\cos 2t + 2\cos(4t + 45^\circ)$
(c) $f_3(t) = 4\sin^2 600\pi t$
(d) $f_4(t) = e^{j10t}$

16.3 Give the Fourier coefficients a_0, a_n, and b_n of the waveform in Fig. 16.47. Plot the amplitude and phase spectra.

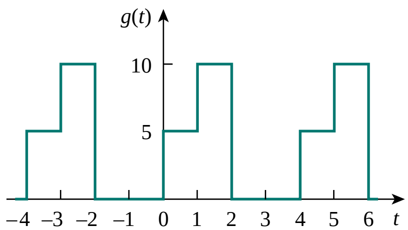


Figure 16.47 For Prob. 16.3.

16.4 Find the Fourier series expansion of the backward sawtooth waveform of Fig. 16.48. Obtain the amplitude and phase spectra.

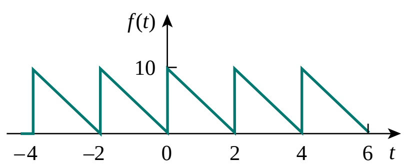


Figure 16.48 For Probs. 16.4 and 16.50.

***16.5** A voltage source has a periodic waveform defined over its period as

$$v(t) = t(2\pi - t)\ \text{V}, \qquad 0 < t < 2\pi$$

Find the Fourier series for this voltage.

16.6 A periodic function is defined over its period as

$$h(t) = \begin{cases} 10\sin t, & 0 < t < \pi \\ 20\sin(t - \pi), & \pi < t < 2\pi \end{cases}$$

Find the Fourier series of $h(t)$.

16.7 Find the quadrature (cosine and sine) form of the Fourier series

$$f(t) = 2 + \sum_{n=1}^{\infty} \frac{10}{n^3 + 1} \cos\left(2nt + \frac{n\pi}{4}\right)$$

16.8 Express the Fourier series

$$f(t) = 10 + \sum_{n=1}^{\infty} \frac{4}{n^2 + 1} \cos 10nt + \frac{1}{n^3} \sin 10nt$$

(a) in a cosine and angle form,
(b) in a sine and angle form.

*An asterisk indicates a challenging problem.

16.9 The waveform in Fig. 16.49(a) has the following Fourier series:

$$v_1(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi t + \frac{1}{9}\cos 3\pi t + \frac{1}{25}\cos 5\pi t + \cdots\right)\ \text{V}$$

Obtain the Fourier series of $v_2(t)$ in Fig. 16.49(b).

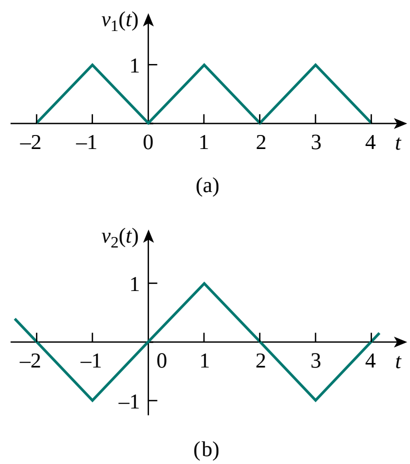
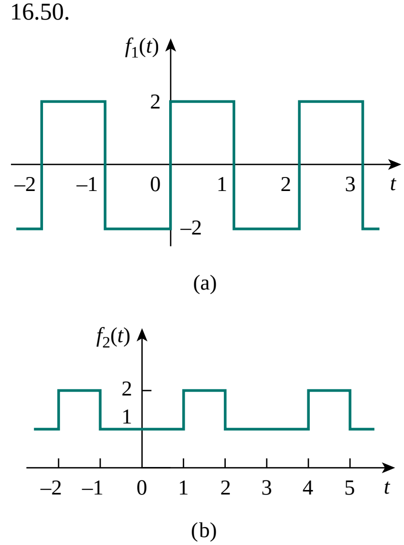


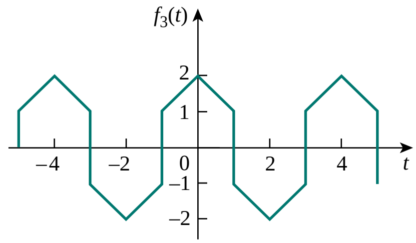
Figure 16.49 For Probs. 16.9 and 16.52.

Section 16.3 Symmetry Considerations

16.10 Determine if these functions are even, odd, or neither.
(a) $1 + t$ (b) $t^2 - 1$ (c) $\cos n\pi t \sin n\pi t$
(d) $\sin^2 \pi t$ (e) e^{-t}

16.11 Determine the fundamental frequency and specify the type of symmetry present in the functions in Fig. 16.50.





(c)

Figure 16.50 For Probs. 16.11 and 16.48.

16.12 Obtain the Fourier series expansion of the function in Fig. 16.51.

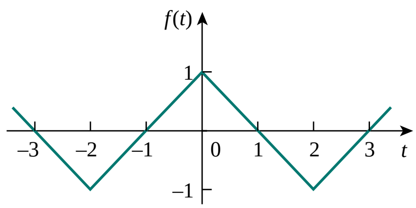


Figure 16.51 For Prob. 16.12.

16.13 Find the Fourier series for the signal in Fig. 16.52. Evaluate $f(t)$ at $t = 2$ using the first three nonzero harmonics.

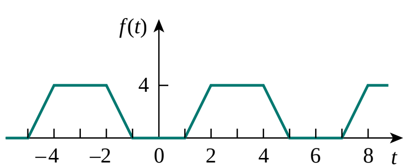


Figure 16.52 For Probs. 16.13 and 16.51.

16.14 Determine the trigonometric Fourier series of the signal in Fig. 16.53.

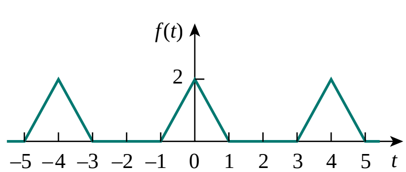


Figure 16.53 For Prob. 16.14.

16.15 Calculate the Fourier coefficients for the function in Fig. 16.54.

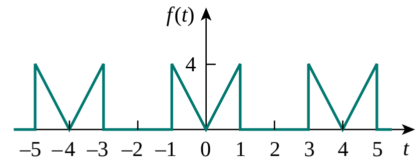


Figure 16.54 For Prob. 16.15.

16.16 Find the Fourier series of the function shown in Fig. 16.55.

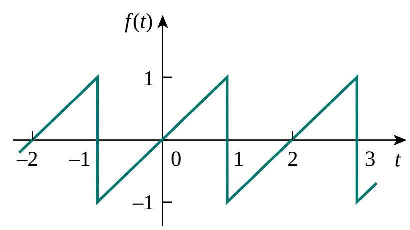


Figure 16.55 For Prob. 16.16.

16.17 In the periodic function of Fig. 16.56,

(a) find the trigonometric Fourier series coefficients a_2 and b_2,

(b) calculate the magnitude and phase of the component of $f(t)$ that has $\omega_n = 10$ rad/s,

(c) use the first four nonzero terms to estimate $f(\pi/2)$,

(d) show that

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

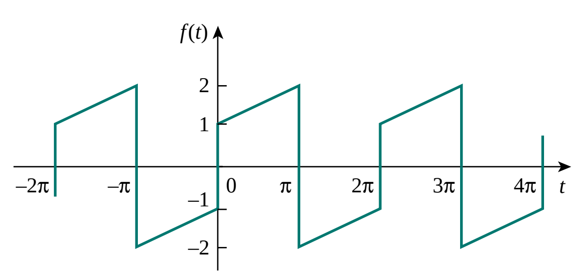


Figure 16.56 For Prob. 16.17.

16.18 Determine the Fourier series representation of the function in Fig. 16.57.

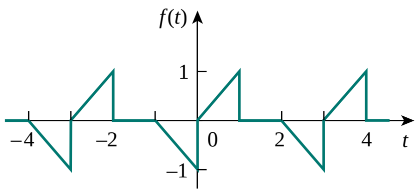


Figure 16.57 For Prob. 16.18.

16.19 Find the Fourier series representation of the signal shown in Fig. 16.58.

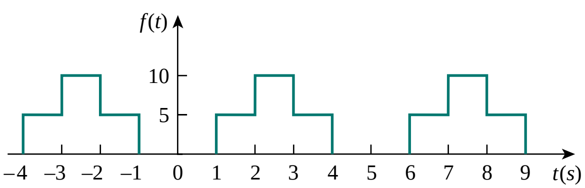


Figure 16.58 For Prob. 16.19.

16.20 For the waveform shown in Fig. 16.59 below,

(a) specify the type of symmetry it has,

(b) calculate a_3 and b_3,

(c) find the rms value using the first five nonzero harmonics.

16.21 Obtain the trigonometric Fourier series for the voltage waveform shown in Fig. 16.60.

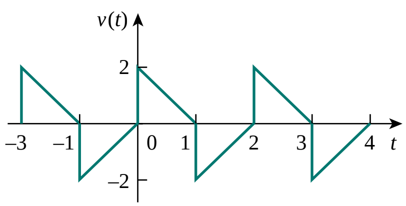


Figure 16.60 For Prob. 16.21.

16.22 Determine the Fourier series expansion of the sawtooth function in Fig. 16.61.

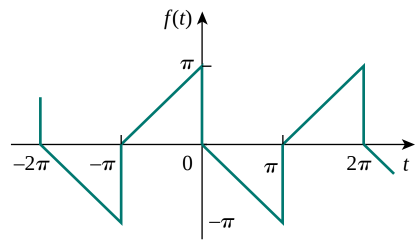


Figure 16.61 For Prob. 16.22.

Section 16.4 Circuit Applications

16.23 Find $i(t)$ in the circuit of Fig. 16.62 given that

$$i_s(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos 3nt \text{ A}$$

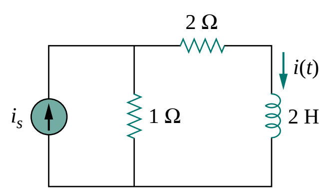


Figure 16.62 For Prob. 16.23.

16.24 Obtain $v_o(t)$ in the network of Fig. 16.63 if

$$v(t) = \sum_{n=1}^{\infty} \frac{10}{n^2} \cos\left(nt + \frac{n\pi}{4}\right) \text{ V}$$

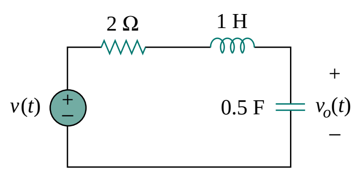


Figure 16.63 For Prob. 16.24.

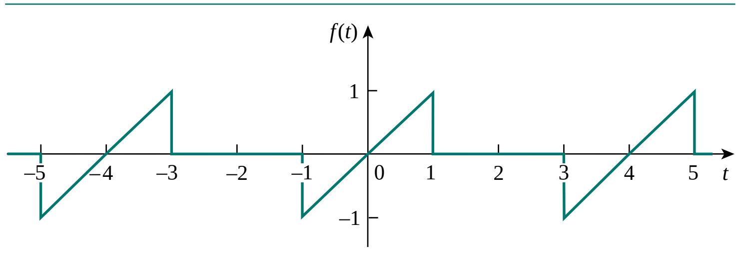


Figure 16.59 For Prob. 16.20.

16.25 If v_s in the circuit of Fig. 16.64 is the same as function $f_2(t)$ in Fig. 16.50(b), determine the dc component and the first three nonzero harmonics of $v_o(t)$.

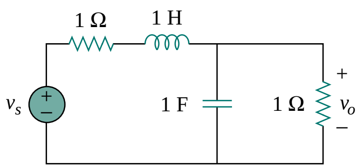


Figure 16.64 For Prob. 16.25.

16.26 Determine $i_o(t)$ in the circuit of Fig. 16.65 if

$$v_s(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \left(\frac{-1}{n\pi} \sin \frac{n\pi}{2} \cos nt + \frac{3}{n\pi} \sin nt \right)$$

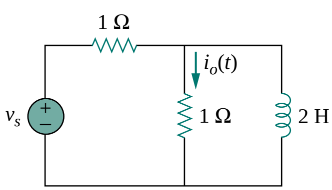


Figure 16.65 For Prob. 16.26.

16.27 The periodic voltage waveform in Fig. 16.66(a) is applied to the circuit in Fig. 16.66(b). Find the voltage $v_o(t)$ across the capacitor.

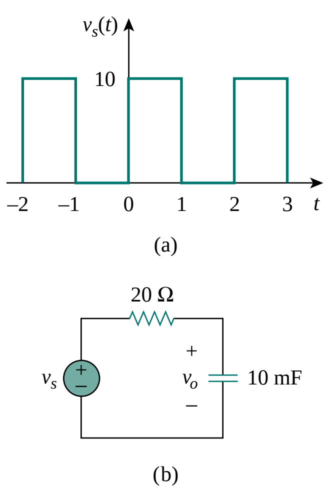


Figure 16.66 For Prob. 16.27.

16.28 If the periodic voltage in Fig. 16.67(a) is applied to the circuit in Fig. 16.67(b), find $i_o(t)$.

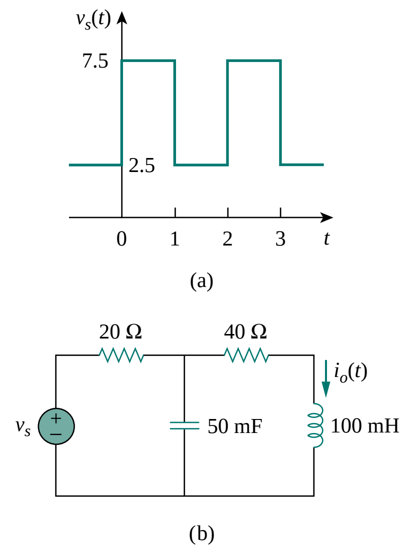


Figure 16.67 For Prob. 16.28.

*__16.29__ The signal in Fig. 16.68(a) is applied to the circuit in Fig. 16.68(b). Find $v_o(t)$.

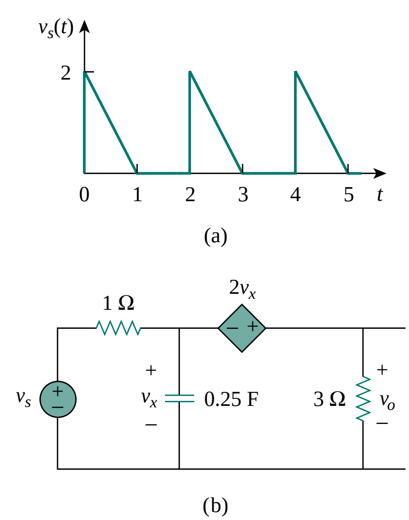


Figure 16.68 For Prob. 16.29.

16.30 The full-wave rectified sinusoidal voltage in Fig. 16.69(a) is applied to the lowpass filter in Fig. 16.69(b). Obtain the output voltage $v_o(t)$ of the filter.

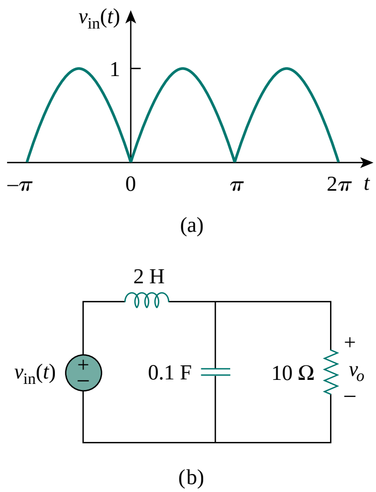


Figure 16.69 For Prob. 16.30.

Section 16.5 Average Power and RMS Values

16.31 The voltage across the terminals of a circuit is

$$v(t) = 30 + 20\cos(60\pi t + 45^\circ) + 10\cos(60\pi t - 45^\circ)\text{ V}$$

If the current entering the terminal at higher potential is

$$i(t) = 6 + 4\cos(60\pi t + 10^\circ) - 2\cos(120\pi t - 60^\circ)\text{ A}$$

find:

(a) the rms value of the voltage,
(b) the rms value of the current,
(c) the average power absorbed by the circuit.

16.32 A series RLC circuit has $R = 10\ \Omega$, $L = 2$ mH, and $C = 40\ \mu$F. Determine the effective current and average power absorbed when the applied voltage is

$$v(t) = 100\cos 1000t + 50\cos 2000t + 25\cos 3000t\text{ V}$$

16.33 Consider the periodic signal in Fig. 16.53. (a) Find the actual rms value of $f(t)$. (b) Use the first five nonzero harmonics of the Fourier series to obtain an estimate for the rms value.

16.34 Calculate the average power dissipated by the 10-Ω resistor in the circuit of Fig. 16.70 if

$$i_s(t) = 3 + 2\cos(50t - 60^\circ) + 0.5\cos(100t - 120^\circ)\text{ A}$$

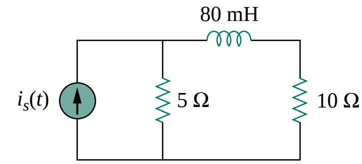


Figure 16.70 For Prob. 16.34.

16.35 For the circuit in Fig. 16.71,

$$i(t) = 20 + 16\cos(10t + 45^\circ) + 12\cos(20t - 60^\circ)\text{ mA}$$

(a) find $v(t)$, and
(b) calculate the average power dissipated in the resistor.

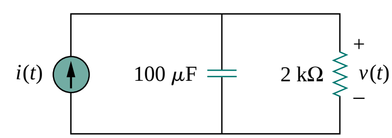


Figure 16.71 For Prob. 16.35.

Section 16.6 Exponential Fourier Series

16.36 Obtain the exponential Fourier series for $f(t) = t$, $-1 < t < 1$, with $f(t + 2n) = f(t)$.

16.37 Determine the exponential Fourier series for $f(t) = t^2$, $-\pi < t < \pi$, with $f(t + 2\pi n) = f(t)$.

16.38 Calculate the complex Fourier series for $f(t) = e^t$, $-\pi < t < \pi$, with $f(t + 2\pi n) = f(t)$.

16.39 Find the complex Fourier series for $f(t) = e^{-t}$, $0 < t < 1$, with $f(t + n) = f(t)$.

16.40 Find the exponential Fourier series for the function in Fig. 16.72.

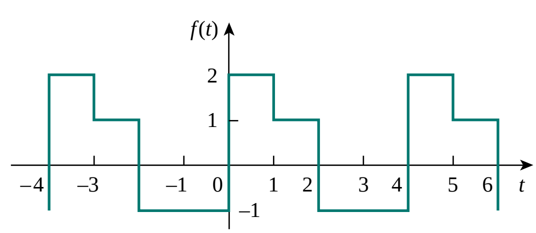


Figure 16.72 For Prob. 16.40.

16.41 Obtain the exponential Fourier series expansion of the half-wave rectified sinusoidal current of Fig. 16.73.

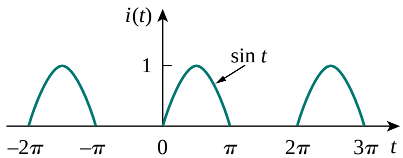


Figure 16.73 For Prob. 16.41.

16.42 The Fourier series trigonometric representation of a periodic function is

$$f(t) = 10 + \sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}\cos n\pi t + \frac{n}{n^2+1}\sin n\pi t\right)$$

Find the exponential Fourier series representation of $f(t)$.

16.43 The coefficients of the trigonometric Fourier series representation of a function are:

$$b_n = 0, \qquad a_n = \frac{6}{n^3-2}, \qquad n = 0, 1, 2, \ldots$$

If $\omega_n = 50n$, find the exponential Fourier series for the function.

16.44 Find the exponential Fourier series of a function which has the following trigonometric Fourier series coefficients

$$a_0 = \frac{\pi}{4}, \qquad b_n = \frac{(-1)^n}{n}, \qquad a_n = \frac{(-1)^n - 1}{\pi n^2}$$

Take $T = 2\pi$.

16.45 The complex Fourier series of the function in Fig. 16.74(a) is

$$f(t) = \frac{1}{2} - \sum_{n=-\infty}^{\infty}\frac{je^{-j(2n+1)t}}{(2n+1)\pi}$$

Find the complex Fourier series of the function $h(t)$ in Fig. 16.74(b).

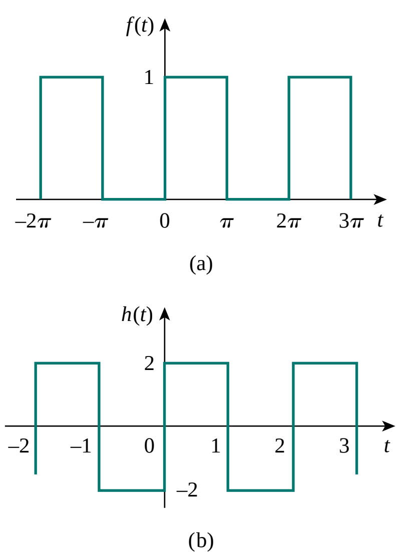


Figure 16.74 For Prob. 16.45.

16.46 Obtain the complex Fourier coefficients of the signal in Fig. 16.56.

16.47 The spectra of the Fourier series of a function are shown in Fig. 16.75. (a) Obtain the trigonometric Fourier series. (b) Calculate the rms value of the function.

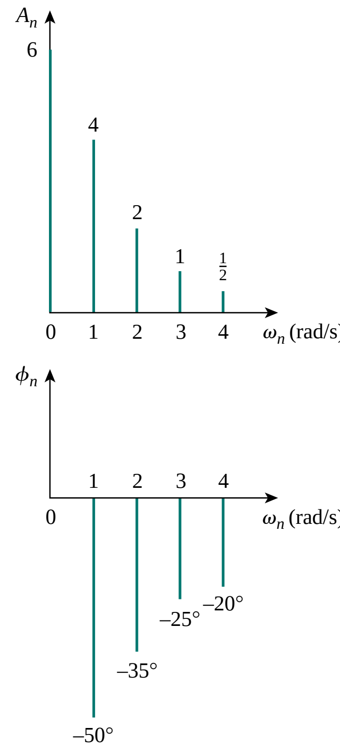


Figure 16.75 For Prob. 16.47.

16.48 Plot the amplitude spectrum for the signal $f_2(t)$ in Fig. 16.50(b). Consider the first five terms.

16.49 Given that

$$f(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty}\left(\frac{20}{n^2\pi^2}\cos 2nt - \frac{3}{n\pi}\sin 2nt\right)$$

plot the first five terms of the amplitude and phase spectra for the function.

Section 16.7 Fourier Analysis with *PSpice*

16.50 Determine the Fourier coefficients for the waveform in Fig. 16.48 using *PSpice*.

16.51 Calculate the Fourier coefficients of the signal in Fig. 16.52 using *PSpice*.

16.52 Use *PSpice* to obtain the Fourier coefficients of the waveform in Fig. 16.49(a).

16.53 Rework Prob. 16.29 using *PSpice*.

16.54 Use *PSpice* to solve Prob. 16.28.

Section 16.8 Applications

16.55 The signal displayed by a medical device can be approximated by the waveform shown in Fig. 16.76. Find the Fourier series representation of the signal.

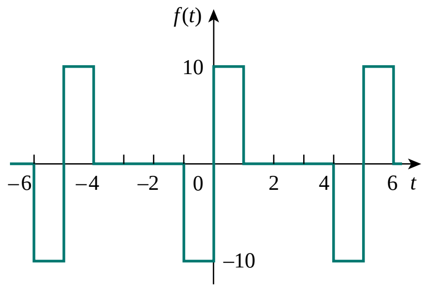


Figure 16.76 For Prob. 16.55.

16.56 A spectrum analyzer indicates that a signal is made up of three components only: 640 kHz at 2 V, 644 kHz at 1 V, 636 kHz at 1 V. If the signal is applied across a 10-Ω resistor, what is the average power absorbed by the resistor?

16.57 A certain band-limited periodic current has only three frequencies in its Fourier series representation: dc, 50 Hz, and 100 Hz. The current may be represented as

$$i(t) = 4 + 6\sin 100\pi t + 8\cos 100\pi t - 3\sin 200\pi t - 4\cos 200\pi t \text{ A}$$

(a) Express $i(t)$ in amplitude-phase form.

(b) If $i(t)$ flows through a 2-Ω resistor, how many watts of average power will be dissipated?

16.58 The signal in Fig. 16.66(a) is applied to the high-pass filter in Fig. 16.77. Determine the value of R such that the output signal $v_o(t)$ has an average power of least 70 percent of the average power of the input signal.

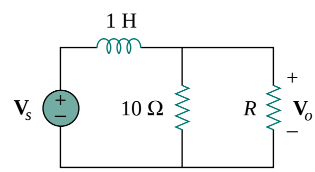


Figure 16.77 For Prob. 16.58.

COMPREHENSIVE PROBLEMS

16.59 The voltage across a device is given by

$$v(t) = -2 + 10\cos 4t + 8\cos 6t + 6\cos 8t - 5\sin 4t - 3\sin 6t - \sin 8t \text{ V}$$

Find:

(a) the period of $v(t)$,

(b) the average value of $v(t)$,

(c) the effective value of $v(t)$.

16.60 A certain band-limited periodic voltage has only three harmonics in its Fourier series representation. The harmonics have the following rms values: fundamental 40 V, third harmonic 20 V, fifth harmonic 10 V.

(a) If the voltage is applied across a 5-Ω resistor, find the average power dissipated by the resistor.

(b) If a dc component is added to the periodic voltage and the measured power dissipated increases by 5 percent, determine the value of the dc component added.

16.61 Write a program to compute the Fourier coefficients (up to the 10th harmonic) of the square wave in Table 16.3 with $A = 10$ and $T = 2$.

16.62 Write a computer program to calculate the exponential Fourier series of the half-wave rectified sinusoidal current of Fig. 16.73. Consider terms up to the 10th harmonic.

16.63 Consider the full-wave rectified sinusoidal current in Table 16.3. Assume that the current is passed through a 1-Ω resistor.

(a) Find the average power absorbed by the resistor.

(b) Obtain c_n for $n = 1, 2, 3$, and 4.

(c) What fraction of the total power is carried by the dc component?

(d) What fraction of the total power is carried by the second harmonic ($n = 2$)?

16.64 A band-limited voltage signal is found to have the complex Fourier coefficients presented in the table below. Calculate the average power that the signal would supply a 4-Ω resistor.

$n\omega_0$	$\|c_n\|$	θ_n
0	10.0	0°
ω	8.5	15°
2ω	4.2	30°
3ω	2.1	45°
4ω	0.5	60°
5ω	0.2	75°