The Factorial

Single Factorial

We can define the factorial function (Single factorial) symboled by (!) as :

$$\boldsymbol{n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 \qquad , for\ all\ n \in Z^{+}}$$

In special case we can define $0! = 1$

$\boldsymbol{4! = 4.3.2.1 = 24}$

$\boldsymbol{5! = 5.4.3.2.1 = 120}$

$\boldsymbol{1! = 1}$

Double Factorial

We can define the double factorial function symboled by (!!) as :

$$\boldsymbol{n!! = \begin{cases} n(n-2)(n-4)\cdots 5\cdot 3\cdot 1 & , \quad n>0\,, odd \\ n(n-2)(n-4)\cdots 4\cdot 2 \quad , & , \quad n>0\,, even \\ 1 & , \quad n=-1,0 \end{cases}}$$

$0!! = 1 \quad , \qquad 1!! = 1 \quad , \quad 2!! = 2 \quad , 3!! = 3.1 = 3$

$5!! = 5.3.1 = 15$

$6!! = 6.4.2 = 48$

Note : the relation between the single and double fac. Can be written as :

$n! = n!!\,(n-1)!!$

Gamma Function

we can define the gamma function as :

$$\Gamma(n) = \lim_{n\to\infty} \int_0^m x^{n-1} e^{-x}\, dx \quad , n > 0$$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad , n > 0$$

So as

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$\Gamma(2) = \int_0^\infty x^{2-1} e^{-x}\, dx = \int_0^\infty x^1 e^{-x}\, dx$$

$$= -\boldsymbol{xe^{-x}} - \int_0^\infty -\boldsymbol{e^{-x}dx} = -\boldsymbol{xe^{-x}} - \boldsymbol{e^{-x}}]_0^\infty = (\mathbf{0}-\mathbf{0}) - (\mathbf{0}-\mathbf{1}) = \mathbf{1}$$

Rules of gamma function

- ❖ $\Gamma(n+1) = \boldsymbol{n}! \quad , \quad \boldsymbol{n} \in \boldsymbol{Z}^+$
- ❖ $\Gamma(n+1) = \boldsymbol{n}\Gamma(n) \quad , \forall n \neq 0$
- ❖ $\Gamma(n) = \frac{\Gamma(n+1)}{\boldsymbol{n}} \quad , \quad n \in Q^-$
- ❖ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\boldsymbol{\pi}}$
- ❖ $\Gamma(n)\Gamma(1-n) = \frac{\boldsymbol{\pi}}{\boldsymbol{sinn\pi}}$
- ❖ $\Gamma(n)\Gamma(1-n) = \int_{\mathbf{0}}^{\boldsymbol{n}} \frac{\boldsymbol{x^{n-1}}}{\mathbf{1}+\boldsymbol{x}} \boldsymbol{dx} \quad , \quad \mathbf{0} < \boldsymbol{n} < \mathbf{1}$
- ❖ $\Gamma(n) = \mp\infty \quad , n \in Z^- \cup \{0\}$
- ❖ $\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2}\, dx \quad , n > 0$

Note: we doesn't find the gamma function for the negative integer values .

$\Gamma(4) = 3! = 6$

$$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2.2!} = \frac{120}{4} = 30$$

$$\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

$$\frac{\Gamma\left(\frac{8}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\Gamma\left(\frac{5}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\Gamma\left(\frac{2}{3}+1\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{10}{9}$$

Example : calculate the following :

$$1\rangle \int_0^\infty t^4 e^{-t} dt = \Gamma 5 = 4! = 24$$

$$2\rangle \int_0^\infty (1+2x^2)^2 e^{-x} dx = \int_0^\infty (1+4x^2+4x^4)\quad e^{-x} dx =$$

$$\int_0^\infty e^{-x} dx + 4\int_0^\infty x^2 e^{-x} dx + \int_0^\infty x^4 e^{-x} dx = \Gamma 1 + 4\Gamma 3 + 4\Gamma 5 = 105$$

$$3\rangle \int_0^1 \sqrt{Ln(\frac{1}{x})}dx = \int_0^1 (Ln\frac{1}{x})^{\frac{1}{2}}dx$$

$$let \quad u = Ln\left(\frac{1}{x}\right) \quad \rightarrow \quad x = e^{-u} \quad \rightarrow \quad dx = -e^{-u}du$$

$$x = 0 \quad \rightarrow \quad u = \infty \quad , \quad x = 1 \quad \rightarrow \quad u = 0$$

$$\int_0^1 \left(Ln\frac{1}{x}\right)^{\frac{1}{2}} dx = \int_\infty^0 -u^{\frac{1}{2}}e^{-u}du = \int_0^\infty u^{\frac{1}{2}}e^{-u}du = \Gamma\frac{3}{2} = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi}$$

$$4\rangle \int_0^1 (Ln(\frac{1}{x}))^{\frac{3}{2}}dx$$

$$let \quad u = Ln\left(\frac{1}{x}\right) \quad \rightarrow \quad x = e^{-u} \quad \rightarrow \quad dx = -e^{-u}du$$

$$x = 0 \quad \rightarrow \quad u = \infty \quad , \quad x = 1 \quad \rightarrow \quad u = 0$$

$$\int_0^1 \left(Ln\frac{1}{x}\right)^{\frac{3}{2}} dx = \int_\infty^0 -u^{\frac{3}{2}}e^{-u}du = \int_0^\infty u^{\frac{3}{2}}e^{-u}du = \Gamma\frac{5}{2} = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2} = \frac{3}{4}\sqrt{\pi}$$

$$5\rangle\ Find \quad \int_0^\infty 3^{-4z^2}dz$$

$$3^{-4z^2} = e^{Ln3^{-4z^2}} = e^{-4z^2Ln3}$$

$$\int_0^\infty 3^{-4z^2}dz = \int_0^\infty e^{-4z^2Ln3}dz$$

$$let\quad u = -4z^2 Ln3 \quad \rightrightarrows \quad z^2 = \frac{u}{4Ln3} \quad \rightrightarrows z = \frac{\sqrt{u}}{2\sqrt{Ln3}}$$

$$dz = \frac{1}{4\sqrt{Ln3}}\frac{1}{\sqrt{u}}du = \frac{1}{4\sqrt{Ln3}}u^{\frac{-1}{2}}du$$

$$z = 0 \ \rightrightarrows u = 0 \quad , \qquad z = \infty \ \rightrightarrows u = \infty$$

$$\int_0^\infty e^{-u} \ \frac{1}{4\sqrt{Ln3}}u^{\frac{-1}{2}}du = \frac{1}{4\sqrt{Ln3}}\int_0^\infty e^{-u}\ u^{\frac{-1}{2}}du$$

$$= \frac{1}{4\sqrt{Ln3}}\Gamma\frac{1}{2} = \frac{\sqrt{\pi}}{4\sqrt{Ln3}}$$

6⟩ ***Prove that*** $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx$$

$$let\quad u = x^2 \quad \rightrightarrows \quad x = \sqrt{u} \quad \rightrightarrows dx = \frac{du}{2\sqrt{u}} = \frac{1}{2}u^{\frac{-1}{2}}du$$

$$x = 0 \ \rightrightarrows u = 0 \quad , x = \infty \ \rightrightarrows u = \infty$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx = 2\int_0^\infty e^{-u}\frac{1}{2}u^{\frac{-1}{2}}du = \int_0^\infty e^{-u}\ u^{\frac{-1}{2}}du$$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

$7\rangle$ ***Prove that*** $\quad 2^n\,\Gamma\left(n+\frac{1}{2}\right) = 1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)\sqrt{\pi}$

$\because\ \Gamma(n+1) = n\Gamma(n)$

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)$$

$$\vdots$$

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\cdots\cdots\frac{5}{2}\frac{3}{2}\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)$$

<u>**By multiply two sides by 2^n**</u> $\quad$ ($\boldsymbol{every\ element\ in\ bracket\ by\ 2}$)

$$2^n\ \Gamma\left(n+\frac{1}{2}\right) = (2n-1)(2n-3)(2n-5)\cdots\cdots 5\cdot 3\cdot 1\ \Gamma\left(\frac{1}{2}\right)$$

$$2^n\ \Gamma\left(n+\frac{1}{2}\right) = 1\cdot 3\cdot 5\cdots\cdots(2n-5)(2n-3)(2n-1)\sqrt{\pi}$$

$8\rangle$ ***Evaluate*** $\quad \displaystyle\int_0^\infty \frac{x^c}{c^x}\,dx \quad , c>1$

$$let\ \ c^x = e^u \ \rightrightarrows\ Ln\, c^x = u \ \rightrightarrows\ x\, Ln\, c = u$$

$$x = \frac{u}{Ln\, c} \quad \rightrightarrows\quad dx = \frac{du}{Ln\, c}$$

$$x = 0\ \rightrightarrows\ u = 0 \quad , x = \infty\ \rightrightarrows\ u = \infty$$

$$\int_{0}^{\infty}\left(\frac{u}{Ln\ c}\right)^{c} e^{-u}\frac{du}{Ln\ c}=\left(\frac{1}{Ln\ c}\right)^{c+1}\int_{0}^{\infty} u^{c}\ e^{-u}du$$

$$\left(\frac{1}{Ln\ c}\right)^{c+1}\ \Gamma\,(c+1)$$

$$9\rangle\ \textit{Evaluate}\quad \int_{0}^{\infty}\frac{x^{\frac{-2}{3}}}{1+x}dx$$

$$\textit{By rule}\ \Gamma(n)\Gamma(1-n)=\int_{0}^{n}\frac{x^{n-1}}{1+x}dx\quad ,\quad 0<n<1$$

$$\int_{0}^{\infty}\frac{x^{\frac{-2}{3}}}{1+x}dx=\int_{0}^{\infty}\frac{x^{\frac{1}{3}-1}}{1+x}dx\qquad n=\frac{1}{3}$$

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)=\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{\sin\frac{\pi}{3}}=\frac{2\pi}{\sqrt{3}}$$

Beta Function

Consider the integral

$$B(m,n)=\int_{0}^{1} x^{m-1}(1-x)^{n-1}dx \quad ,m,n\ >0$$

This integral is convergent if m , n are greater than zero , and the value B(m,n) or B(n,m) is a function of m , n (not of x) which is called the Beta function and symboled by B(m,n)

Rules of Beta Function

1⟩ $B(m,n)=\int_{0}^{1} x^{m-1}(1-x)^{n-1}dx$

2⟩ $B(m,n)=\int_{0}^{1} \frac{x^{m-1}}{(1+x)^{m+n}}dx$

3⟩ $B(m,n)=2\int_{0}^{\frac{\pi}{2}} sin^{2m-1}(\theta)cos^{2n-1}\theta d\theta$

The connection between Beta function and Gamma function is :

4⟩ $B(m,n)\quad=\frac{\Gamma\,(\mathrm{m})\Gamma\,(\mathrm{n})}{\Gamma\,(\mathrm{m}+\mathrm{n})}$

5⟩ $B(m,n)\quad=B(n,m)$

Examples :

1) *Calculate* $B\left(\frac{1}{2},\frac{1}{2}\right)$

$$B\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}=\sqrt{\pi}\sqrt{\pi}=\pi$$

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2) *Calculate* $B(7,9)$

$$B(7,9)=\frac{\Gamma(7)\Gamma(9)}{\Gamma(16)}=\frac{6!\,.\,\Gamma(9)}{15.14.13\ldots..9\Gamma(9)}=\frac{6!}{15.14.13\ldots..9}$$

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3) *Calculate* $B\left(\frac{1}{3},\frac{2}{3}\right)$

$$B\left(\frac{1}{3},\frac{2}{3}\right)=\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}=\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{sin\frac{\pi}{3}}=\frac{2\pi}{\sqrt{3}}$$

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4) *Show that* $B(x,1)=x^{-1}$

$$B(x,1)=\frac{\Gamma(x)\Gamma(1)}{\Gamma(x+1)}$$

$$B(x,1)=\frac{\Gamma(x)}{x\Gamma(x)}=\frac{1}{x}=x^{-1}$$

5⟩ *Show that* $B(x+1,y)+B(x,y+1)=B(x,y)$

$$B(x+1,y)+B(x,y+1)=\frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)}+\frac{\Gamma(x)\Gamma(y+1)}{\Gamma(x+y+1)}$$

$$B(x+1,y)+B(x,y+1)=\frac{x\,\Gamma(x)\Gamma(y)}{\Gamma(x+y+1)}+\frac{y\,\Gamma(x)\Gamma(y)}{\Gamma(x+y+1)}$$

$$B(x+1,y)+B(x,y+1)=\frac{\Gamma(x)\Gamma(y)\,.\,\{x+y\}}{\Gamma(x+y+1)}$$

$$B(x+1,y)+B(x,y+1)=\frac{\Gamma(x)\Gamma(y)\,.\,\{x+y\}}{(x+y)\Gamma(x+y)}$$

$$B(x+1,y)+B(x,y+1)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=B(x,y)$$

6⟩ *Show that* $B(x+1,y+1)=\dfrac{x!\,y!}{(x+y+1)!}$

$$B(x+1,y+1)=\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}=\frac{x!\,y!}{(x+y+1)!}$$

7⟩ *Show that* $B(x+1,y)=\dfrac{x!\,y!}{(x+y+1)!}$

$$B(x+1,y)=\frac{x}{x+y}B(x,y)$$

$$B(x+1,y)=\frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+y+1)}$$

$$B(x+1,y)=\frac{x\,\Gamma(x)\Gamma(y)}{(x+y)\,\Gamma(x+y)}$$

$$B(x+1,y)=\frac{x}{(\mathrm{x}+\mathrm{y})}B(x,y)$$

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8) $Find \int_0^1 x^3(1-x)^7dx$

$$\int_0^1 x^3(1-x)^7dx = B(4,8)$$

$$\int_0^1 x^3(1-x)^7dx = \frac{\Gamma(4)\Gamma(8)}{\Gamma(5)}=\frac{1}{1320}$$

9) $Find \int_0^{\frac{\pi}{2}} sin^3(\theta)cos^5\theta d\theta$

By comparing with integral $\int_0^{\frac{\pi}{2}} sin^{2m-1}(\theta)cos^{2n-1}\theta d\theta$

$2m-1=3 \Rightarrow \ m=2 \quad ,2n-1=5 \ \Rightarrow n=3$

$$\int_0^{\frac{\pi}{2}} sin^3(\theta)cos^5\theta d\theta = \frac{1}{2}B(2,3)$$

$$\int_0^{\frac{\pi}{2}} sin^3(\theta)cos^5\theta d\theta = \frac{1}{2}\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}=\frac{1}{2}\frac{1\ (2)}{24}=\frac{1}{24}$$

$$10\rangle\ prove \quad \int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{4}{5}\frac{\sqrt{2}\,\pi^{\frac{3}{2}}}{\Gamma\left(\frac{1}{4}\right)^2}$$

By comparing with integral $\int_0^{\frac{\pi}{2}} sin^{2m-1}(\theta)cos^{2n-1}\theta d\theta$

$$2m-1=2 \Rightarrow \ m=\frac{3}{2} \quad ,2n-1=\frac{1}{2} \Rightarrow n=\frac{3}{4}$$

$$\int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{1}{2}B\left(\frac{3}{2},\frac{3}{4}\right)$$

$$\int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{9}{4}\right)} = \frac{1}{2}\,\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\ \Gamma\left(\frac{3}{4}\right)}{\frac{5}{4}\Gamma\left(\frac{5}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{1}{4}\frac{4}{5}\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)} \quad , \qquad since\ \ \Gamma(n+1)=\boldsymbol{n}\Gamma(n)$$

$$\int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{4}{5}\sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \quad ,\because \quad \Gamma(n+1)=\boldsymbol{n}\Gamma(n)$$

$$\because \quad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{sin\frac{\pi}{4}} = \sqrt{2}\pi \ \ \Rightarrow \Gamma\left(\frac{3}{4}\right) = \frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{4}{5}\sqrt{\pi}\,\frac{\frac{\sqrt{2}\pi}{\Gamma\left(\frac{1}{4}\right)}}{\Gamma\left(\frac{1}{4}\right)}$$

$$\int_0^{\frac{\pi}{2}} sin^2(\theta)cos^{\frac{1}{2}}\theta d\theta = \frac{4\sqrt{2}}{5}\,\frac{\pi^{\frac{3}{2}}}{\Gamma\left(\frac{1}{4}\right)^2}$$

11⟩ *prove* $\int_0^{\frac{\pi}{2}} \sqrt{tan\theta}d\theta = \frac{\pi}{\sqrt{2}}$

$$\int_0^{\frac{\pi}{2}} \sqrt{tan\theta}d\theta = \int_0^{\frac{\pi}{2}} \frac{\sqrt{sin\theta}}{\sqrt{cos\theta}}d\theta = \int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta\ cos^{-\frac{1}{2}}\theta\ d\theta$$

By comparing with integral $\int_0^{\frac{\pi}{2}} sin^{2m-1}(\theta)cos^{2n-1}\theta d\theta$

$$2m-1 = \frac{1}{2} \Rightarrow\ m = \frac{3}{4}\quad ,2n-1 = \frac{-1}{2} \Rightarrow n = \frac{1}{4}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{tan\theta}d\theta = \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{tan\theta}d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$\because\quad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{sin\frac{\pi}{4}} = \sqrt{2}\pi$$

$$\int_0^{\frac{\pi}{2}} \sqrt{tan\theta}d\theta = \frac{1}{2}\sqrt{2}\pi = \frac{\pi}{\sqrt{2}}$$

$$12\rangle\ prove \quad \int_0^1 (1-x^n)^{\frac{1}{n}}\,\mathrm{dx} = \frac{\Gamma\left(\frac{1}{n}\right)^2}{2\mathrm{n}\Gamma\left(\frac{2}{n}\right)}$$

$$let \quad u = x^n \quad \rightrightarrows \quad x = u^{\frac{1}{n}} \quad \rightrightarrows dx = \frac{1}{n}u^{\frac{1}{n}-1}du$$

$$\mathbf{x = 0 \ \rightrightarrows\ \boldsymbol{u} = 0 \quad , \boldsymbol{x} = 1 \ \rightrightarrows\ \boldsymbol{u} = 1}$$

$$\int_0^1 (1-u)^{\frac{1}{n}}\frac{1}{n}u^{\frac{1}{n}-1}du = \frac{1}{n}\int_0^1 u^{\frac{1}{n}-1}(1-u)^{\frac{1}{n}}\,du$$

$$= \frac{1}{\mathrm{n}}B\left(\frac{1}{n}, \frac{1}{n}+1\right)$$

$$= \frac{1}{\mathrm{n}}B(\frac{1}{n}, \frac{1}{n}+1)$$

$$= \frac{1}{\mathrm{n}}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}+1\right)}{\Gamma\left(\frac{2}{n}+1\right)} = \frac{1}{\mathrm{n}}\frac{\frac{1}{\mathrm{n}}\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)}{\frac{2}{n}\Gamma\left(\frac{2}{n}\right)}$$

$$\therefore \quad \int_0^1 (1-x^n)^{\frac{1}{n}}\,\mathrm{dx} = \frac{\Gamma\left(\frac{1}{n}\right)^2}{2\mathrm{n}\Gamma\left(\frac{2}{n}\right)}$$

HOME WORK

$1\rangle$ ***Show that*** : $\Gamma(n)=\int_0^1\left(Ln\left(\frac{1}{x}\right)\right)^{n-1}dx \quad ,n>0$

$2\rangle$ ***Show that if*** $n>0$, $\quad m=\frac{1}{2}(n-2)$, ***then*** : $\int_0^\infty \frac{x^m}{e^{x^n}}dx=\frac{\sqrt{\pi}}{2m+2}$

$3\rangle$ ***Prove that*** $\frac{3^n\,\Gamma\left(n+\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}=2\cdot5\cdot8\cdot11\cdots(3n-1)\sqrt{\pi}$

$4\rangle$ ***Find*** $\int_{-\infty}^{\infty}\sqrt{x}e^{-x^3}dx$

$5\rangle$ ***Evaluate*** $\int_0^\infty x^{2n-1}e^{-kx^2}dx$

$6\rangle$ ***prove that*** $\int_0^\infty e^{-y^{\frac{1}{m}}}dy=m\Gamma(m)$

$7\rangle$ ***Evaluate*** $\int_0^1 x^m\left(Ln\left(\frac{1}{x}\right)\right)^{n-1}dx$

$8\rangle$ ***Evaluate*** $\int_0^1\frac{dx}{\sqrt{-Ln(x)}}$

$9\rangle$ ***Evaluate*** $\int_0^\infty\frac{1}{\sqrt[4]{x}(1+x)}dx$

$10\rangle$ ***evaluate*** $\int_{-2}^{\infty}(x+2)^5e^{-(x+2)}\,dx$

$11\rangle$ ***Show that*** $\int_0^2 x\sqrt[3]{8-x^3}\,dx=\frac{16\pi}{9\sqrt{3}}$

1: Functions in Several Variables

Def: A function of two variables is a rule that assigns a real number f(x,y) to each ordered pair of real (x,y) in the domain of the function , For a function f defined on the domain $D \subset R^2$, we some times write , $f: D \subset R^2 \rightarrow R$ to indicate that f maps points in two dimensions to real number ·

Likewise A function of three variables is a rule that assigns a real number f(x,y,z) to each ordered triple of real (x,y,z) in the domain of the function , For a function f defined on the domain $D \subset R^3$, we some times write , $f: D \subset R^3 \rightarrow R$ to indicate that f maps points in three dimensions to real number · as examples :

$$f(x,y) = xy^2 \quad , \qquad g(x,y) = x^2 - e^y$$

$$f(x,y,z) = xy^2 cosz \quad , \qquad g(x,y,z) = 3zx^2 - e^y$$

2: Domain of a Functions in Several Variables

Example 1 : Find the Domain of :

1) $f(x,y) = x\, Lny$

x may be take any real value, but y takes positive values only ·

then the domain of f is : $D = \{(x,y) : (x,y) \in R^2, y > 0\}$

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2) $g(x,y) = \frac{2x}{y-x^2}$

$y - x^2 = 0 \ , \rightarrow y = x^2 \ \ (\, unavailable\,)$

$D = \{(x,y) : (x,y) \in R^2, y \neq x^2\}$

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3) $f(x,y,z) = \frac{\cos(x+y+z)}{xy}$

The denominator equal zero when xy=0 , and that which occurs if x=0 or y=0 , then :

$D = \{(x,y,z) : (x,y,z) \in R^3, x, y \neq 0\}$

4) $g(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$

The function is an even root then the expression inside the root may be greater than or equal zero

$9 - x^2 - y^2 - z^2 \geq 0 \quad \rightarrow \quad x^2 + y^2 + z^2 \leq 9$

$D = \{(x,y,z) : (x,y,z) \in R^3 \ , \ x^2 + y^2 + z^2 \leq 9\}$

Example H.W :

CHECK POINT

Describe the domain of :

1) $f(x,y) = \frac{1}{x+y}$, 2) $f(x,y) = Ln(2 + x + y)$

3) $f(x,y,z) = \dfrac{3xy}{z - \sqrt{x}}$, 4) $f(x,y,z) = \sqrt{1 - x^2 - y^2 - z^2}$

3: The Limit of a Functions in Several Variables

First we remind you that the concept of limit is fairly simple , for a function of single variable ,

If we write $\lim_{x\to a} f(x) = L$ *, we mean that as x gets closer and closer to " a" , f(x) gets closer and closer to the number L .for a functions of several variables the idea is very simillar , when we write*

$$\lim_{(x,y)\to(0,0)} f(x,y) = L$$

we mean that as (x,y) gets closer and closer to (a,b) , f(x,y) gets closer and closer to the number L .

for instance ,

$$\lim_{(x,y)\to(2,3)} (xy-2) = 2(3)-2 = 4$$

$$\lim_{(x,y)\to(-1,\pi)} (\sin(xy) - x^2y) = \sin(-\pi) - \pi = -\pi$$

CHECK POINT

$$Evaluate: \lim_{(x,y)\to(2,1)} (x^2y - 2Lny)$$

Notes : in other words for many nice functions we can compute limits simply by subistituting into the function , unfortunately as will functions of single variables the limits we're most interested in cannot be completed by simply subistituting values for x and y , for instance :

$$\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$$

Subistitute x=1 , y=0 in equation gives the intermidate form $\frac{0}{0}$ to evaluate this limit we must investigate further .

Example 1 : for the function $f(x,y) = \frac{-xy}{x^2+y^2}$ at the following paths :

1) The X- axis , 2) the Y-axis , 3) Line y=x ,

4) Line y=-x , 5) Line $y=x^2$

Solution:

1) $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2} = \lim\limits_{(x,0)\to(0,0)} \frac{-0}{x^2} = \lim\limits_{x\to 0} 0 = 0$

2) $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2} = \lim\limits_{(0,y)\to(0,0)} \frac{-0}{y^2} = \lim\limits_{y\to 0} 0 = 0$

3) $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2} = \lim\limits_{(x,x)\to(0,0)} \frac{-x^2}{x^2+x^2} = \lim\limits_{x\to 0} \frac{-1}{2} = \frac{-1}{2}$

4) $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2} = \lim\limits_{(x,-x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim\limits_{x\to 0} \frac{1}{2} = \frac{1}{2}$

5) $\lim\limits_{(x,y)\to(0,0)} \frac{-xy}{x^2+y^2} = \lim\limits_{(x,x^2)\to(0,0)} \frac{-x.x^2}{x^2+x^4} = \lim\limits_{x\to 0} \frac{-x^3}{x^2+x^4} = \lim\limits_{x\to 0} \frac{-x}{1+x^2} = 0$

Note :

To evaluate the limit at (x,y) →(a,b) we consider :

A) The vertical line path along the line x=a , and compute the limit as y approaches to " b " .

B) The horizontal line path along the line y=b, and compute the limit as x approaches to " a " .

C) another path that which from relation of the values of x , y that gives in limit .

D) if any values from A , B and C is different then the limit doesn't exists .

Example 2: Find $\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$

A) $\lim_{(1,y)\to(1,0)} \frac{y}{x+y-1} = \lim_{y\to 0} \frac{y}{1+y-1} = \lim_{y\to 0} \frac{y}{y} = 1$

B) $\lim_{(x,0)\to(1,0)} \frac{y}{x+y-1} = \lim_{x\to 1} \frac{0}{x-1} = \lim_{x\to 0} 0 = 0$

Since the function is approaching to two different values along two different paths to the point (1,0) , then the limit doesn't exists .

Example 3: Find $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

$$A) \quad \lim_{(0,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = \lim_{y\to 0} 0 = 0$$

$$B) \quad \lim_{(x,0)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0$$

C) Try the path y=x (since x=y=0)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Since the limit along the last path doesn't match the limit along the first two paths , then the limit doesn't exists .

Example 4: **Find** $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

$$A) \quad \lim_{(0,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0}{0+y^4} = \lim_{y\to 0} 0 = 0$$

$$B) \quad \lim_{(x,0)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0$$

C) Try the path $x=y^2$ (since x=y=0)

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(y^2,y)\to(0,0)} \frac{y^2.y^2}{y^4+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

Since the limit doesn't agree with the first two paths , then the limit doesn't exists

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Example 5: **Find** $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

$$A) \quad \lim_{(0,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = \lim_{y\to 0} 0 = 0$$

$$B) \quad \lim_{(x,0)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{x\to 0} \frac{0}{x^2+0} = \lim_{x\to 0} 0 = 0$$

C) Try the path y=x (since x=y=0)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^3}{x^2+x^2} = \lim_{x\to 0} \frac{x^3}{2x^2} = \lim_{x\to 0} \frac{x}{2} = 0$$

Since the limit along all path takes the same values , then the limit exists .

Example 6: **Evaluate** $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 Lnx}{(x-1)^2+y^2}$

$$A)\quad \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 Lnx}{(x-1)^2+y^2} = \lim_{y\to0} \frac{0}{0+y^2} = \lim_{y\to0} 0 = 0$$

$$B)\quad \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 Lnx}{(x-1)^2+y^2} = \lim_{x\to1} \frac{(x-1)^2 Lnx}{(x-1)^2+0} = \lim_{x\to1} Lnx = 0$$

C) Try the path y=x-1

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 Lnx}{(x-1)^2+y^2} = \lim_{(x,x-1)\to(1,0)} \frac{(x-1)^2 Lnx}{(x-1)^2+(x-1)^2} = \lim_{x\to1} \frac{Lnx}{2} = 0$$

Since the limit along all path takes the same values, then the limit exists.

Example 7: **Evaluate** $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2-z^2}{x^2+y^2+z^2}$

$$A)\quad \lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2-z^2}{x^2+y^2+z^2} = \lim_{(0,0,z)\to(0,0,0)} \frac{0+0-z^2}{0+0+z^2} = \lim_{z\to0}(-1) = -1$$

$$B)\quad \lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2-z^2}{x^2+y^2+z^2} = \lim_{(0,y,0)\to(0,0,0)} \frac{0+y^2-0}{0+y^2+0} = \lim_{z\to0}(1) = 1$$

Since the limit along first two paths takes different values , then the limit doesn't exists .

CHECK POINT

1) Show that the limit doesn't exists :

1) $\lim_{(x,y)\to(0,0)} \frac{3x^2}{x^2+y^2}$, 2) $\lim_{(x,y)\to(0,0)} \frac{3x^2\sqrt{y}}{x^2+y^2}$

3) $\lim_{(x,y)\to(0,0)} \frac{ysinx}{x^2+y^2}$, 4) $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2yz}{x^4+y^4+z^4}1$

2) Show that the limit exists :

1) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$, 2) $\lim\limits_{(x,y)\to(0,0)} \frac{2x^2 \sin y}{2x^2+y^2}$, 3) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{3x^3}{x^2+y^2+z^2}$

4: The Partial Derivative

Def: the partial derivative of $f(x,y)$ with respect to x written as $\frac{\partial f}{\partial x}$ is defined by :

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

For any values of x , y which the limit exists .

Def: the partial derivative of $f(x,y)$ with respect to y written as $\frac{\partial f}{\partial y}$ is defined by :

$$\frac{\partial f}{\partial y} = \lim_{\Delta y\to 0} \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

For any values of x , y which the limit exists .

Example 1: By using the definition of the partial derivative , Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ to $f(x,y) = 3x^2y^2$

Solution :

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{3(x+\Delta x)^2y^2-3x^2y^2}{\Delta x}$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{3(x^2 + 2x\Delta x + \Delta^2 x)y^2 - 3x^2y^2}{\Delta x} \Longrightarrow\Longrightarrow \frac{\partial f}{\partial x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2y^2 + 6xy^2\Delta x + 3\Delta^2 xy^2 - 3x^2y^2}{\Delta x}$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta x(6xy^2 + 3\Delta xy^2)}{\Delta x} = 6xy^2$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{3x^2(y+\Delta y)^2 - 3x^2y^2}{\Delta y}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta x \to 0} \frac{3x^2(y^2 + 2y\Delta y + \Delta^2 y) - 3x^2y^2}{\Delta y} \Longrightarrow\Longrightarrow \frac{\partial f}{\partial y}$$

$$= \lim_{\Delta y \to 0} \frac{3x^2y^2 + 6yx^2\Delta y + 3x^2\Delta^2 y - 3x^2y^2}{\Delta y}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta x(6yx^2 + 3x^2\Delta y)}{\Delta y} = 6yx^2$$

Def :

Higher order partial derivative for functions of two variables they are four different second order partial derivatives , the partial derivative with respect to x of $\frac{\partial f}{\partial x}$ is $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$ usually written as $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

Simillary taking two successive partial derivatives with respect to y gives as $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

usually written as $\frac{\partial^2 f}{\partial y^2}$ or f_{yy} .

for mixed second order partial derivatives on derivative is taken with respect to each variable , if the first partial derivative is taken with respect to x ,

we written as : $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

if the first partial derivative is taking with respect to y , we have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$

Example 2: **Find all second order partial derivatives of $f(x,y) = x^2y - y^3 + Lnx$**

Solution :

$$\frac{\partial f}{\partial x} = 2xy + \frac{1}{x}$$

$$\frac{\partial f}{\partial y} = x^2 - 3y^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2y - \frac{1}{x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x$$

Note : if f_{xy} , f_{yx} are continous fun. in domain of f then $f_{xy} = f_{yx}$

Check point

Example H.W: for $f(x,y) = \cos(xy) - x^3 + y^4$ compute f_{xyy} , f_{xxyy}

Example 3: for $f(x,y,z) = \sqrt{xy^3z} + 4x^2y$, for $x,y,z \geq 0$, Compute f_x, f_{xy}, f_{xyz}

Solution :

$f(x,y,z) = x^{\frac{1}{2}}\ y^{\frac{3}{2}}\ z^{\frac{1}{2}} + 4x^2y$

$f_x = \frac{1}{2}x^{\frac{-1}{2}}\ y^{\frac{3}{2}}\ z^{\frac{1}{2}} + 8xy$

$f_{xy} = \frac{3}{4}x^{\frac{-1}{2}}\ y^{\frac{1}{2}}\ \ z^{\frac{1}{2}} + 8x$

$f_{xyz} = \frac{3}{8}x^{\frac{-1}{2}}\ y^{\frac{1}{2}}\ \ z^{\frac{-1}{2}} = \frac{3}{8}\sqrt{\frac{y}{xz}}$

5: tangent plane and normal line :

DEF: (Tangent Plane)

suppose that f(x,y) has continous first partial derivative at (a,b) , A normal Vector to the tangent plane to $z = f(x,y)$at $(a,b)$is then $(f_x(a,b), f_y(a,b), -1\ )$, further an equation of the tangent plane is given by

$$f_x(a,b)[x-a] + f_y(a,b)[y-b] - [z - f(a,b)] = 0$$

OR

$$[z - f(a,b)] = f_x(a,b)[x-a] + \ f_y(a,b)[y-b]$$

DEF: (Normal Line)

Observe that since we know a normal vector to tangent plane , a line orthogonal to tangent plane called the normal line and given by the following equations :

$$x = a + f_x(a,b)t$$

$$y = b + f_y(a,b)t$$

$$z = f(a,b) - t$$

Example 1: for $z = 6 - x^2 - y^2$, Find the equation of the tangent plane and the normal line to the function z at the point (1,2,1)

Solution :

$f(x,y) = 6 - x^2 - y^2$, $\rightrightarrows$ $f(1,2) = 1$
$f_x = -2x$, $\rightrightarrows$ $f_x(1,2) = -2$
$f_y = -2y$, $\rightrightarrows$ $f_y(1,2) = -4$

Normal vector is (-2,-4,-1) , then the equation of the tangent plane is :

$z - 1 = -2(x-1) - 4(y-2)$

And the normal line

$x = 1 - 2t$
$y = 2 - 4t$
$z = 1 - t$

Example 2: for $z = x^3 + y^3 + \frac{x^2}{y}$, Find the equation of the tangent plane and the normal line to the function z at the point (2,1,3)

Solution :

$f(x,y) = x^3 + y^3 + \frac{x^2}{y}$, $\rightrightarrows$ $f(2,1) = 13$

$f_x = 3x^2 + \frac{2x}{y}$, $\rightrightarrows$ $f_x(2,1) = 16$

$f_y = 3y^2 - \frac{x^2}{y^2}$, $\quad \Rightarrow \; f_y(2,1) = -1$

Normal vector is (16,-1,-1) , then the equation of the tangent plane is :

$z - 13 = 16(x-2) - (y-1)$

And the normal line

$x = 2 + 16t$

$y = 1 - t$

$z = 13 - t$

Example H.W : Find equations of the tangent plane and normal line to the functions :

CHECK POINT

1) $z = x^2 + y^2 - 1$, at $(2,1,4)$

2) $z = e^{-x^2-y^2}$, at $(1,1,e^{-2})$

3) $z = sinx\, cosy$, at $(0,\pi,0)$

6: CHAIN RULE :

If $z = f(x,y)$ where x=x(t) , y=y(t) are differentiable and f(x,y) is a differentiable function of x and y then :

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 1: for $f(x,y) = x^2e^y$, $x(t) = t^2 - 1$, $y(t) = sint$, Find $\frac{df}{dt}$

Solution :

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{df}{dt} = 2xe^{y}(2t) + x^2e^{y}\,cost$$

$$\frac{df}{dt} = 2(t^2-1)e^{sint}(2(t^2-1)) + (t^2-1)^2e^{sint}\,cost$$

Theorem : suppose that z=f(x,y) , where f is differentiable function of x and y and where x=x(u,v) , y=y(u,v) , both have first order partial derivative , then we have the following chain rules :

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

Example 2: suppose that $f(x,y) = e^{xy}$, $x = 3u\,sinv$, $y = 4v^2u$, Find $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$

Solution :

$$\frac{\partial f}{\partial x} = ye^{xy} \quad , \frac{\partial f}{\partial y} = xe^{xy} \quad , \frac{\partial x}{\partial u} = 3sinv \quad , \frac{\partial x}{\partial v} = 3ucosv \quad , \frac{\partial y}{\partial u} = 4v^2 \quad , \frac{\partial y}{\partial v} = 8vu$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial u} = (ye^{xy})(3sinv) + (xe^{xy})(4v^2)$$

Subistituting for x and y we get :

$$\frac{\partial f}{\partial u} = \left(4v^2ue^{3u\,sinv.4v^2u}\right)(3sinv) + (3u\,sinve^{3u\,sinv.4v^2u})(4v^2)$$

$$\frac{\partial f}{\partial u}=\left(4v^2u\,e^{12v^2u^2\,sinv}\right)(3sinv)+(3u\,sinv\,e^{12v^2u^2\,sinv})(4v^2\,)$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial f}{\partial v}=(ye^{xy})(3u\,cosv)+(xe^{xy})(8uv\,)$$

Subistituting for x and y we get :

$$\frac{\partial f}{\partial u}=\left(4v^2u\,e^{12v^2u^2\,sinv}\right)(3u\,cosv)+(3u\,sinv\,e^{12v^2u^2\,sinv})(8uv\,)$$

Example 3: *suppose that* $f(x,y)=4x^2y^3$, $x=u^3-v\,sinu$, $y=4u^2$,

Find $\frac{\partial f}{\partial u},\frac{\partial f}{\partial v}$

Solution :

$$\frac{\partial f}{\partial x}=8xy^3\quad,\frac{\partial f}{\partial y}=12x^2y^2\quad,\frac{\partial x}{\partial u}=3u^2-vcosu\quad,\frac{\partial x}{\partial v}=-sinu\quad,\frac{\partial y}{\partial u}=8u\quad,\frac{\partial y}{\partial v}=0$$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial u}=(8xy^3)(3u^2-vcosu)+(12x^2y^2)(8u\,)$$

Subistituting for x and y we get :

$$\frac{\partial f}{\partial u}=(8(u^3-v\,sinu)(4u^2)^3)(3u^2-vcosu)+(12(u^3-v\,sinu)^2(4u^2)^2)(8u\,)$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial f}{+\partial u} = (8xy^3)(-sinu) + (12x^2y^2)(0\,)$$

Subisitituting for x and y we get :

$$\frac{\partial f}{\partial u} = (8(u^3 - v\,sinu)(4u^2)^3)(-sinu) + (12(u^3 - v\,sinu)^2(4u^2)^2)(8u\,)$$

CHECK POINT

Example H·W : use the chain rule to find $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$:

1⟩ $f(x,y) = x^2y - siny \;\; , x = \sqrt{t^2+1}\,, \quad y = e^{-t}$, find $\frac{df}{dt}$

2⟩ $f(x,y) = xy^3 - 4x^2 \;\; , \;\; x = e^{u^2} \;\; , \quad y = \sqrt{v^2+1}sinv$

7: IMPLICIT DIFFERENTIATION :

If $f(x,y,z) = c$ we define z implicitly as differentiable function of x and y then :

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \qquad , \qquad f_z \neq 0$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} \qquad , \qquad f_z \neq 0$$

Example 1: suppose that $f(x,y,z) = xy^2 + z^3 + \sin(xyz) = 0$, Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Solution :

$f_x = y^2 + yz\cos(xyz)$

$f_y = 2xy + xz\cos(xyz)$

$f_z = 3z^2 + xy\cos(xyz)$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{y^2 + yz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$

Example H.W : use the Implicit Differentition to find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$:

1) $3yz^2 - e^{4x}\cos 4z - 3y^2 = 4$

2) $xyz - 4x^2z^2 + cosxy = 0$

CHECK POINT

8: DIRECTIONAL DERIVATIVE :

Suppose that f is a differentiable at (a,b) and $u=(u_1,u_2)$ is any unit vector then we can write :

$$D_uf(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

Example 1: suppose that $f(x,y) = x^2y - 4y^3$, compute $D_uf(2,1)$ for the direction of

A) $u = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, B) $u =$ is the direction $(2,1)$to $(4,0)$

Solution :

$f_x = 2xy \rightrightarrows f_x(2,1) = 4$, $f_y = x^2 - 12y^2 \rightrightarrows f_y(2,1) = -8$

$A)\ u = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$D_u f(2,1) = f_x(2,1)u_1 + f_y(2,1)u_2$$

$$D_u f(2,1) = 4\frac{\sqrt{3}}{2} - 8\frac{1}{2} \cong -0.5$$

B) we must first find the unit vector u in the indicated direction .

Observe that the vector from (2,1) to (4,0) corrosponds to the position vector (2,-1) ,

and so the unit vector in the direction is $u = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$ then

$$D_u f(2,1) = f_x(2,1)u_1 + f_y(2,1)u_2$$

$$D_u f(2,1) = 4\frac{2}{\sqrt{5}} - 8\left(\frac{-1}{\sqrt{5}}\right) = \frac{16}{\sqrt{5}}.$$

DEF:

The gradient of f(x,y) in the vector valued function is :

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

Provided both partial derivatives exists .

DEF:

If f is a differentiable function of x and y, and u is any unit vector then :

$$D_u f(a,b) = \nabla f(a,b) \cdot u$$

Example 2: suppose that $f(x,y)=x^2+y^2$, compute $D_u f(1,-1)$ for the direction of

$A)\ v=(-3,4)$, $B)\ v=(3,-4)$

Solution :

$$\nabla f(x,y)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=(2x,2y)$$

$$\nabla f(1,-1)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=(2,-2)$$

$$A)\quad u=\left(\frac{-3}{5},\frac{4}{5}\right)$$

$$D_u f(1,-1)=\nabla f(1,-1)\cdot u=(2,-2)\cdot\left(\frac{-3}{5},\frac{4}{5}\right)=\frac{-6-8}{5}=\frac{-14}{5}$$

$$B)\quad u=\left(\frac{3}{5},\frac{-4}{5}\right)$$

$$D_u f(1,-1)=\nabla f(1,-1)\cdot u=(2,-2)\cdot\left(\frac{3}{5},\frac{-4}{5}\right)=\frac{6+8}{5}=\frac{14}{5}$$

DEF: Suppose that f is a differentiable function of x, y and z at the point (a,b,c) in the direction of unit vector $u=(u_1,u_2,u_3)$ is given by :

$$D_u f(a,b,c)=f_x(a,b,c)u_1+f_y(a,b,c)u_2+f_z(a,b,c)u_3$$

And the gradient of f(x,y,z) is the vector valued function

$$\nabla f(x,y,z)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}j+\frac{\partial f}{\partial z}k$$

Provided all the partial derivatives are defined

DEF:

If f is a differentiable function of x,y ,z and u is any unit vector then :

$$D_u f(a,b,c) = \nabla f(a,b,c) \cdot u$$

Example 3: suppose that $f(x,y,z) = x^3yz^2 - 4xy$, compute $D_u f(1,-4,2)$ for the direction of $u = (1,1,-2)$

Solution :

$f_x = 3x^2yz^2 - 4y \rightrightarrows f_x(1,-4,2) = -32$,

$f_y = x^3z^2 - 4x \quad \rightrightarrows f_y(1,-4,2) = 20$

$f_z = 2x^3yz \qquad \rightrightarrows f_z(1,-4,2) = -16$

Unit vector is $u = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

$\nabla f(1,-4,2) = (-32,20,-16)$

$D_u f(1,-4,2) = \nabla f(1,-4,2) \cdot u$

$$D_u f(2,1) = (-32,20,-16) \cdot \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$$

$$D_u f(2,1) = \frac{-32 + 20 + 32}{\sqrt{6}} = \frac{20}{\sqrt{6}}$$

Example H·W : compute the directional derivative of f at the given point in the direction of the indicated vector :

CHECK POINT

1) $f(x,y)=\sqrt{xy}-y^2$, $(1,4)$, u is direction $(1,4)$

2) $f(x,y)=e^{4x^2-y}$, $(1,4)$, u is direction $(-2,-1)$

3) $f(x,y)=\cos(2x-y)$, $(\pi,0)$, u is direction from $(\pi,0)$ to $(2\pi,\pi)$

4) $f(x,y,z)=\sqrt{x^2+y^2+z^2}$, $(1,-2,2)$, u is direction of $(1,2,-3)$

1: Direct Integral : A) Integration Respect to x

Def: Suppose that f(x,y) is continous function on the region R , define :

$$R = \{(x,y): a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$$

For continous function $g_1(x), g_2(x)$ then

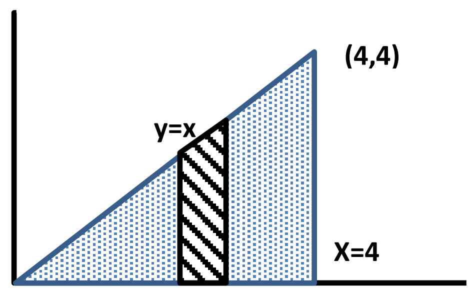
$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

Example 1 : let R be the Region bounded by the graphs of

$y = x, y = 0, x = 4$ **Evaluate :** $\iint_R (4e^{x^2} - 5\sin y)dA$

Solution :

first we draw the region R



$$I = \iint_R f(x,y)dA = \int_0^4 \int_0^x 4e^{x^2} - 5\sin y \, dydx$$

$$I = \int_0^4 4ye^{x^2} + 5\cos y\Big]_0^x dx$$

$$I = \int_0^4 4xe^{x^2} + 5\cos x - 5 \, dx$$

$$I = 2e^{x^2} + 5\sin x - 5x\Big]_0^4 = 1.78 \times 10^7$$

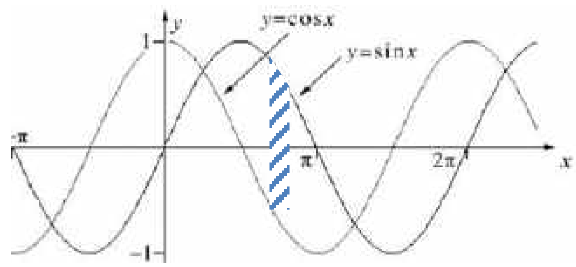
Example 2 : let R be the Region bounded by the graphs of

$y = \cos x$ **and** $y = \sin x$ **Evaluate :** $\iint_R (-2xy)dA$

Solution :

first we draw the region R

$\cos x = \sin x \rightarrow \tan x = 1 \rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$



$$I = \iint_R f(x,y)dA = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_{\cos x}^{\sin x} -2xy \; dydx$$

$$I = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} -xy^2\Big]_{\cos x}^{\sin x} dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} x\cos^2 x - x\sin^2 x \; dx$$

$$I = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} x\cos 2x \; dx$$

$u = x \quad , \quad dv = \cos 2x$

$du = dx \quad , \quad v = \frac{1}{2}\sin 2x$

$$I = \frac{1}{2}x\sin 2x - \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin 2x \, dx$$

$$I = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{\pi}{2}$$

1: Direct Integral : B) Integration Respect to y

Def: Suppose that f(x,y) is continous function on the region R , define :

$$R = \{(x, y): c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$$

For continous function $h_1(y), h_2(y)$ then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

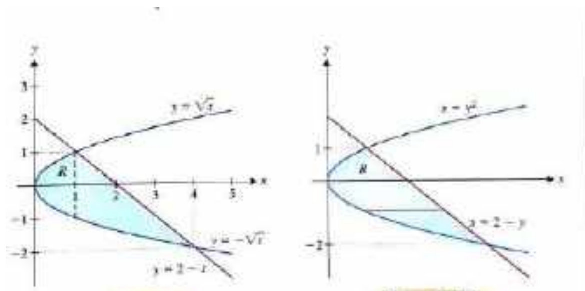
Example 3 : let R be the Region bounded by the graphs of

$x = y^2$, $x = 2 - y$ **Evaluate :** $\iint_R (2x)dA$

Solution :

if we take the region about the x axis

We will find two sub-regions , so that



We will take the region about y axis(one region)

$$\overset{x=y^2}{\underset{x=2-y}{}} \rightarrow y^2 + y - 2 = 0 \rightarrow y = -2,1$$

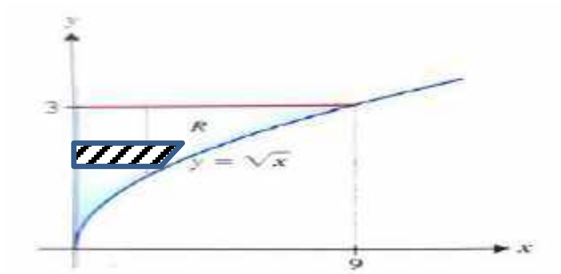
$$I = \iint_R f(x,y)dA = \int_{-2}^{1}\int_{y^2}^{2-y} \mathbf{2x}\ dxdy = \int_{-2}^{1} \mathbf{x^2}\Big]_{y^2}^{2-y} dy =$$

$$I = \int_{-2}^{1} (2-y)^{\mathbf{2}} - y^4\ dy = \int_{-2}^{1} \mathbf{4 - 4y + y^2} - y^4\ dy = 4y - 2y^2 + \frac{y^3}{3} - \frac{y^5}{5}\Big]_{-2}^{1} = ?$$

Example 4 : let R be the Region bounded by the graphs of $y=\sqrt{x}$, $x=0$ and $y=3$ Evaluate : $\iint_R (2xy^2+2y\cos x)dA$

Solution :

$y=\sqrt{x} \rightarrow x=y^2$

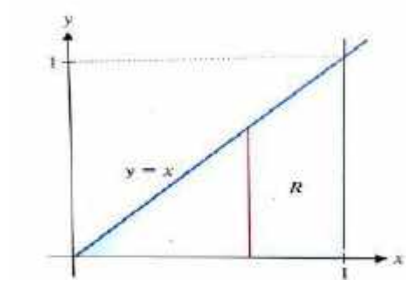


$$I=\iint_R f(x,y)dA=\int_0^3\int_0^{y^2} 2xy^2+2y\cos x \; dxdy$$

$$I=\int_0^3 x^2y^2+2y\sin x\Big]_0^{y^2} dy=\int_0^3 y^6+2y\sin y^2 dy=\frac{y^7}{7}-\cos y^2\Big]_0^3=314.3$$

Example 5 : let R be the Region bounded by the line $y=\sqrt{x}$, $y=0$ and $x=1$ Evaluate : $\iint_R (e^{x^2})dA$

Solution :



$I=\iint_R f(x,y)dA=\int_0^1\int_y^1 e^{x^2} dxdy$ (this integration don't available)

$$I=\iint_R f(x,y)dA=\int_0^1\int_0^{x} e^{x^2} \; dydx$$

$$I = \int_0^1 \left. ye^{x^2} \right]_0^x dx = \int_0^1 xe^{x^2} dx = \frac{1}{2} \left. e^{x^2} \right]_0^1 = \frac{1}{2}(e-1)$$

Example H.W :

CHECK POINT

1) $Re-solve\ ex(4) by\ using\ dA = dydx$

2) $Evaluate \iint_R e^{x^2} dA$, $R is\ bounded\ by\ y = x^2, y = 1$

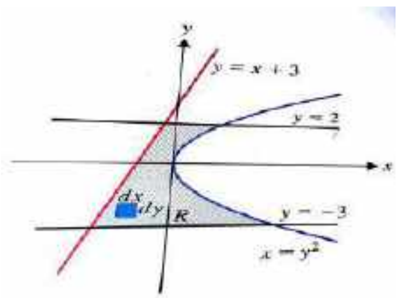
3) $Evaluate \iint_R \sqrt{y^2+1}\, dA$, $R is\ bounded\ by\ x = 4 - y^2, x = 0$

2: The Area

Let $R = \{(x, y): a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$ then the area can be written as :

$$A = \iint_R dA$$

Example 6 : Find the area of the plane region bounded by the graphs $x = y^2$, $y - x = 3$, $y = -3$ and $y = 2$



Solution

$$A = \iint_R dA \; A = \int_{-3}^{2}\int_{y-3}^{y^2} dxdy = \int_{-3}^{2} x]_{y-3}^{y^2} dy$$

$$A = \int_{-3}^{2} y^2 - y + 3 \; dy = \frac{y^3}{3} - \frac{y^2}{2} + 3y\Big]_{-3}^{2} = 29.16$$

Example H.W :

CHECK POINT

Use the double integral to compute the area of the region bounded by :

1) $y = x^2$, $x = y^2$

2) $y = 2x$, $y = 3 - x$, $y = 0$

3: The Volumes

Let $R = \{(x, y): a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$ then the volume of the region R can be written as :

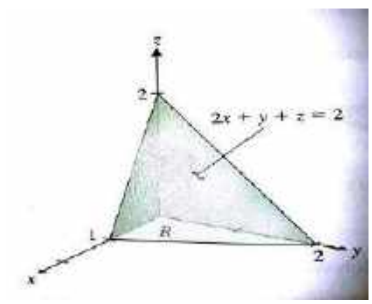
$$V = \iint_R f(x,y)dA = \iint_R Z\, dA \quad , \qquad Z = f(x,y)$$

Such that :

$dA = dxdy \;\; or \; dydx$

Example 7 : Find the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and the three coordinates planes .

Solution



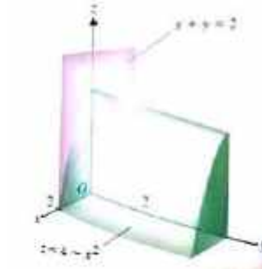
$$V = \iint_R zdA = \int_0^1 \int_0^{2-2x} 2 - 2x - y \; dydx$$

$$V = \int_0^1 2y - 2xy - \frac{y^2}{2}\Big]_0^{2-2x} dx$$

$$V = \int_0^1 4 - 4x - 4x + 4x^2 - \frac{1}{2}(2-2x)^2 dx = \frac{2}{3}$$

Example 8 : Find the volume of the solid lying in the first octant and bounded by the graphs $z = 4 - x^2$, $x + y = 2$, $x = y = z = 0$.

Solution



$$V = \iint_R z\,dA = \int_0^2 \int_0^{2-y} 4 - x^2 \; dx\,dy$$

$$V = \int_0^2 4x - \frac{x^3}{3}\Big]_0^{2-y} \, dy$$

$$V = \int_0^2 4(2-y) - \frac{1}{3}(2-y)^3 dy = \frac{20}{3}$$

Example H.W :

CHECK POINT

Find the volume of solid lying in the first octant bounded by the graphs of :

$$z = 4 - x^2 - y^2 \ , \ y = 2 - 2x^2 \ , \ x = y = z = 0$$

4: Moments & Center of Mass

We will discussing a physical application of double integral consider a thin flat plate (a Lamina) in the shape of the region R whose density (mass per unit area) varies through out the plate , from an engineering stand point its often important to determine where you could place a support to balanced the plate · we call this point the center of Mass of the Lamina , the mass density given by the function $\rho(x,y)$ ·

Let m be the total mass of lamina is than given as :

$$m = \iint_R \rho(x,y)dA$$

And we define the moment about the x axis as :

$$M_x = \iint_R y\,\rho(x,y)dA$$

and the moment about the y axis as :

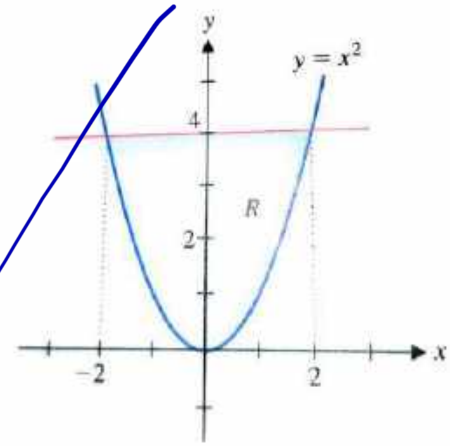
$$M_y = \iint_R x\,\rho(x,y)dA$$

then the center of mass is the point $(\bar{x},\bar{y})$ defined by :

$$\bar{x} = \frac{M_y}{m} \quad , \quad \bar{y} = \frac{M_x}{m}$$

Example 9 : Find the Center of Mass of the lamina in the shape of region bounded by the graphs of : $y = x^2$, $y = 4$ having Mass density given by : $\boldsymbol{\rho(x, y) = 1 + 2y + 6x^2}$.

Solution



$$m = \iint_R \rho(x,y)dA = \int_{-2}^{2}\int_{x^2}^{4} 1 + 2y + 6x^2 \; dydx$$

$$m = \int_{-2}^{2} y + y^2 + 6x^2y]_{x^2}^{4} dy$$

$$m = \int_{-2}^{2} -7x^4 + 23x^2 + 20 \, dx = 113.1$$

Now we compute the Moment :

$$M_y = \iint_R x\,\rho(x,y)dA = \int_{-2}^{2}\int_{x^2}^{4} x(1 + 2y + 6x^2) \; dydx$$

$$M_y = \int_{-2}^{2}\int_{x^2}^{4} (x + 2xy + 6x^3) \; dydx$$

$$M_y = \int_{-2}^{2} xy + xy^2 + 6x^3y]_{x^2}^{4} dx$$

$$M_y = \int_{-2}^{2} -7x^5 + 23x^3 + 20x \, dx = 0$$

$$\therefore M_y = 0$$

$$M_x = \iint_R y\,\rho(x,y)dA = \int_{-2}^{2}\int_{x^2}^{4} y(1+2y+6x^2)\ dydx$$

$$M_x = \int_{-2}^{2}\int_{x^2}^{4} (y+2y^2+6x^2y)\ dydx$$

$$M_x = \int_{-2}^{2} \frac{y^2}{2}+\frac{2y^3}{3}+3x^2y^2\Big]_{x^2}^{4} dx$$

$$M_x = \int_{-2}^{2} \{(8+\frac{128}{3}+48x^2)-(\frac{x^4}{2}+\frac{2x^6}{3}+3x^6)\}dx = 318.2$$

$\therefore M_x = 318.2$

$$\bar{x} = \frac{M_y}{m} = \frac{0}{113.1} = 0$$

$$\bar{y} = \frac{M_x}{m} = \frac{318.2}{113.1} = 2.8$$

$\therefore$ *Center of Mass is* :

$(\bar{x},\bar{y}) = (0,2.8)$

Example H.W :

CHECK POINT

Find the Center of Mass of Lamina bounded bythe graphs of :

$x = y^2$, $x = 1$ **and** $\rho(x,y) = y^2 + x + 1$

5: The Second Moment

DEF:

The Second Moment about the y axis often called (Moment of Inertia about the y axis of Lamina) in the shape of the region R with density function $\rho(x,y)$ is defined by :

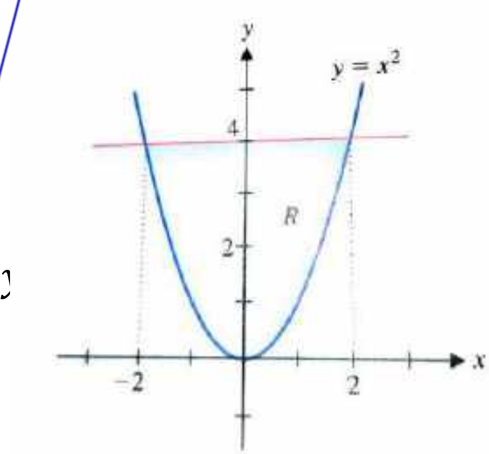
$$I_y = \iint_R x^2 \, \rho(x,y) dA$$

Simillary the second moment about the x-axis

$$I_x = \iint_R y^2 \, \rho(x,y) dA$$

Example 10 : Find the moments of inertia I_x, I_y for the lamina in the shape of region bounded by the graphs of : $y = x^2$, $y = 4$ having Mass density given by : $\rho(x,y) = 1 + 2y + 6x^2$.

Solution



$$I_y = \iint_R x^2 \, \rho(x,y) dA = \int_{-2}^{2} \int_{x^2}^{4} x^2(1 + 2y + 6x^2) \, dy$$

$$I_y = \int_{-2}^{2} 20x^2 + 23x^4 - 7x^6 \, dx = 145.7$$

$$I_x = \iint\limits_R y^2\,\rho(x,y)dA = \int_{-2}^{2}\int_{x^2}^{4} y^2(1+2y+6x^2)\ dydx$$

$$I_x = \int_{-2}^{2}\int_{x^2}^{4}(y^2+2y^3+6x^2y^2)\ dydx$$

$$I_x = \int_{-2}^{2} \frac{y^3}{3}+\frac{y^4}{2}+2x^2y^3\Big]_{x^2}^{4}\ dx = 983.37$$

CHECK POINT

Example H·W :

Find the Moment inertia I_x , I_y of Lamina in the shape

Of the region bounded by :

$y = x^2$, $y = 4$ **with density function** $\rho(x,y) = 1$

6: Double Integral in polar Coordinates

DEF:

Suppose that $f(r,\theta)$ is continous function in the region R such that :

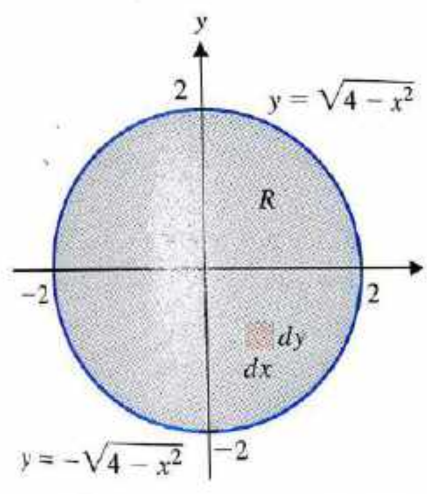
$R = (r,\theta): \alpha \le \theta \le \beta$, $g_1(\theta) \le r \le g_2(\theta)$, then :

$$\iint_R f(r,\theta)dA = \int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, rdrd\theta$$

Example 11 : Evaluate $\iint_R x^2 + y^2 + 3\ dA$, Where R is the circle with center is the origin and radius is 2 .

Solution

$$I = \iint_R x^2 + y^2 + 3\, dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^2 + y^2 + 3\ dydx$$



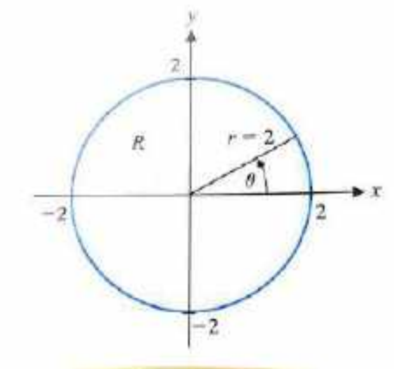
$$I = \int_{-2}^{2} x^2 y + \frac{y^3}{3} + 3y\Big]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} =$$

$$I = 2\int_{0}^{2} (x^2+3)\sqrt{4-x^2} + \frac{1}{3}\sqrt{(4-x^2)^3} =$$

This integration is hard to solve directly , we will search about shortly method by re-write this integral by using polar coordinates , such that :

$x = r\cos\theta \;,\; y = r\sin\theta \;,\; r = \sqrt{x^2+y^2} \;, dA = r\,dr\,d\theta$

Re solve previous question by polar



$$I = \iint_R x^2 + y^2 + 3\,dA$$

$$I = \int_0^{2\pi}\int_0^{2} (r^2+3)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} (r^3+3r)\,dr\,d\theta$$

$$I = \int_0^{2\pi} \frac{r^4}{4} + \frac{3r^2}{2}\Big]_0^2 d\theta = \int_0^{2\pi} 10\,d\theta = 10\theta]_0^{2\pi} = 20\pi$$

Example 12 : Evaluate the iterated integral by converting to polar

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2(x^2+y^2)^2\,dy\,dx$$

Solution

$$y = 0 \rightarrow r\sin\theta = 0 \rightarrow r = 0$$

$$y = \sqrt{1-x^2} \rightarrow x^2+y^2 = 1 \rightarrow r^2 = 1 \rightarrow r = 1$$

$$x = 1 \rightarrow \cos\theta = \frac{x}{r} = \frac{1}{1} \rightarrow \theta = 0$$

$$x = -1 \rightarrow \cos\theta = \frac{x}{r} = \frac{-1}{1} \rightarrow \theta = \pi$$

$$I=\int_{0}^{\pi}\int_{0}^{1} r^2\cos^2\theta\ (r^2)^2 r\,dr\,d\theta=\int_{0}^{\pi}\int_{0}^{1} r^7\cos^2\theta\ dr\,d\theta$$

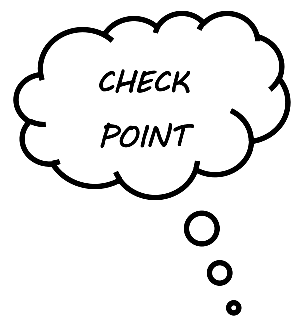
$$I=\int_{0}^{\pi}\left.\frac{r^8}{8}\cos 2\theta\right]_{0}^{1} d\theta=\frac{1}{16}\int_{0}^{\pi} 1+\cos 2\theta\ d\theta$$

$$I=\frac{1}{16}\left.\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_{0}^{\pi}=\frac{\pi}{16}$$

Example H.W :

1)Evaluate the interated integral by converting to polar coordinates:

A) $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\sqrt{x^2+y^2}\,dydx$



B) $\displaystyle\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sin(x^2+y^2)dydx$

2)use the polar coordinates to evaluate the double integral :

A) $\displaystyle\iint_R \sqrt{x^2+y^2}\ dA$, R is the disk $x^2+y^2\le 9$

B) $\displaystyle\iint_R y\,dA$, R is bounded by $r=2-\cos\theta$

7: Area in polar Coordinates

DEF:

Let R be the region defined as : $R = (r,\theta) : \alpha \le \theta \le \beta \;\; , \;\; g_1(\theta) \le r \le g_2(\theta)$

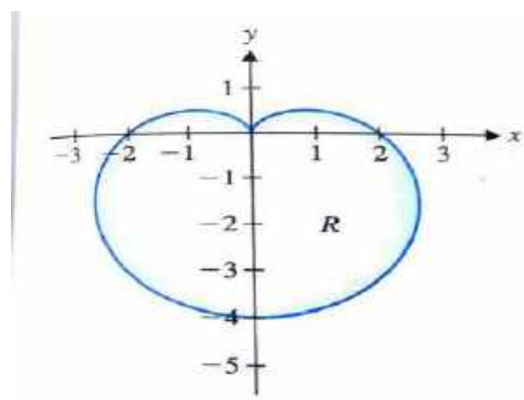
, then the area can written as :

$$A = \iint_R dA \quad , \quad such\ that \quad dA = r dr d\theta$$

Example 13 :

Find the area inside the curve defined by : $r = 2 - 2\sin\theta$

Solution :



$$A = \iint_R dA = \int_0^{2\pi} \int_0^{2-2\sin\theta} r dr d\theta$$

$$A = \int_0^{2\pi} \frac{r^2}{2}\Bigg]_0^{2-2\sin\theta} = \frac{1}{2}\int_0^{2\pi} (2-2\sin\theta)^2 \, d\theta = 6\pi$$

Example H·W :

CHECK POINT

Find the Area of the region bounded by the given curves :

1) $r = 3 + 2\sin\theta$

2) $r = 2 - 2\cos\theta$

8: Volumes in polar Coordinates

DEF:

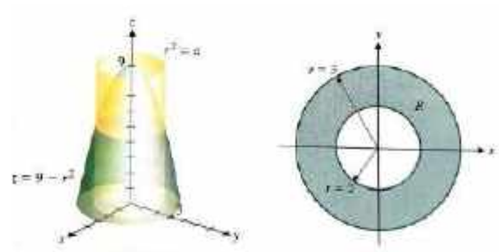
Let R be the region defined as : $R = (r,\theta): \alpha \le \theta \le \beta \ , \ \ g_1(\theta) \le r \le g_2(\theta)$

, then the volume can be written as :

$$V = \iint\limits_R z\, dA \quad , \ such\ that \ \ z = f(x,y), \ \ dA = r dr d\theta$$

Example 14 :

Find the volume inside the paraboloid $z = 9 - x^2 - y^2$ and the cylinder $x^2 + y^2 = 4$ and above the xy plane (z=0)



Solution :

$x^2 + y^2 = 4 \to r^2 = 4 \ \to r = 2$

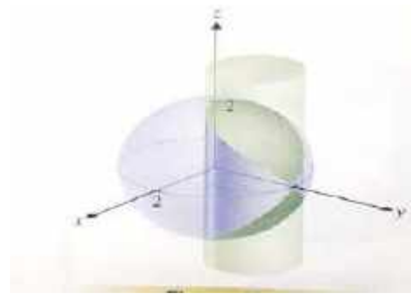
$z = 9 - x^2 - y^2 \to 0 = 9 - x^2 - y^2 \ \to 0 = 9 - r^2 \to r = 3$

$$V = \iint\limits_R f(x,y) dA = \int_0^{2\pi}\int_2^3 (9 - r^2) r dr d\theta = \int_0^{2\pi}\int_2^3 (9r - r^3) dr d\theta$$

$$V = \int_0^{2\pi} \frac{9r^2}{2} - \frac{r^4}{4}\Big]_2^3 \, d\theta = \frac{25}{2}\pi$$

Example 15 :

Find the volume cut out the sphare $x^2+y^2+z^2=4$ by the cylinder $x^2+y^2=2y$.



Solution :

$z^2=4-x^2-y^2 \rightarrow z=\mp\sqrt{4-x^2-y^2}$

$$V=2\iint\limits_R \sqrt{4-x^2-y^2}\,dA$$

$\theta : 0\,,\pi$

$\boldsymbol{x^2+y^2=2y \rightarrow r^2=2r\sin\theta \rightarrow r=2\sin\theta}$

$$V=2\int_0^{\pi}\int_0^{2\sin\theta}\sqrt{4-r^2}\,r\,dr\,d\theta=4\int_0^{\frac{\pi}{2}}\int_0^{2\sin\theta} r\sqrt{4-r^2}\,dr\,d\theta$$

$$V=-\frac{4}{3}\int_0^{\frac{\pi}{2}}(4-r^2)^{\frac{3}{2}}\Big]_0^{2\sin\theta}\,d\theta=-\frac{4}{3}\int_0^{\frac{\pi}{2}}(4-4\sin^2\theta)^{\frac{3}{2}}-(4)^{\frac{3}{2}}\,d\theta$$

$$V=-\frac{32}{3}\int_0^{\frac{\pi}{2}}(\cos^2\theta)^{\frac{3}{2}}-1\,d\theta=-\frac{32}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta-1\,d\theta=\frac{-64}{9}+\frac{16}{3}\pi$$

CHECK POINT

Example H·W :

Use the polar coordinates to evaluate volumes :

1) Below $z = x^2 + y^2$ above z=0 , inside $x^2 + y^2 = 9$

2) Below $z = \sqrt{1 - x^2 - y^2}$ inside $x^2 + y^2 = \frac{1}{4}$ $above\ the\ xy\ plane$

الموضوع : هندسة تقطيع وتكرير / العام الدراسي ٢٠٢٠ - ٢٠٢١

Tangent plane and normal line

normal vector $= (f_x(a,b), f_y(a,b), -1)$

equation of tangent plane

$$f_x(a,b)[x-a] + f_y(a,b)[y-b] - [z - f(a,b)] = 0$$

normal line are given by

$$\left.\begin{aligned} x &= a + f_x(a,b)t \\ y &= b + f_y(a,b)t \\ z &= f(a,b) - t \end{aligned}\right\}$$

Example(1): $f(x,y) = 6 - x^2 - y^2$, find

1- equation of tangent

2- normal vector. at $(1,2,1)$

3- normal line.

ans

$f(x,y) = 6 - x^2 - y^2$, $f(1,2) = 1$

$f_x(x,y) = -2x$, $f_x(1,2) = -2$

$f_y(x,y) = -2y$, $f_y(1,2) = -4$

1) $z-1 = -2(x-1) - 4(y-2)$

2 - normal vector is $(-2, -4, -1)$

3) normal line

$x = 1 - 2t$
$y = 2 - 4t$
$z = 1 - t$

Example(2): $z = x^3 + y^3 + \frac{x^2}{y}$ at $(2,1,3)$
find
① tangent plane
② normal vector
③ normal line.

Solution

$f(2,1) = (2^3) + (1)^3 + \frac{(2)^2}{1} = 8 + 1 + 4 = 13$

$f_x(x,y) = 3x^2 + \frac{2x}{y}$, $f_x(2,1) = 3 \times 2^2 + 2 \times 2 = 16$

$f_y(x,y) = 3y^2 - \frac{x^2}{y^2}$, $f_y(2,1) = 3 \times 1 - \frac{4}{1} = -1$

① tangent plane $z - 13 = 16(x-2) - (y-1)$

② normal vector $(16, -1, -1)$

③ $x = 2 + 16t$
$y = 1 - t$, $z = 13 - t$

Chain rule   قاعدة السلسلة

معرفة لمتغيرين   معرفة لمتغير آخر   معرفة لمتغير آخر

if $Z = f(x,y)$, $x = x(t)$, $y = y(t)$

دوال قابلة للاشتقاق فان $f(x,y)$ دالة قابلة

للاشتقاق ايضا بالشكل

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

EX(1): $f(x,y) = x^2 e^y$, $\left.\begin{matrix} x(t) = t^2 - 1 , \\ y(t) = \sin t \end{matrix}\right\}$ find $\frac{df}{dt}$

Solution

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x e^y)(2t) + (x^2 e^y) \cdot \cos t$$

نعوض عن كل من x و y من السؤال

$$= 2(t^2-1) \cdot e^{\sin t} \cdot (2t) + (t^2-1)^2 e^{\sin t} \cdot \cos t$$

نبسط المقدار

$$\frac{df}{dt} = 4(t^3 - t) e^{\sin t} + (t^2-1)^2 \cos t \, e^{\sin t}$$

تحل بقية الاسئلة والواجبات بنفس الطريقة

theorem : suppose $z = f(x,y)$ دالة بمتغيرين
وهي دالة قابلة للاشتقاق بالنسبة للمتغيرين x, y وحيث أن
$x = x(u,v)$, $y = y(u,v)$ وهما دوال تمتلكان مشتقة جزئية
من الرتبة الأولى . لذلك فإن قاعدة السلسلة لمشتقات
الدالة f بالنسبة لـ u, v تكتب بالشكل .

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot \partial y$$

Ex(2):- $f(x,y) = e^{xy}$, $x = 3u\sin v$,
$y = 4v^2 u$

find $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$

solution

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (y e^{xy})\cdot(3\sin v) + x e^{xy}(4v^2)$$

$$= (4v^2 u) e^{(3u\sin v)\cdot 4v^2 u} + 3u\sin v\, e^{3u\sin v\cdot 4v^2 u}(4v^2)$$

$$12uv^2 u\, e^{12u^2v^2\sin v}\cdot \sin v + 12v^2 u\sin v\, e^{12v^2u^2\sin v}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= y e^{xy} \cdot (3u\cos v) + x e^{xy} \cdot 8vu$$

ثم نبسط الحل

7) implicit Differentiation الاشتقاق الضمني

$f(x,y,z) = c$

يكون اشتقاق Z ضمنياً بالنسبة لـ x و y بالشكل التالي

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}, \quad f_z \neq 0$$

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}, \quad f_z \neq 0$$

نستخدم الاشتقاق الضمني إذا كانت لدينا دالة ما صورة فقط معرفة لديه متغيرات وكان المطلوب اشتقاق Z بالنسبة لـ x و y

Ex(1):- $f(x,y,z) = xy^2 + z^3 + \sin(xyz) = 0$

find (1) $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

solution.

$f_x = y^2 + yz\cos(xyz)$

$f_y = 2xy + xz\cos(xyz)$

$f_z = 3z^2 + xz\cos(xyz)$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \frac{-(y^2 + yz\cos(xyz))}{3z^2 + xy\cos(xyz)}$$

$$\frac{\partial z}{\partial y} = -\frac{f_y}{\partial z} = -\frac{2xy + xz\cos(xyz)}{3z^2 + xy\cos(xyz)}$$

8 - Directinal derevative المشتقة الاتجاهية

اذا كانت f دالة قابلة للاشتقاق عند النقطة (a,b) وكانت $U=(U_1, U_2)$ هو متجه الوحدة فإنه تعرف المشتقة الاتجاهية عند النقطة (a,b) بالشكل

$$D_u f(a,b) = f_x(a,b)U_1 + f_y(a,b)U_2$$

Example:- $f(x,y) = x^2y - 4y^3$, find $D_u f(2,1)$ for the direction

(A) $u = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, B) $u =$ is the direction $(2,1)$ to $(4,0)$

solution

$f_x = 2xy \rightarrow f_x(2,1) = 4$

$f_y = x^2 - 12y^2 \rightarrow f_y(2,1) = 4 - 12 = -8$

A) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$D_u f(2,1) = f_x(2,1)u_1 + f_y(2,1)u_2 = 4 \cdot \frac{\sqrt{3}}{2} - 8 \cdot \frac{1}{2}$

الموضوع :   التاريخ : / / ٢٠

الجديد - القديم $(4,0) - (2,1)$

$(4-2), (0-1) = (2,-1)$

$$\text{unit vector} = \left(\frac{u_1}{\sqrt{u_1^2+u_2^2}}, \frac{u_2}{\sqrt{u_1+u_2}}\right) = \left(\frac{2}{\sqrt{4+1}}, \frac{-1}{\sqrt{4+1}}\right) = \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$$

$$\therefore D_u f(2,-1) = f_x u_1 + f_y u_2$$

$$D_u f(2,-1) = 4 \cdot \frac{2}{\sqrt{5}} - 8\left(\frac{-1}{\sqrt{5}}\right) = \frac{16}{\sqrt{5}}$$

Def. (gradient) الانحدار او التدرج

يعرف الانحدار للدالة $f(x,y)$ بالشكل

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

ملاحظة : اذا كانت $f(x,y)$ دالة قابلة للاشتقاق لكل من x و y وكان u هو اي متجه وحدة فان

$$D_u f(a,b) = \nabla f(a.b)\, u$$

مشتقة اتجاهية   الانحدار

Example:- suppse $f(x,y)=x^2+y^2$, compute $D_u f(1,-1)$ for the Direction of

A) $v=(-3,4)$, B) $v=(3,-4)$

Solution

$$\nabla f(x,y)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=(2x,2y)$$

$$\nabla f(+1,-1)=(2,-2)$$

A) $u=\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}},\frac{v_2}{\sqrt{v_1^2+v_2}}\right)=\left(\frac{-3}{5},\frac{4}{5}\right)$

$$\therefore D_u f(1,-1)=\nabla f(+1,-1)\cdot u=(2,-2)\cdot\left(\frac{-3}{5},\frac{4}{5}\right)$$

$$=\frac{-6}{5}-\frac{8}{5}=\frac{-14}{5}$$

B) $v=\left(\frac{v_1}{\sqrt{v_1^2+v_2^2}},\frac{v_2}{\sqrt{v_1^2+v_2^2}}\right)=\left(\frac{3}{5},-\frac{4}{5}\right)$

$$D_u f(1,-1)=\nabla f(+1,-1)\cdot u=(2,-2)\left(\frac{3}{5},-\frac{4}{5}\right)$$

$$=\frac{6}{5}+\frac{8}{5}=\frac{14}{5}$$

اما اذا كانت f دالة بثلاث متغيرات x و y و z وهي دالة قابلة للاشتقاق لـ x و y و z ومعرفة عند النقطة (a,b,c) نعرف المشتقة الاتجاهية بالنسبة لمتجه الوحدة $u=(u_1,u_2,u_3)$ بالشكل التالي

$$D_u f(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)\cdot u_3$$

ويعرف التدرج او الانحدار بالشكل التالي

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

م/ اذا كانت f قابلة للاشتقاق بالنسبة لـ x و y و z وكان u متجه وحدة فان

$$D_u f(a,b,c) = \nabla f(a,b,c)\cdot u$$

Example $f(x,y,z) = x^3yz^2 - 4xy$ - compute

$D_u f(1,-4,2)$ for the Direction of $u = (1,1,-2)$

solution :-

$3\times1.4.4$

$f_x = 3x^2yz^2 - 4y$, $f_x(1,-4,2) = -84+16 = 32$

$f_y = x^3z^2 - 4x$, $f_y(1,-4,2) = 1\times4 - 4 = 0$

$f_z = 2x^3yz$ $\quad f_z(1,-4,2) = 2(1)(-4)(2) = -16$

$$\text{unit vector} = \left(\frac{u_1}{\sqrt{u_1^2+u_2^2+u_3^2}}, \frac{u_2}{\sqrt{u_1^2+u_2^2+u_3^2}}, \frac{u_3}{\sqrt{u_1^2+u_2^2+u_3^2}}\right)$$

$u = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right)$

$\because \nabla f(a,b,c) = \nabla f(1,-4,2) = (-32, 0, -16)$

$D_u f(1,-4,2) = \nabla f(1,-4,2)\, u$

$\therefore D_u f(1,-4,2) = (-32, 0, -16) \cdot \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right)$

$= \frac{-32}{\sqrt{6}} + 0 + \frac{32}{\sqrt{6}} = 0$

Triple Integral :

Def: Suppose that f(x,y,z) is continous on the region Q defined by

$Q = \{(x, y, z): a \le x \le b\ ,\ c \le y \le d\ ,\ e \le z \le f\}$, then we can write the triple integral over Q as a triple iterated integral :

$$\iiint_Q f(x,y,z)\, dv = \int_e^f \int_c^d \int_a^b f(x,y,z) dxdydz$$

Note : dv may be dxdydz , dxdzdy , dzdydx , ⋯

Example 1 : Evaluate the triple integral $\iiint_Q 2xe^y sinz\, dv$, Where Q is the rectangle defined by :

$$Q = \{(x, y, z): 1 \le x \le 2\ , 0 \le y \le 1\ , 0 \le z \le \pi\}$$

Solution :

$$\iiint_Q 2xe^y sinzdv = \int_0^\pi \int_0^1 \int_1^2 2xe^y sinz\, dxdydz$$

$$\int_0^\pi \int_0^1 e^y sinz\, x^2\Big|_1^2\, dydz$$

$$3\int_0^\pi \int_0^1 e^y sinz\ dydz = 3\int_0^\pi e^y sinz\big|_0^1 dz = 3(e-1)\int_0^\pi sinz\, dz$$

$$= 3(e-1)(-cosz)\big|_0^\pi = 6(e-1)$$

Note :the triple integral may be transform to double integral by :

$$\iiint\limits_{Q} f(x,y,z)\,dv = \iint\limits_{R} \int\limits_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dzdA$$

Example 2 : Evaluate the triple integral $\iiint_Q 6xy\,dv$, Where Q is the tetrahedron bounded by the planes x=0 , y=0 , z=0 and 2x+y+z=4 :

Solution :

$$\iiint\limits_{Q} 6xydv = \iint\limits_{R} \int\limits_{0}^{4-2x-y} 6xydzdA$$

$$\iint\limits_{R} 6xyz\Big|_0^{4-2x-y} dA = \iint\limits_{R} 6xy(4-2x-y)dA$$

Now this integral is double integral and lies in xy-plane (z=0)

$$z = 4-2x-y \ \gg\gg z=0 \rightarrow y = 4-2x \ \gg\gg y=0 \rightarrow x=2$$

$$\int\limits_0^2 \int\limits_0^{4-2x} 24xy - 12x^2y - 6xy^2 \ dydx = \int\limits_0^2 12xy^2 - 6x^2y^2 - 2xy^3\Big|_0^{4-2x} dx$$

$$= \int\limits_0^2 12x(4-2x)^2 - 6x^2(4-2x)^2 - 2x(4-2x)^3 dx = \frac{64}{5}$$

Try this Question use integrating respect to x at first ?

Volumes Using Triple Integral

The volume on the region Q is defined by :

$$V=\iiint_Q dv$$

Example 3 : use the triple integral to find the volume of the solid Q bounded by the graph of $y=4-x^2-z^2$ and the xz-plane

Solution :

$$V=\iiint_Q dv=\int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}} dzdydx$$

$$V=\int_{-2}^{2}\int_{0}^{4-x^2} z|_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}}\, dydx=2\int_{-2}^{2}\int_{0}^{4-x^2}\sqrt{4-x^2-y}\,dydx$$

$$V=\frac{-4}{3}\int_{-2}^{2}\int_{0}^{4-x^2}(4-x^2-y)^{\frac{3}{2}}\Big|_{0}^{4-x^2} dydx=\frac{4}{3}(4-x^2)^{\frac{3}{2}}dx=8\pi$$

Example 4 : use the triple integral to find the volume of the solid Q bounded by the graph of $z=4-y^2$ and $x+z=4$ x=0,z=0

Solution :

$$V=\iiint\limits_{Q} dv=\iint\limits_{R}\int_{0}^{4-z} dxdA=\int_{-2}^{2}\int_{0}^{4-y^2}\int_{0}^{4-z} dxdzdy$$

$$V=\int_{-2}^{2}\int_{0}^{4-y^2} x\Big|_{0}^{4-z} dzdy=\int_{-2}^{2}\int_{0}^{4-y^2}(4-z)\,dzdy$$

$$V=\int_{-2}^{2} 4z-\frac{z^2}{2}\Big|_{0}^{4-y^2}=\int_{-2}^{2} 4(4-y^2)-\frac{(4-y^2)^2}{2}dy=\frac{128}{5}$$

Example H.W :

CHECK POINT

1) Evaluate the triple integral $\iiint_Q f(x,y,z)dv$:

A) $f(x,y,z)=2x+y-z$, $Q=\{(x,y,z):0\le x,z\le 2,-2\le y\le 2\}$

B) $f(x,y,z)=3y^2-2z$, Q is the tetrahedron bounded 3x+2y-z=6 , and the coordinate planes .

2) compute the volume to :

A) $z=x^2\,,z=1\,,y=0\ and\ y=2$

B) $z=y^2\,,\ z=1\,,2x+z=4\,,x=0$

Mass and Center of Mass

let m be the total mass and given by :

$$m = \iiint_Q \rho(x,y,z)dv$$

And the moment in the yz-plane is :

$$M_{yz} = \iiint_Q x\,\rho(x,y,z)dv$$

And the moment in the xz-plane is :

$$M_{xz} = \iiint_Q y\,\rho(x,y,z)dv$$

And the moment in the xy-plane is :

$$M_{xy} = \iiint_Q z\,\rho(x,y,z)dv$$

And the center of mass given by the point $(\bar{x},\bar{y},\bar{z})$ *such that :*

$$\bar{x} = \frac{M_{yz}}{m} \quad , \bar{y} = \frac{M_{xz}}{m} \quad , \bar{z} = \frac{M_{xy}}{m}$$

Example 5 : Find the center of mass of the solid of constant mas density " 1 " bounded by the graphs of the right circular cone $z=\sqrt{x^2+y^2}$ and the plane z=4 .

Solution :

$$m=\iiint_Q \rho(x,y,z)dv=\iint_R \int_{\sqrt{x^2+y^2}}^{4} 1\, dzdA$$

$$m=\iint_R 4-\underbrace{\sqrt{x^2+y^2}}_{r}\ \underbrace{dA}_{rdrd\theta}=\int_0^{2\pi}\int_0^4 (4-r)rdrd\theta=\int_0^{2\pi}\int_0^4 (4r-r^2)drd\theta$$

$$m=\int_0^{2\pi} 2r^2-\frac{r^3}{3}\Big|_0^4 d\theta=\frac{64}{3}\pi$$

$$M_{xy}=\iiint_Q z\rho(x,y,z)dv=\iint_R \int_{\sqrt{x^2+y^2}}^{4} z\, dzdA=\iint_R \frac{z^2}{2}\Big|_{\sqrt{x^2+y^2}}^{4} dA$$

$$M_{xy}=\frac{1}{2}\iint_R 16-\underbrace{(x^2+y^2)}_{r^2}\ \underbrace{dA}_{rdrd\theta}=\frac{1}{2}\int_0^{2\pi}\int_0^4 (16-r^2)rdrd\theta$$

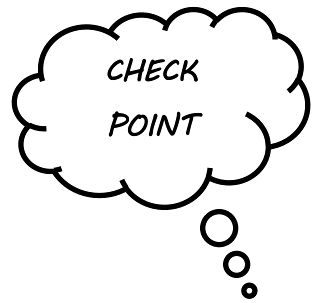
$$M_{xy}=\frac{1}{2}\int_0^{2\pi}\int_0^4 (16r-r^3)drd\theta=\frac{1}{2}\int_0^{2\pi} 8r^2-\frac{r^4}{4}\Big|_0^4 d\theta=64\pi$$

As same we find $M_{xz}, M_{yz} = 0$

$$\bar{x} = \frac{M_{yz}}{m} = 0 \ , \bar{y} = \frac{M_{xz}}{m} = 0 \ , \bar{z} = \frac{M_{xy}}{m} = 3$$

Then the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3)$

Example H.W :



1) Evaluate the mass center for $\rho(x, y, z)$

A) $\rho(x, y, z) = 4$,solid bounded by $z = x^2 + y^2$, $z = 4$

A) $\rho(x, y, z) = 10 + x$,solid bounded by tetrahedron $x + 3y + z = 6$ and the coordinate planes .

Euler Formula :

if θ is the angle (Argument) of the components of the complex function then Euler formula can written as :

$$e^{i\theta} = \cos\theta + i\sin\theta \quad , \quad 0 \le \theta \le 2\pi$$

$$\overline{e^{i\theta}} = \cos\theta - i\sin\theta \quad , \quad 0 \le \theta \le 2\pi$$

Note : $z = x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$|z| = |re^{i\theta}| = r \text{ , such that } |e^{i\theta}| = 1$$

The graph of function $|z| = r$ is a circle with center is the origin and radius is r , while the graph of function $|z - z_o| = r$ is a circle with center is the $z_o = (x_o, y_o)$ and radius is r .

Direct Integral :

Let $f(t) = u(t) + iv(t)$ where u and v be real valued function , the definite integral of f(t) on the interval $a \le t \le b$ is defined as :

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Example 1 : Evaluate $\int_0^1 (1+it)^2 dt$

Solution :

$$I = \int_0^1 (1 + 2it - t^2)dt$$

$$I = \int_0^1 (1 - t^2)dt + i\int_0^1 2tdt$$

$$I = t - \frac{t^3}{3}\Big|_0^1 + i\, t^2\Big|_0^1 = \frac{2}{3} + i$$

Example 2 : Evaluate $\int_0^{\frac{\pi}{4}} e^{it}\, dt$

Solution :

$$I = \int_0^{\frac{\pi}{4}} e^{it}\, dt = \int_0^{\frac{\pi}{4}} cost dt + i\int_0^{\frac{\pi}{4}} sint\ \ dt =$$

$$I = sint\Big|_0^{\frac{\pi}{4}} - icost\Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i(1 - \frac{1}{\sqrt{2}})$$

Example H·W :

CHECK POINT

1) Evaluate the following integral :

A) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$, B) $\int_0^{\frac{\pi}{6}} e^{2it} dt$, c) $\int_0^{\pi} e^{(1+i)t}\, dt$

Integral on Regions

Example 3 : if C is the region defined by a circle $z - z_o = re^{i\theta}$ such that $0 \le \theta \le 2\pi$, and z_o is the center of the circle , r is the radius , find

$$\int_C \frac{dz}{z - z_o}$$

Solution :

$$z - z_o = re^{i\theta} \;\; \rightarrow dz = ire^{i\theta} d\theta$$

$$\int_C \frac{dz}{z - z_o} = \int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = \int_0^{2\pi} i \, d\theta = i\theta|_0^{2\pi} = 2\pi i$$

Example 4 : if C is the region defined by right half of a circle $|z| = 1$, find $\int_C |z| \, dz$

Solution :

$$|z| = 1 \, , z = e^{i\theta} \;\; \rightarrow dz = ie^{i\theta} d\theta$$

$$\int_C |z| dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1. ie^{i\theta} d\theta = e^{i\theta}\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$

$$\int_C |z| dz = e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}} = 2i$$

(check another method by Euler formula expansion integral)

Example 5 : if C is the portion of circle defined by 1^{st} and 3^{rd} Quarter of circle $|z| = 2$, find $\int_C |z|\, dz$

Solution :

The region C divided into two sub-regions $C{:}C_1{+}C_2$

$|z| = 2\,, z = 2e^{i\theta} \;\; \rightarrow dz = 2ie^{i\theta}d\theta$

$$\int_C |z|dz = \int_{C_1} |z|\, dz + \int_{C_2} |z|\, dz$$

$$\int_{C_1} |z|dz = \int_0^{\frac{\pi}{2}} 2.\,2ie^{i\theta}d\theta = 4e^{i\theta}\Big|_0^{\frac{\pi}{2}} = 4(i-1)$$

$$\int_{C_2} |z|dz = \int_{\pi}^{\frac{3\pi}{2}} 2.\,2ie^{i\theta}d\theta = 4e^{i\theta}\Big|_{\pi}^{\frac{3\pi}{2}} = 4(1-i)$$

$$\int_C |z|dz = 4i - 4 + 4 - 4i = 0$$

Example 5 : if C is the upper half of circle $|z| = 2$, find $\int_C z^2\,dz$

Solution :

$|z| = 2\,, z = 2e^{i\theta} \;\; \rightarrow dz = 2ie^{i\theta}d\theta$

$$\int_C z^2 dz = \int_0^{\pi} z^2\,dz$$

$$\int_C z^2 dz = \int_0^{\pi} (2e^{i\theta})^2 . 2ie^{i\theta} d\theta = 4e^{i\theta}\Big|_0^{\frac{\pi}{2}} = 4(i-1)$$

$$\int_C z^2 dz = 8\int_0^{\pi} ie^{3i\theta} d\theta = \frac{8}{3}e^{3i\theta}\Big|_0^{\pi} = \frac{8}{3}\left(e^{3\pi i} - 1\right) = \frac{-16}{3}$$

Example 6 : if C is the upper half of circle $|z| = 1$, find $\int_C \frac{1}{\sqrt{z}}dz$

Solution :

$|z| = 1\,, z = e^{i\theta} \;\; \rightarrow dz = ie^{i\theta}d\theta$

$$\int_C \frac{1}{\sqrt{z}}dz = \int_0^{\pi} \frac{ie^{i\theta}d\theta}{e^{i\theta/2}} = \int_0^{\pi} ie^{i\frac{\theta}{2}}d\theta$$

$$\int_C \frac{1}{\sqrt{z}}dz = 2e^{i\frac{\theta}{2}}\Big|_0^{\pi} = 2(i-1)$$

Example 7 : if C is the upper half of circle $|z| = 1$, find $\int_C z^n\,dz$

Solution :

$|z| = 1\,, z = e^{i\theta} \;\; \rightarrow dz = ie^{i\theta}d\theta$

$$\int_C z^n dz = \int_0^{2\pi} z^n\,dz$$

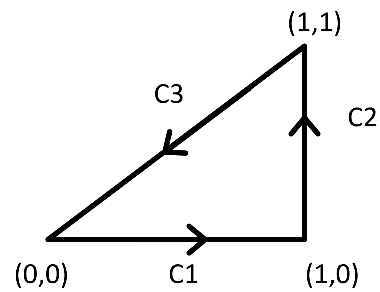
$$\int_C z^n dz = \int_0^{2\pi} \left(e^{i\theta}\right)^n . ie^{i\theta} d\theta$$

$$\int_C z^n dz == \int_0^{2\pi} i\, e^{i(n+1)\theta} d\theta = \left.\frac{e^{i(n+1)\theta}}{n+1}\right|_0^{\pi} = \frac{1}{n+1}(1-1) = 0$$

Notes :

1) If the given region represent straight line (horizontal) then the value of y stay constant (dy=0) and x is changing
2) If the given region represent straight line (vertical) then the value of x stay constant (dx=0) and y is changing
3) If the given region represent straight line (italic) then x and y changing and we will find a relation connecting x with y by slope and point rule .

Example 8 : Find the numerical value to $\int_C xdz$, if $C{:}C_1{+}C_2{+}C_3$ defined in the given figure :



Solution :

C1: (0,0) → (1,0)

y=0 →dy=0 , x:0→1

$$\int_{C1} xdz = \int_{C1} x(dx + idy)$$

$$\int_{C1} xdz = \int_0^1 xdx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

C2: (1,0) → (1,1)

x=1 →dx=0 , y:0→1

$$\int_{C2} xdz = \int_{C2} idy$$

$$\int_{C2} xdz = iy|_0^1 = i$$

C3: (1,1) → (0,0)

$$\frac{y-y1}{x-x1}=m \quad , \qquad such\ that\ \ m=\frac{\Delta y}{\Delta x}$$

$$\frac{y-0}{x-0}=\frac{0-1}{0-1}\rightarrow y=x\ \rightarrow dy=dx$$

$$\int_{C3} xdz=\int_{C3} x(dx+idy)$$

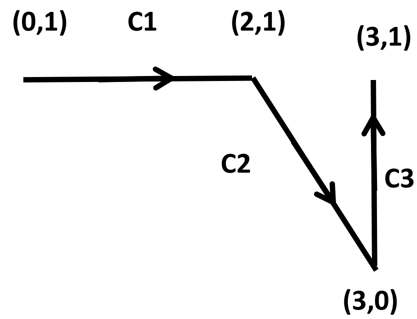
$$\int_{C3} xdz=\int_{1}^{0} x(dx+idx)=(1+i)\int_{1}^{0} x\,dx$$

$$\int_{C3} xdz=(1+i)\left.\frac{x^2}{2}\right|_{1}^{0}=\frac{-1}{2}(1+i)$$

$$\therefore\ \int_{C} xdz=\int_{C1} xdz+\int_{C2} xdz+\int_{C3} xdz$$

$$\therefore\ \int_{C} xdz=\frac{1}{2}+i-\frac{1}{2}(1+i)=\frac{i}{2}$$

Example 9 : Find the numerical value to $\int_C \bar{z}dz$, if $C: C_1+C_2+C_3$ defined in the given figure :



Solution :

$z = x + yi \rightarrow \bar{z} = x - yi \,, dz = dx + idy$

C1: (0,1) → (2,1)

y=1 →dy=0 , x:0→2

$$\int_{C1} \bar{z}dz = \int_{C1} (x - yi)(dx + idy)$$

$$\int_{C1} \bar{z}dz = \int_0^2 (x - i)dx = \frac{(x-i)^2}{2}\bigg|_0^2 = \frac{(2-i)^2}{2} + \frac{1}{2}$$

C2: (2,1) → (3,0)

$$\frac{y-1}{x-2} = \frac{0-1}{3-2} \rightarrow y - 1 = 2 - x \rightarrow y = 3 - x \rightarrow dy = -dx$$

$$\int_{C2} \bar{z}dz = \int_{C2} (x - yi)(dx + idy)$$

$$\int_{C2} \bar{z}dz = \int_{C2} (x - (3-x)i)(dx - idx)$$

$$\int_{C2} \bar{z}dz = (1-i)\int_{2}^{3}(x-3i+xi)dx$$

$$\int_{C2} \bar{z}dz = (1-i)\left\{\frac{x^2}{2} - 3ix + i\frac{x^2}{2}\Big|_2^3\right\} = ?$$

C3: (3,0) → (3,1)

x=3 →dx=0 , y:0→1

$$\int_{C3} \bar{z}dz = \int_{0}^{1}(3-yi)idy$$

$$\int_{C3} \bar{z}dz = \int_{0}^{1}(3i+y)dy$$

$$\int_{C3} \bar{z}dz = 3iy + \frac{y^2}{2}\Big|_0^1 = ?$$

$$\therefore \int_{C} \bar{z}dz = \int_{C1} \bar{z}dz + \int_{C2} \bar{z}dz + \int_{C3} \bar{z}dz$$

1) The Mathematical Induction :

Mathematical induction means use a concluded method to prove some thing and to do this method we follows these three steps :

1) We suppose presume that the relation is true when n=1
2) We presume that the relation is true when n=k
3) We prove that the relation is true when n=k+1

Example 1 : use the mathematical induction to prove the series

$$\mathbf{1+2+3+\cdots+n=\frac{n(n+1)}{2}}$$

Solution :

1) $n=1 \Rightarrow 1=\frac{1(1+1)}{2} \Rightarrow 1=1$ is true .
2) We assume the series is true when n=k

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

3) We prove the series is true when n=k+1

$$\underbrace{1+2+3+\cdots+k}_{step\ 2}+(k+1)\overset{?}{=}\frac{k(k+1)}{2}$$

$$\frac{k(k+1)}{2}+(k+1)=(k+1)\left(\frac{k}{2}+1\right)$$

$$\frac{(k+1)(k+2)}{2}=R.H.S$$

Example 2 : use the mathematical induction to prove the series

$$3+9+27+\cdots+3^n=\frac{3^{n+1}-3^n}{2}$$

Solution :

1) $n=1 \Longrightarrow 3=\frac{9-3}{2} \Longrightarrow 3=3$ is true .

2) We assume the series is true when n=k

$$3+9+27+\cdots+3^k=\frac{3^{k+1}-3}{2}$$

3) We prove the series is true when n=k+1

$$\underbrace{3+9+27+\cdots+3^k}_{step\ 2}+3^{k+1}\stackrel{?}{=}\frac{3^{k+2}-3}{2}$$

$$\frac{3^{k+1}-3}{2}+3^{k+1}=\frac{3^{k+1}-3+2(3^{k+1})}{2}$$

$$\frac{3(3^{k+1})-3}{2}=\frac{3^{k+2}-3}{2}=R.H.S$$

Example 3 : use the mathematical induction to prove the series

$$4+16+64+\cdots+4^n=\frac{4^{n+1}-4}{3}$$

Solution :

1) $n=1 \Longrightarrow 4=\frac{16-4}{3} \Longrightarrow 4=4$ is true .

2) We assume the series is true when n=k

$$4+16+64+\cdots+4^k=\frac{4^{k+1}-4}{3}$$

3) We prove the series is true when n=k+1

$$\underbrace{4+16+64+\cdots+4^k}_{step\ 2}+4^{k+1}\stackrel{?}{=}\frac{4^{k+2}-4}{3}$$

$$\frac{4^{k+1}-4}{3}+4^{k+1}=\frac{4^{k+1}-4+3(4^{k+1})}{3}$$

$$\frac{4(4^{k+1})-4}{3}=\frac{4^{k+2}-4}{3}=R.H.S$$

Example 4 : use the mathematical induction to prove the series

$$\mathbf{(1)^2-(2)^2+(3)^2-(4)^2+\cdots+(-1)^{n+1}(n)^2=\frac{(-1)^{n+1}n(n+1)}{2}}$$

Solution :

1) $n=1 \Longrightarrow (\mathbf{1})^2=\frac{(-1)^2(1)(2)}{2} \Longrightarrow 1=1$ is true .

2) We assume the series is true when n=k

$$\mathbf{(1)^2-(2)^2+(3)^2-(4)^2+\cdots+(-1)^{k+1}(k)^2=\frac{(-1)^{k+1}k(k+1)}{2}}$$

3) We prove the series is true when n=k+1

$$\mathbf{\underbrace{(1)^2-(2)^2+(3)^2-(4)^2+\cdots+(-1)^{k+1}(k)^2}_{step\ 2}+(-1)^{k+2}(k+1)^2\stackrel{?}{=}\frac{(-1)^{k+2}(k+1)(k+2)}{2}}$$

$$\mathbf{\frac{(-1)^{k+1}k(k+1)}{2}+(-1)^{k+2}(k+1)^2=(-1)^{k+1}(k+1)\left[\frac{k}{2}-(k+1)\right]}$$

$$\mathbf{(-1)^{k+1}(k+1)\left[\frac{k-2k-2}{2}\right]=(-1)^{k+1}(k+1)\left[\frac{-k-2}{2}\right]=(-1)^{k+2}\frac{(k+1)(k+2)}{2}=R.H.S}$$

Example 5 : use the mathematical induction to prove

$$5^{n+2} - 5^n \text{ is divided by } 3 \text{ , } \forall n \in \mathbb{N}^+$$

Solution :

We shall prove that $\frac{5^{n+2}-5^n}{3} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{125-5}{3} = 40 \in \mathbb{N}^+ \Longrightarrow$ is true .

2) We assume the expression is true when n=k

$$\frac{5^{k+2} - 5^k}{3} \in \mathbb{N}^+$$

3) We prove the expression is true when n=k+1

$$\frac{5^{k+3} - 5^{k+1}}{3} \in \mathbb{N}^+ \,?$$

$$\frac{5^{k+3}-5^{k+1}}{3} = 5\underbrace{\frac{5^{k+2}-5^k}{3}}_{\in\mathbb{N}^+} \in \mathbb{N}^+$$

$\therefore\ 5^{n+2} - 5^n$ is divided by 3

Example 6 : use the mathematical induction to prove

$$5^n - 2^n \text{ is divided by } 3 \text{ , } \forall n \in \mathbb{N}^+$$

Solution :

We shall prove that $\frac{5^n-2^n}{3} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{5-2}{3} = 1 \in \mathbb{N}^+ \Longrightarrow$ is true .

2) We assume the expression is true when n=k

$$\frac{5^k - 2^k}{3} \in \mathbb{N}^+$$

3) We prove the expression is true when n=k+1

$$\frac{5^{k+1} - 2^{k+1}}{3} \in \mathbb{N}^+ \text{ ?}$$

$$\frac{5^{k+1}-2^{k+1}}{3} = \frac{5.5^k-2^{k+1}}{3} = \frac{(3+2)5^k-2^{k+1}}{3}$$

$$\frac{5^{k+1}-2^{k+1}}{3} = \frac{3.5^k+2.5^k-2^{k+1}}{3}$$

$$\frac{5^{k+1}-2^{k+1}}{3} = \frac{3.5^k}{3} + \frac{2.5^k-2^{k+1}}{3}$$

$$\frac{5^{k+1}-2^{k+1}}{3} = \underbrace{5^k}_{\in\mathbb{N}^+} + 2\underbrace{\frac{(5^k-2^k)}{3}}_{\in\mathbb{N}^+}$$

$$\frac{5^{k+1}-2^{k+1}}{3} \in \mathbb{N}^+$$

$\therefore\ 5^n - 2^n$ is divided by 3

Example 7 : use the mathematical induction to prove

$9^{n+1} - 1$ is divided by 8 , $\forall n \in \mathbb{N}^+$

Solution :

We shall prove that $\frac{9^{n+1}-1}{8} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{81-1}{8} = 10 \in \mathbb{N}^+ \Longrightarrow$ is true .

2) We assume the expression is true when n=k

$$\frac{9^{k+1} - 1}{8} \in \mathbb{N}^+$$

3) We prove the expression is true when n=k+1

$$\frac{9^{k+2} - 1}{8} \in \mathbb{N}^+ ?$$

$$\frac{9^{k+2}-1}{8} = \frac{9.9^{k+1}-1}{8} = \frac{(8+1)9^{k+1}-1}{8}$$

$$\frac{9^{k+2}-1}{8} = \frac{8.9^{k+1}+9^{k+1}-1}{8}$$

$$\frac{9^{k+2}-1}{8} = \frac{8.9^{k+1}}{8} + \frac{9^{k+1}-1}{8}$$

$$\frac{9^{k+2}-1}{8} = \underbrace{9^{k+1}}_{\in\mathbb{N}^+} + \underbrace{\frac{9^{k+1}-1}{8}}_{\in\mathbb{N}^+}$$

$$\frac{9^{k+2}-1}{8} \in \mathbb{N}^+$$

$\therefore$ $9^{n+1} - 1$ is divided by 8

Example 8 : use the mathematical induction to prove

$$8^n - 1 \text{ is divided by } 7 , \forall n \in \mathbb{N}^+$$

Solution :

We shall prove that $\frac{8^n-1}{7} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{8-1}{7} = 1 \in \mathbb{N}^+ \Longrightarrow$ is true .

2) We assume the expression is true when n=k

$$\frac{8^k - 1}{7} \in \mathbb{N}^+$$

3) We prove the expression is true when n=k+1

$$\frac{8^{k+1} - 1}{7} \in \mathbb{N}^+ ?$$

$$\frac{8^{k+1}-1}{7} = \frac{8.8^k-1}{7} = \frac{(7+1)8^k-1}{7}$$

$$\frac{8^{k+1}-1}{7} = \frac{7.8^k+8^k-1}{7}$$

$$\frac{8^{k+1}-1}{7} = \frac{7.8^k}{7} + \frac{8^k-1}{7}$$

$$\frac{8^{k+1}-1}{7} = \underbrace{8^k}_{\in\mathbb{N}^+} + \underbrace{\frac{8^k-1}{7}}_{\in\mathbb{N}^+}$$

$$\frac{8^{k+1}-1}{7} \in \mathbb{N}^+$$

$\therefore$ $8^n - 1$ is divided by 7

Example 9 : *use the mathematical induction to prove*

$$13^n - 6^n \text{ is divided by } 7 \ , \ \forall n \in \mathbb{N}^+$$

Solution :

We shall prove that $\frac{13^n-6^n}{7} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{13-6}{7} = 1 \in \mathbb{N}^+ \Longrightarrow$ *is true* .

2) *We assume the expression is true when n=k*

$$\frac{13^k - 6^k}{7} \in \mathbb{N}^+$$

3) *We prove the expression is true when n=k+1*

$$\frac{13^{k+1} - 6^{k+1}}{7} \in \mathbb{N}^+ \ ?$$

$$\frac{13^{k+1}-6^{k+1}}{7} = \frac{13.13^k-6^{k+1}}{7} = \frac{(7+6)13^k-6^{k+1}}{7}$$

$$\frac{13^{k+1}-6^{k+1}}{7} = \frac{7.13^k+6.13^k-6^{k+1}}{7}$$

$$\frac{13^{k+1}-6^{k+1}}{7} = \frac{7.13^k}{7} + \frac{6.13^k-6^{k+1}}{7}$$

$$\frac{13^{k+1}-6^{k+1}}{7} = \underbrace{13^k}_{\in\mathbb{N}^+} + 6\underbrace{\frac{(13^k-6^k)}{7}}_{\in\mathbb{N}^+}$$

$$\frac{13^{k+1}-6^{k+1}}{7} \in \mathbb{N}^+$$

$\therefore \ 13^n - 6^n$ *is divided by 7*

Example 10 : use the mathematical induction to prove

$7^n - 5^n$ is an even number , $\forall n \in \mathbb{N}^+$

Solution :

We shall prove that $\frac{7^n - 5^n}{2} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{7-5}{2} = 1 \in \mathbb{N}^+ \Longrightarrow$ is true .

2) We assume the expression is true when n=k

$$\frac{7^k - 5^k}{2} \in \mathbb{N}^+$$

3) We prove the expression is true when n=k+1

$$\frac{7^{k+1} - 5^{k+1}}{2} \in \mathbb{N}^+ ?$$

$$\frac{7^{k+1} - 5^{k+1}}{2} = \frac{7.7^k - 5^{k+1}}{2} = \frac{(2+5)7^k - 5^{k+1}}{2}$$

$$\frac{7^{k+1} - 5^{k+1}}{2} = \frac{2.7^k + 5.7^k - 5^{k+1}}{2}$$

$$\frac{7^{k+1} - 5^{k+1}}{2} = \frac{2.7^k}{2} + \frac{5.7^k - 5^{k+1}}{2}$$

$$\frac{7^{k+1} - 5^{k+1}}{2} = \underbrace{7^k}_{\in \mathbb{N}^+} + 5\underbrace{\frac{(7^k - 5^k)}{2}}_{\in \mathbb{N}^+}$$

$$\frac{7^{k+1} - 5^{k+1}}{2} \in \mathbb{N}^+$$

$\therefore\ 7^n - 5^n$ is divided by 7

$\therefore\ 7^n - 5^n$ is an even number

Example 11 : use the mathematical induction to prove

$$(10)^{n+1} - 9n - 10 \text{ is divided by } 9 \text{ , } \forall n \in \mathbb{N}^+$$

Solution :

We shall prove that $\frac{(10)^{n+1}-9n-10}{9} \in \mathbb{N}^+$?

1) $n = 1 \Longrightarrow \frac{100-9-10}{9} = 9 \in \mathbb{N}^+ \Longrightarrow$ is true ·

2) We assume the expression is true when n=k

$$\frac{(10)^{k+1}-9k-10}{9} \in \mathbb{N}^+$$

3) We prove the expression is true when n=k+1

$$\frac{(10)^{k+2}-9(k+1)-10}{9} \in \mathbb{N}^+ \text{ ?}$$

$$\frac{(10)^{k+2}-9(k+1)-10}{9} = \frac{(10)^{k+2}-9k-9-10}{9} = \frac{(9+1)10^{k+1}-5^{k+1}}{9}$$

$$\frac{(10)^{k+2}-9(k+1)-10}{9} = \frac{10(10)^{k+1}-9k-9-10}{9} = \frac{(9+1)10^{k+1}-9k-9-10}{9}$$

$$\frac{(10)^{k+2}-9(k+1)-10}{9} = \frac{9.10^{k+1}+10^{k+1}-9k-9-10}{9}$$

$$\frac{(10)^{k+2}-9(k+1)-10}{9} = \frac{9.10^{k+1}}{9} + \frac{10^{k+1}-9k-9-10}{9}$$

$$\frac{(10)^{k+2}-9(k+1)-10}{9} = 10^{k+1} + \frac{10^{k+1}-9k-10}{9} - \frac{9}{9} = \underbrace{10^{k+1}}_{\in\mathbb{N}^+} + \underbrace{\frac{10^{k+1}-9k-10}{9}}_{\in\mathbb{N}^+} - \underbrace{1}_{\in\mathbb{N}^+}$$

$$\frac{(10)^{k+2}-9(k+1)-10}{9} \in \mathbb{N}^+$$

$$\therefore (10)^{n+1} - 9n - 10 \text{ is divided by } 9$$

Example 12 : use the mathematical induction to prove

$$2^n < n! \quad , \; \forall n \in \mathbb{N}^+ , n \geq 4$$

Solution :

1) $n = 4 \Longrightarrow 2^4 < 4! \Longrightarrow 16 < 24$ is true ·

2) We assume the expression is true when n=k

$2^k < k! \quad , \; k > 4$

3) We prove the expression is true when n=k+1

$2^{k+1} < (k+1)!$?

From step (2) we have

$2^k < k!$, multiply two sides by (k+1) we conclude

$(k+1)2^k < (k+1)k!$

$(k+1)2^k < (k+1)! \; \ldots\ldots\ldots\ldots (1)$

$\because 2 < k+1$, multiply two sides by 2^k we conclude

$2^k 2 < 2^k(k+1)$

$2^{k+1} < 2^k(k+1) \; \ldots\ldots\ldots\ldots\ldots (2)$

From equation 1 and 2 conclude :

$2^{k+1} < (k+1)2^k < (k+1)!$, $\quad A < B < C \rightarrow A < C$

$\therefore \; 2^{k+1} < (k+1)!$

CHECK POINT

Example H.W :

1) use the mathematical induction to prove :

A) $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1$

B) $2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{n(6n^2+3n-1)}{2}$

C) $1^4 + 2^4 + 3^4 + \cdots n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

2) use the mathematical induction to prove :

A) $4^n - 1$ divided by 3 , $\forall n \in \mathbb{N}^+$

B) $5^n - 4^n - 1$ divided by 4 , $\forall n \in \mathbb{N}^+$

C) $7^n - 2^n$ divided by 5 , $\forall n \in \mathbb{N}^+$

D) $9^n + 7$ divided by 8 , $\forall n \in \mathbb{N}^+$

E) $4^n + 6n - 1$ divided by 9 , $\forall n \in \mathbb{N}^+$

F) $3^{4n+2} + 5^{2n+1}$ divided by 14 , $\forall n \in \mathbb{N}^+$

G) $2^{5n+1} + 5^{n+2}$ divided by 27 , $\forall n \in \mathbb{N}^+$

H) $5^{2n} - 1$ divided by 24 , $\forall n \in \mathbb{N}^+$

I) $49^n + 16n - 1$ divided by 64 , $\forall n \in \mathbb{N}^+$

J) $11^{n+2} + 12^{2n+1}$ divided by 133 , $\forall n \in \mathbb{N}^+$

K) $28^n - 2^n$ divided by 26 , $\forall n \in \mathbb{N}^+$

3) use the mathematical induction to prove :

$2^n < n! < 3^n$, $\forall n \in \mathbb{N}^+, n \geq 4$

1- Combinations:

We can define the combination at the numbers of method to choosing r things from n things , such that n is greater than or equal zero , and combination can be written as :

$$C_r^n = \frac{n!}{r!\,(n-r)!} \quad ,n \geq r \quad ,n \in N^+, r \in N$$

$$C_1^3 = \frac{3!}{1!\,2!} = \frac{6}{2} = 3$$

$$C_3^5 = \frac{5!}{3!\,2!} = \frac{20}{2} = 10$$

$$C_0^6 = \frac{6!}{0!\,6!} = \frac{6!}{6!} = 1$$

$$C_2^7 = \frac{7!}{2!\,5!} = \frac{42}{2} = 21$$

$$C_6^4 = Not\ available$$

Notes:

- $C_0^n = C_n^n = 1$
- $C_1^n = C_{n-1}^n = n$
- $n! = n(n-1)(n-2)\dots 3.2.1$
- $0! = 1$
- $C_r^n = C_{n-r}^n$

2-The Binomial Theorem :

You must have multiplid a binomial by itself , or by another binomial , let us use this knowledge to do some expansions ,

consider the binomial $(x+y)$:

$$(x+y)^1 = x+y$$

$$(x+y)^2 = (x+y)(x+y) = x^2+2xy+y^2$$

$$(x+y)^3 = (x+y)(x+y)^2 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = (x+y)(x+y)^3 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = (x+y)(x+y)^4 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

$$\vdots$$

$$(x+y)^n = x^n+nx^{n-1}y+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}y^3+\cdots+y^n$$

The last expression (n is positive integer) can be written as :

$$\boxed{(x+y)^n = C_0^n x^n + C_1^n x^{n-1}y + C_2^n x^{n-2}y^2 + C_3^n x^{n-3}y^3 + \cdots + C_n^n y^n}$$

$$\because\ C_0^n = C_n^n = 1$$

$$\boxed{(x+y)^n = x^n + C_1^n x^{n-1}y + C_2^n x^{n-2}y^2 + C_3^n x^{n-3}y^3 + \cdots + y^n}$$

Notes

- in each equation above , the right hand side is called the binomial expansion of the left hand side ·

- The numbers of terms in the expansion is one more than the exponent of the binomial(n+1) , for example , in expansion of $(x+y)^4$ the numbers of terms is 5=(4+1) .
- The exponent of x in the first term is the same of the exponent (n) of the binomial , and the exponent decreases by 1 in each successive term of the expansion .
- The exponent of y in the first term is zero (as $y^0=1$) , and the exponent of y in the 2^{nd} term is 1 , and it increases by 1 in each successive term till it becomes the exponent of the binomial in the last term of the expansion.
- The sum of the exponent of x and y in each term is equal to the exponent of the binomial.
- The coffecient of the 1^{st} term is 1 (as C_0^n) always , and the 2^{nd} coffecient is n (as C_1^n) and so on , the term before last have the coffecient n (as C_{n-1}^n) , and the coffecient of the last term is 1 (as C_n^n) .
- The value of the middle terms (combinations) are equal and so on till the 1^{st} and the last terms .
- If the signal between the terms in expansion is negative then the odd order terms positive always , and the even order terms is negative , i.e whenthe terms are alternating .

Example 1 : write the binomial expansion of $(x+3y)^5$.

Solution :

$$(x+3y)^5 = \underbrace{C_0^5}_{=1} x^5 + \underbrace{C_1^5}_{=5} x^4(3y) + \underbrace{C_2^5}_{=10} x^3(3y)^2 + \underbrace{C_3^5}_{=10} x^2(3y)^3 + \underbrace{C_4^5}_{=5} x(3y)^4 + \underbrace{C_5^5}_{=1} (3y)^5$$

$$(x+3y)^5 = x^5 + 5x^4(3y) + 10x^3(3y)^2 + 10x^2(3y)^3 + 5x(3y)^4 + (3y)^5$$

$$(x+3y)^5 = x^5 + 15x^4y + 90x^3y^2 + 270x^2y^3 + 405xy^4 + 243y^5$$

Example 2 : write the binomial expansion of $(2a-3b)^4$.

Solution :

$$(2a-3b)^4 = \underbrace{C_0^4}_{=1} (2a)^4 - \underbrace{C_1^4}_{=4} (2a)^3(3y)^1 + \underbrace{C_2^4}_{=6} (2a)^2(3y)^2 - \underbrace{C_3^4}_{=4} (2a)^1(3y)^3 \ + \underbrace{C_4^4}_{=1} (3y)^4$$

$$(2a-3b)^4 = (2a)^4 - 4(2a)^3(3y)^1 + 6(2a)^2(3y)^2 - 4(2a)^1(3y)^3 \ + (3y)^4$$

$$(2a-3b)^4 = 16a^4 - 4(8)a^3(3y)^1 + 6(4)a^2(9)y^2 - 4(2a)(27)y^3 \ + (81)y^4$$

$$(2a-3b)^4 = 16a^4 - 96a^3y + 216a^2b^2 - 216ay^3 \ + 81\, y^4$$

Example 3 : write the expansion of $(\frac{y}{x}+\frac{1}{y})^4$ $, x, y \neq 0$.

Solution :

$$(\frac{y}{x}+\frac{1}{y})^4 = \underbrace{C_0^4}_{=1} (\frac{y}{x})^4 + \underbrace{C_1^4}_{=4} \left(\frac{y}{x}\right)^3 \left(\frac{1}{y}\right)^1 + \underbrace{C_2^4}_{=6} \left(\frac{y}{x}\right)^2 \left(\frac{1}{y}\right)^2 + \underbrace{C_3^4}_{=4} \left(\frac{y}{x}\right)^1 \left(\frac{1}{y}\right)^3 \ + \underbrace{C_4^4}_{=1} \left(\frac{1}{y}\right)^4$$

$$(\frac{y}{x}+\frac{1}{y})^4 = \frac{y^4}{x^4} + 4\frac{y^3}{x^3}\frac{1}{y} + 6\frac{y^2}{x^2}\frac{1}{y^2} + 4\frac{y}{x}\frac{1}{y^3} \ + \frac{1}{y^4}$$

$$(\frac{y}{x}+\frac{1}{y})^4=\frac{y^4}{x^4}+\frac{4y^2}{x^3}+\frac{6}{x^2}+\frac{4}{xy^2}+\frac{1}{y^4}$$

Example 4 : By using the binomial theorem , evaluate $(101)^3$.

Solution :

$$(101)^3=(100+1)^3=\underbrace{C_0^3}_{=1}(100)^3+\underbrace{C_1^3}_{=3}(100)^2(1)^1+\underbrace{C_2^3}_{=3}(100)^1(1)^2+\underbrace{C_3^3}_{=1}(1)^3$$

$$(100+1)^3=(100)^3+3(100)^2(1)^1+3(100)^1(1)^2+(1)^3$$

$$(100+1)^3=1000000+30000+300+1=1030301$$

Example 5 : By using the binomial theorem , evaluate $(0.99)^3$.

Solution :

$$(0.99)^3=(1-0.01)^3=\underbrace{C_0^3}_{=1}(1)^3-\underbrace{C_1^3}_{=3}(1)^2(0.01)^1+\underbrace{C_2^3}_{=3}(1)^1(0.01)^2-\underbrace{C_3^3}_{=1}(0.01)^3$$

$$(0.99)^3=1-3(0.01)+3(0.0001)-(0.000001)$$

$$(0.99)^3=1-0.03+0.0003-0.000001=0.970299$$

CHECK POINT

Example H.W :

1) write the expansion of : $(3a+2b)^5$, $(4z-w)^6$, $(\frac{y}{x}-\frac{x}{y})^4$

2) By using the binomial theorem , evaluate : $(102)^4$, $(97)^5$, $(1.01)^3$

3- Pascal's Triangle :

We start to generate pascal's triangle by writing down the number 1 , then we write a new row with the number 1 twice .

We then generate new rows to build a triangle of numbers .

Each new row must begin and end at 1 , the remaining numbers in each row above which lie above left and above right .

So adding the two 1's in the 2^{nd} row gives 2, and this number goes in the vacant space in the 3^{rd} row , and so on pascal's tringle is :

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Example 6 : use pascal triangle to find $(\mathbf{x}+\mathbf{y})^4$

Solution :

$\because n = 4$, the row is the 5th row

The pascal numbers are : 1 , 4 , 6 , 4 , 1

$\therefore (x+y)^4 = 1(x^4) + 4(x^3)(y) + 6(x^2)(y^2) + 4(x)(y^3) + 1(y^4)$

$\therefore (x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Example 7 : use pascal triangle to find $(\mathbf{x}-\mathbf{2y})^3$

Solution :

$\because n = 3$, the row is the 4th row

The pascal numbers are : 1 , 3 , 3 , 1

$\therefore (x-2y)^3 = 1(x^3) - 3(x^2)(2y) + 3(x)(2y)^2 - 1(2y)^3$

$\therefore (x-2y)^3 = x^3 - 6x^2y + 12xy^2 - 8y^3$

CHECK POINT

Example H.W :

1) By using Pascal's Triangle find the expansion of :
   - $(3x-y)^4$
   - $(2x+3)^5$
   - $(6)^7$

4- General Terms and the Middle Terms :

If we want to find one term of the expansion , this term called as general term and denoted by T_r , and calculate from the following relation :

$$T_r = C^n_{r-1}\,(X)^{n-r+1}(Y)^{r-1}$$

Where X is the first term in binomial (not expansion) , Y is the second term in binomial (not expansion) , r is the order of the required term , n is the power of the binomial .

Example 8 : Find the 3rd Term of the expansion $(2x + 3y)^6$

Solution :

$n = 6, \quad r = 3, X = 2x, Y = 3y$

$T_3 = C^6_2\,(2x)^{6-3+1}(3y)^{3-1}$

$T_3 = 15\,(2x)^4(3y)^2$

$T_3 = 15\,(16)(9)(x)^4(y)^2 = 2160x^4y^2$

Example 9 : Find the 5th Term of the expansion $(1 - \frac{2}{3}x^3)^6$

Solution :

$n = 6, r = 5, X = 1, Y = -\frac{2}{3}x^3$

$T_5 = C_4^6\ (1)^{6-5+1}(-\frac{2}{3}x^3)^{5-1}$

$T_5 = 15\left(\frac{-2}{3}\right)^4 (x^3)^4$

$T_5 = 15\left(\frac{16}{81}\right)x^{12} = \frac{80}{27}x^{12}$

Notes :

- To find the middle term in any expansion we depended on the power
- If the exponent of the expansion is even (the number of terms is odd) , then he middle term is the term of order $\frac{n}{2}+1$.
- If the exponent of the expansion is odd (the number of terms is even) , then he middle terms is the terms of order $\frac{n+1}{2}$, $\frac{n+3}{2}$.
- Calculate the middle term by formula of general terms .

Example 10 : Find the Middle Term of the expansion $(\mathbf{x^2-y^2})^8$

Solution :

Since n=8 (even) , then the number of terms is 9

The middle term is : $r = \frac{8}{2}+1 = 5$

$n = 8\,, r = 5\ \ , X = x^2\ \ , Y = y^2$

$T_5 = C_4^8\ (x^2)^{8-5+1}(y^2)^{5-1}$

$T_5 = 70\ (x^2)^4(y^2)^4$

$T_5 = 70\ x^8y^8$

Example 11 : **Find the Middle Terms of the expansion of** $(2\mathbf{x}^2+\frac{1}{x})^9$

Solution :

Since n=9 (odd) , then the number of terms is 10

The middle terms is : $r=\frac{9+1}{2},\frac{9+3}{2}=5,6$

$n=9\,,r=5,6\ ,X=2x^2\ ,Y=\dfrac{1}{x}$

$T_5=C_4^9\,(2x^2)^{9-5+1}(\dfrac{1}{x})^{5-1}$

$T_5=126\,(2x^2)^5(x^{-1})^4$

$T_5=126\,(32)x^{10}x^{-4}=4032x^6$

$T_6=C_5^9\,(2x^2)^{9-6+1}(\dfrac{1}{x})^{6-1}$

$T_6=126\,(2x^2)^4(x^{-1})^5$

$T_6=126\,(16)x^8x^{-5}=2016x^3$

Then the middle terms are $4032x^6\,,2016x^3$.

Note

Sometimes required a coffecient of determinate term without its order then we apply the formula of general term without subistituting the value of r , and later we equalize the powers of required term with the power of resulting term.

Example 12 : Find the Term that includes x^4 of the expansion $(x+3)^6$

Solution :

Let the order of the required term is r ,

Then the term is Tr

$T_r = C_{r-1}^{n}\,(X)^{n-r+1}(Y)^{r-1}$

$n = 6\,, r = ?\;\;, X = x\;\;, Y = 3$

$T_r = C_{r-1}^{6}\,(x)^{6-r+1}(3)^{r-1}$

$T_r = C_{r-1}^{6}\,(x)^{7-r}(3)^{r-1}$

Required term is x^4 , result term is x^{7-r}

$x^4 = x^{7-r} \Longrightarrow 4 = 7 - r \Longrightarrow r = 3$

Then the required term is the 3^{rd}

$T_3 = C_2^6\,(x)^4(3)^2 = 15x^4(9) = 135x^4$

Example 13 : Find the Term that includes a^8 of the expansion $(3+a^2)^8$

Solution :

Let the order of the required term is r ,

Then the term is Tr

$T_r = C_{r-1}^{n}\,(X)^{n-r+1}(Y)^{r-1}$

$n = 8\,, r = ?\;\;, X = 3\;\;, Y = a^2$

$T_r = C_{r-1}^{8}\ (3)^{8-r+1}(a^2)^{r-1}$

$T_r = C_{r-1}^{8}\ (3)^{9-r}a^{2r-2}$

Required term is a^8 , result term is a^{2r-2}

$a^8 = a^{2r-2} \Rightarrow 8 = 2r - 2 \Rightarrow r = 5$

Then the required term is the 5^{th}

$T_5 = C_4^8\ (3)^4 a^8 = 70(81)a^8 = 5670a^8$

Example 14 : Find the Term that free from x of the expansion $(x^2 - \frac{1}{x})^{15}$

Solution :

Let the order of the required term is r ,

Then the term is Tr

$T_r = C_{r-1}^{n}\ (X)^{n-r+1}(Y)^{r-1}$

$n = 15\ , r =?\ \ , X = x^2\ \ , Y = \frac{-1}{x}$

$T_r = C_{r-1}^{15}\ (x^2)^{15-r+1}(\frac{-1}{x})^{r-1}$

$T_r = C_{r-1}^{15}\ (x^2)^{16-r}(-x^{-1})^{r-1}$

$T_r = C_{r-1}^{15}\ \ x^{32-2r}\ (-1)^{r-1}\ \ x^{-r+1}$

$T_r = C_{r-1}^{15}\ \ x^{33-3r}\ (-1)^{r-1}$

Required term is x^0 , result term is x^{33-3r}

$x^0 = x^{33-3r} \Rightarrow 0 = 33 - 3r \Rightarrow r = 11$

Then the required term is the 11^{th}

$$T_{11} = C_{10}^{15}\ x^0\,(-1)^{10} = 3003$$

CHECK POINT

Example H.W :

1) Find the specified terms in each the following expansions :
   - ❖ $\left(x^3 - \frac{1}{x^2}\right)^7$, 5th term
   - ❖ $\left(x + \frac{1}{x}\right)^6$, 4th term
   - ❖ $(5x - 2y)^5$, 6th term
2) Find the Middle term(s) in each the following expansions :
   - ❖ $\left(x + \frac{1}{x^2}\right)^8$
   - ❖ $(3x^3 - 2y^2)^7$
3) Find the Term that includes x^2 of the expansion $(x^3 + \frac{2}{x^2})^9$
4) Find the Term that free from x of the expansion $(x^2 + \frac{2}{x^3})^{10}$

5- Conjugate Terms Binomials

If we want to find the sum of difference of two binomial , the two binomial have the same terms but are different at the middle signal as $(x+y)^n$, $(x-y)^n$, then we can every binomial seperately

$$(x+y)^n = T_1 + T_2 + T_3 + T_4 + \cdots \mp T_{n+1} \quad \ldots\ldots\ldots\ldots (1)$$

$$(x-y)^n = T_1 - T_2 + T_3 - T_4 + \cdots \mp T_{n+1} \quad \ldots\ldots\ldots\ldots (2)$$

Case 1:

If the require binomial is the sum of two conjugate binomials then from the equations (1) & (2) we have :

$$(x+y)^n + (x-y)^n = 2T_1 + 2T_3 + 2T_5 + \cdots + 2T_{(n+1)or(n)}$$

i.e the result of the sum of the binomials is twice the sum of odd order terms in binomial .

Case 2:

If the require binomial is the difference of two conjugate binomials then from the equations (1) & (2) we have :

$$(x+y)^n - (x-y)^n = 2T_2 + 2T_4 + 2T_6 + \cdots + 2T_{(n+1)or(n)}$$

i.e the result of the difference of the binomials is twice the sum of even order terms in binomial .

Example 15 : simplify the expression : $(2+a)^4+(2-a)^4$

Solution :

$(2+a)^4 = T_1 + T_2 + T_3 + T_4 + T_5$

$(2-a)^4 = T_1 - T_2 + T_3 - T_4 + T_5$

$(2+a)^4 + (2-a)^4 = 2T_1 + 2T_3 + 2T_5 = 2(T_1 + T_3 + T_5)$

Now we must find the terms T_1 , T_3 , T_5

$T_r = C^n_{r-1}\,(X)^{n-r+1}(Y)^{r-1}$

$n = 4 , r = 1,3,5 \;, X = 2 \;, Y = a$

$T_1 = C^4_0\,(2)^{4-1+1}(a)^{1-1}$

$T_1 = (1)(\,16)(1) = 16$

$T_3 = C^4_2\,(2)^{4-3+1}(a)^{3-1}$

$T_3 = (6)(\,4)(a^2) = 24a^2$

$T_5 = C^4_4\,(2)^{4-5+1}(a)^{5-1}$

$T_3 = (1)(\,1)(a^4) = a^4$

$\therefore\; (2+a)^4 + (2-a)^4 = 2(16 + 24a^2 + a^4)$

Example 16 : Find the value of : $(102)^4 - (98)^4$

Solution :

$(102)^4 = (100+2)^4 = T_1 + T_2 + T_3 + T_4 + T_5$

$(98)^4 = (100-2)^4 = T_1 - T_2 + T_3 - T_4 + T_5$

$(102)^4 - (98)^4 = 2T_2 + 2T_4 = 2(T_2 + T_4)$

Now we must find the terms T_2 , T_4

$T_r = C^n_{r-1}\,(X)^{n-r+1}(Y)^{r-1}$

$n = 4\,, r = 2{,}4\ ,\qquad X = 100\ , Y = 2$

$T_2 = C^4_1\,(100)^{4-2+1}(2)^{2-1}$

$T_2 = (4)(\,1000000)(2) = 8000000$

$T_4 = C^4_3\,(100)^{4-4+1}(2)^{4-1}$

$T_4 = (4)(\,100)(8) = 3200$

$$\therefore\ (102)^4 - (98)^4 = 2(8000000 + 3200) = 16006400$$

Example H.W :

CHECK POINT

1) Find the value of the following expansions :

- ✓ $\left(2+\sqrt{3}\right)^7 - \left(2-\sqrt{3}\right)^7$
- ✓ $6^9 - 4^9$
- ✓ $(101)^5 + (99)^5$

6- Binomial Theorem for Rational Exponent

So far you have applied the binomial theorem only when the binomial has been raised to a power which is a natural number . what happend if the exponent is negative integer or if it is a fraction ?

We will stat result that allows us to still have a binomial expansion , but it will had infinite terms in this case .

The result is a generalised version of the earlier binomial theorem which you have studied .

If n is a rational number and $\left|\frac{y}{x}\right| < 1$ *, then*

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots$$

Example 17 : *Write the expansion :* $(1+x)^{-1}$ *, when* $|x| < 1$

Solution :

$$(1+x)^{-1} = (1)^{-1} + (-1)(1)^{-2}(x) + \frac{(-1)(-2)}{2!}(1)^{-3}(x)^2 + \frac{(-1)(-2)(-3)}{3!}(1)^{-4}(x)^3 + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

Example 18 : ***Write the expansion*** **:** $(\mathbf{x}+\mathbf{y})^{-2}$ ***, when*** $|x|>|y|$

Solution :

$\because\ |x|>|y| \rightarrow \left|\frac{y}{x}\right|<1$

$$(\mathrm{x}+\mathrm{y})^{-2}=(x)^{-2}+(-2)(x)^{-3}(y)+\frac{(-2)(-3)}{2!}(x)^{-4}(y)^{2}+\frac{(-2)(-3)(-4)}{3!}(x)^{-5}(y)^{3}+\cdots$$

$$(\mathrm{x}+\mathrm{y})^{-2}=x^{-2}-2x^{-3}y+3x^{-4}y^{2}-4x^{-5}y^{3}+\cdots$$

$$(\mathrm{x}+\mathrm{y})^{-2}=\frac{1}{x^{2}}-2\frac{y}{x^{3}}+3\frac{y^{2}}{x^{4}}-4\frac{y^{3}}{x^{5}}+\cdots$$

Example 19 : ***Write the expansion*** **:** $(\mathbf{3}+\mathbf{5p})^{\frac{2}{5}}$ ***, when*** $|p|<\frac{3}{5}$

Solution :

$\because\ |p|<\frac{3}{5} \xrightarrow{\times\frac{5}{3}} \left|\frac{5p}{3}\right|<1$

$$(3+5\mathrm{p})^{\frac{2}{5}}=(3)^{\frac{2}{5}}+(\frac{2}{5})(3)^{\frac{-3}{5}}(5p)+\frac{\left(\frac{2}{5}\right)\left(\frac{-3}{5}\right)}{2!}(3)^{\frac{-7}{5}}(5p)^{2}+\frac{\left(\frac{2}{5}\right)\left(\frac{-3}{5}\right)\left(\frac{-7}{5}\right)}{3!}(3)^{\frac{-12}{5}}(5p)^{3}+\cdots$$

$$(3+5\mathrm{p})^{\frac{2}{5}}=(3)^{\frac{2}{5}}+(3)^{\frac{-3}{5}}(2p)-(3)^{\frac{-2}{5}}p^{2}+(3)^{\frac{-12}{5}}7p^{3}+\cdots$$

Example 20 : Write the expansion : $\frac{1}{3}$

Solution :

$$\because \ \frac{1}{3} = (3)^{-1} = (2+1)^{-1}$$

$$(2+1)^{-1} = (2)^{-1} + \frac{(-1)}{1!}(2)^{-2}(1) + \frac{(-1)(-2)}{2!}(2)^{-3}(1)^2 + \frac{(-1)(-2)(-3)}{3!}(2)^{-4}(1)^3 + \cdots$$

$$(2+1)^{-1} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$$

Example H.W :

CHECK POINT

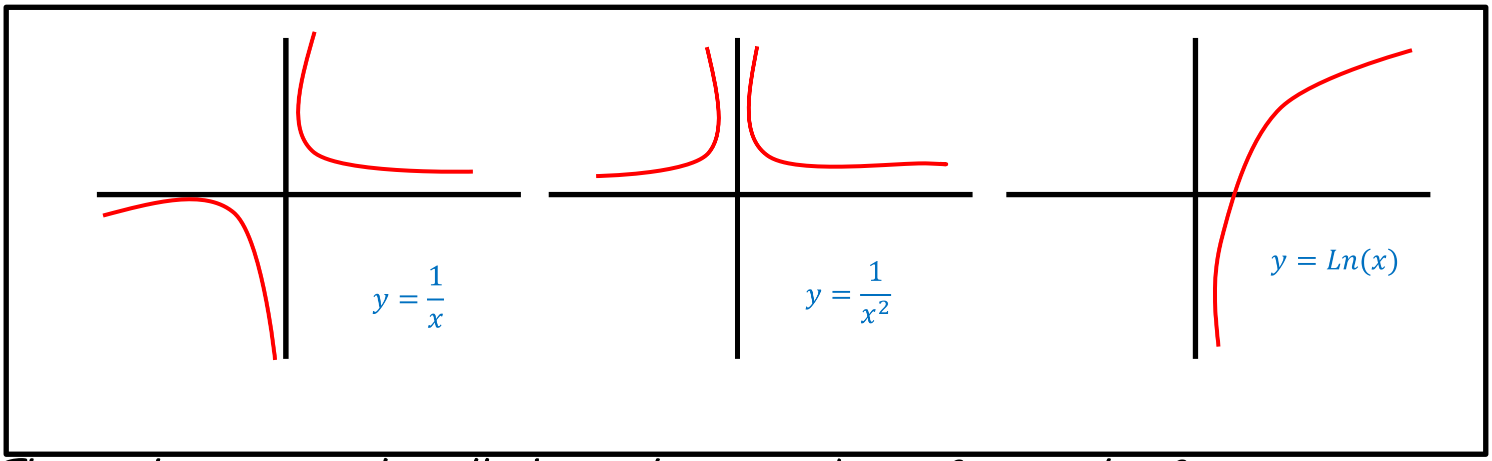
1) Write all the following as expansions :

- 0.2
- $\sqrt[3]{26}$
- $\sqrt[5]{35}$
- $(0.99)^{-4}$
- $\sqrt[3]{(4-2y)^2} \quad , y < 2$
- $(\frac{1}{2x-1})^4 \quad , x > \frac{1}{2}$

Principles:

Vertical Asymptotes

We've had several occasions to encounter acertain specific type of singularity called a vertical asymptote , the following three functions are common examples of this type of behavior , all have a vertical asymptote



These three examples all show the same basic fact : the function is discontinous at x=0 because it blows up nearby but they do have different behavior since they blow up in different directions .

Singular Point

Is the point that makes the value of the function is undefined so as $\sqrt{x}$ when x is negative number , and logarithim function when x=0 and fractional function when denominator equal zero at any point .

One sided infinite limit

We know introduce the following notations to distinguish situations like this :

$\lim_{x \to c^+} f(x) = \infty$: will mean that f(x) becomes arbitrary large for values $x > c$ sufficiently close to c .

$\lim_{x \to c^+} f(x) = -\infty$: will mean that f(x) becomes arbitrary negative for values $x > c$ sufficiently close to c .

$\lim_{x \to c^-} f(x) = \infty$: will mean that f(x) becomes arbitrary large for values $x < c$ sufficiently close to c .

$\lim_{x \to c^-} f(x) = -\infty$: will mean that f(x) becomes arbitrary negative for values $x < c$ sufficiently close to c .

Arithemetic of infinite limits

The following are some informal rules for computing limits involving at inifinity , after i state them i will give a more formal version of each statement .

$\frac{k}{0^+} = \infty , \frac{k}{0^-} = -\infty , \frac{k}{\infty} = 0 , \frac{k}{-\infty} = 0 , \infty + \infty , \infty . \infty$, since k is +ive constant.

Example 1 : Find the following limits :

1) $\lim_{x\to 0^+} \frac{1}{x} = \lim_{x\to 0^+} \frac{1}{0^+} = \infty$

2) $\lim_{x\to 0^-} \frac{1}{x} = \lim_{x\to 0^+} \frac{1}{0^-} = -\infty$

3) $\lim_{x\to 0} \frac{1}{x^2} = \lim_{x\to 0} \frac{1}{0} = \infty$ (for both sides)

4) $\lim_{x\to 0^+} Ln(x) = -\infty$

5) $\lim_{x\to 0^-} Ln(x)$ has no meaning since Ln(x) is undefined for $x < 0$.

Note :

Students are often tempted to do much more arithemtic with ∞ than is possible for example all seven of the following expressions cannot have any meaning : $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty, \infty^0, 1^\infty, 0^0$ these are called Indeterminate Forms (I.F)

Example 2 : Find the following limits from two sides :

1) $\lim_{x\to -1^+} \frac{x^2-3x+2}{x+1} = \lim_{x\to -1^+} \frac{6}{0^+} = \infty$

$$\lim_{x\to -1^-} \frac{x^2-3x+2}{x+1} = \lim_{x\to -1^-} \frac{6}{0^-} = -\infty$$

We note that the limit for f(x) at x=-1 doesn't exists because the limit from the right don't equal the limit from the left .

2) $\lim_{x\to -2^+} \frac{e^x}{(x+2)^2} = \lim_{x\to -2^+} \frac{e^{-2}}{0^+} = \infty$

$$\lim_{x\to -2^-} \frac{e^x}{(x+2)^2} = \lim_{x\to -2^-} \frac{e^{-2}}{0^+} = \infty$$

we note that the limit for f(x) at x=-2 equal to ∞

3) $\lim_{x\to 6^+} \frac{x^2-13x+42}{x^2-12x+36}$

$$= \lim_{x\to 6^+} \frac{(x-6)(x-7)}{(x-6)^2} = \lim_{x\to 6^+} \frac{(x-7)}{(x-6)} = \frac{-1}{0^+} = -\infty$$

$\lim_{x\to 6^-} \frac{x^2-13x+42}{x^2-12x+36}$

$$= \lim_{x\to 6^-} \frac{(x-6)(x-7)}{(x-6)^2} = \lim_{x\to 6^+} \frac{(x-7)}{(x-6)} = \frac{-1}{0^-} = \infty$$

we note that the limit for f(x) at x=6 doesn't exists .

Note :

$e^{i\theta} = cos\theta + isin\theta$

$e^{\theta} = cosh\theta + sinh\theta$

$cosh2n\pi = 1 \quad , n \in Z$

4) $\lim_{x \to 2n\pi^+} \frac{e^x}{1-cosx}$

$$= \lim_{x \to 2n\pi^+} \frac{e^{2n\pi}}{0^+} = \frac{1}{0^+} = \infty$$

$\lim_{x \to 2n\pi^-} \frac{e^x}{1-cosx}$

$$= \lim_{x \to 2n\pi^-} \frac{e^{2n\pi}}{0^+} = \frac{1}{0^+} = \infty$$

we note that the limit for cosx at $x = 2n\pi$ is exists because cosx is an even function .

5) $\lim_{x \to 0^+} e^{\frac{1}{x}}$

$$= \lim_{x \to 0^+} e^{\infty} = \infty$$

$\lim_{x \to 0^-} e^{\frac{1}{x}}$

$$= \lim_{x \to 0^-} e^{-\infty} = 0$$

we note that the limit for f(x) at x=0 doesn't exists .

CHECK POINT

Example H·W :

Find the following limit from two sides if available :

1) $\lim\limits_{x\to1} \dfrac{x^3-7x+7}{x^2-3x+2}$

2) $\lim\limits_{x\to2} \dfrac{x^3-7x+7}{x^2-3x+2}$

3) $\lim\limits_{x\to2} \dfrac{cosx}{Ln(x^2+1)}$

Limit at infinity :

- let f(x) be a function in its domain , then if f(x) near to ∞ when x near to ∞ , then we write $\lim_{x\to\infty} f(x)=\infty$
- let f(x) be a function in its domain , then if f(x) near to ∞ when x near to −∞ , then we write $\lim_{x\to-\infty} f(x)=\infty$
- let f(x) be a function in its domain , then if f(x) near to −∞ when x near to ∞ , then we write $\lim_{x\to\infty} f(x)=-\infty$
- let f(x) be a function in its domain , then if f(x) near to −∞ when x near to −∞ , then we write $\lim_{x\to-\infty} f(x)=-\infty$

Classical Limits

$$\lim_{x\to\infty} x^n = \infty$$

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty & n \text{ is even} \\ -\infty & n \text{ is odd} \end{cases}$$

$$\lim_{x\to\infty} \sqrt{x} = \infty$$

$$\lim_{x\to\mp\infty} \frac{1}{x^n} = 0$$

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

$$\lim_{x\to0^+} \frac{1}{\sqrt{x}} = \infty$$

$$\lim_{x\to0^+} \frac{1}{x^n} = \infty$$

$$\lim_{x\to0^-} \frac{1}{x^n} = \begin{cases} \infty & n \text{ even} \\ -\infty & n \text{ odd} \end{cases}$$

$$\lim_{x\to0^+} \sqrt{x} = 0$$

$$\lim_{x\to0^-} \sqrt{x} = not\ \ exists$$

$$\lim_{x\to-\infty} \sqrt{x} = not\ \ exists$$

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x}} = not\ \ exists$$

$$\lim_{x\to0^-} \frac{1}{\sqrt{x}} = not\ \ exists$$

Notes :

- ❖ $\frac{1}{x}$ is small and positive when x is large and positive .
- ❖ $\frac{1}{x}$ is large and positive when x is small and positive .
- ❖ $\frac{1}{x}$ is large and negative when x is small and negative .
- ❖ $\frac{1}{x}$ is small and negative when x is large and negative .

We summarize these facts by saying :

- ❖ As x tends to ∞ then $\frac{1}{x}$ approaches 0 .
- ❖ As x approaches 0 from the right , then $\frac{1}{x}$ tends to ∞ .
- ❖ As x approaches 0 from the left , then $\frac{1}{x}$ tends to $-\infty$.
- ❖ As x tends to $-\infty$ then $\frac{1}{x}$ approaches 0 .

Theorem :

if f(x) and g(x) are polynomials $A, B, \cdots$, $a, b, \cdots$ are constant such that :

$$f(x) = Ax^n \mp Bx^{n-1} \mp Cx^{n-2} \mp \cdots$$

$$g(x) = ax^m \mp bx^{m-1} \mp cx^{m-2} \mp \cdots$$

Then :

$$\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \begin{cases} \frac{A}{a} & n = m \\ \infty & n > m \\ 0 & n < m \end{cases}$$

Example 3 : Find the following limits:

1) $\lim\limits_{x\to\infty} \frac{4x^2+3x-5}{3x^2-5x+1}$

$$\lim_{x\to\infty} \frac{4x^2+3x-5}{3x^2-5x+1} = \qquad \lim_{x\to\infty} \frac{4\frac{x^2}{x^2}+3\frac{x}{x^2}-\frac{5}{x^2}}{3\frac{x^2}{x^2}-5\frac{x}{x^2}+\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{4+\frac{3}{x}-\frac{5}{x^2}}{3-\frac{5}{x}+\frac{1}{x^2}} = \frac{4+0-0}{3-0+0} = \frac{4}{3}$$

2) $\lim\limits_{x\to-\infty} \frac{5x^3-4x^2+3x-1}{4x^2+3x-2}$

$$\lim_{x\to-\infty} \frac{5x^3-4x^2+3x-1}{4x^2+3x-2} = \lim_{x\to-\infty} \frac{5\frac{x^3}{x^3}-4\frac{x}{x^3}^{2}+3\frac{x}{x^3}-\frac{1}{x^3}}{4\frac{x^2}{x^3}+3\frac{x}{x^3}-\frac{2}{x^3}}$$

$$\lim_{x\to-\infty} \frac{5-\frac{4}{x}+\frac{3}{x^2}-\frac{1}{x^3}}{\frac{4}{x}+\frac{3}{x^2}-\frac{2}{x^3}} = \frac{5-0+0-0}{0+0-0} = \frac{5}{0} = \infty$$

3) $\lim_{x\to\infty} \frac{2x^2-4x+7}{4x^3-3x^2+5x+2}$

$$\lim_{x\to\infty} \frac{2x^2-4x+7}{4x^3-3x^2+5x+2} = \lim_{x\to\infty} \frac{2\frac{x^2}{x^3} - \frac{4x}{x^3} + 7\frac{1}{x^3}}{4\frac{x^3}{x^3} - 3\frac{x^2}{x^3} + 5\frac{x}{x^3} + \frac{2}{x^3}}$$

$$\lim_{x\to\infty} \frac{\frac{2}{x} - \frac{4}{x^2} + \frac{7}{x^3}}{4 - \frac{3}{x} + \frac{5}{x^2} + \frac{2}{x^3}} = \frac{0-0+0}{4-0+0+0} = \frac{0}{4} = 0$$

Theorem :

For the function $f(x) = a^x$, then :

$$\lim_{x\to\infty} a^x = \infty \quad , \quad if \;\; a > 1$$
$$\lim_{x\to-\infty} a^x = 0 \quad , \quad if \;\; a > 1$$
$$\lim_{x\to\infty} a^x = 0 \quad , \quad if \;\; 0 < a < 1$$
$$\lim_{x\to-\infty} a^x = \infty \quad , \quad if \;\; 0 < a < 1$$

Example 4 : Find the following limits :

1) $\lim_{x\to\infty} \left(\frac{3}{2}\right)^x = \infty$

2) $\lim_{x\to\infty} \left(\frac{4}{5}\right)^x = 0$

CHECK POINT

Example H·W :

Find the following limits :

1) $\lim\limits_{x\to\infty} \dfrac{2^{x+3}}{3^{x-1}}$

2) $\lim\limits_{x\to-\infty} \dfrac{2^{x+3}}{3^{x-1}}$

3) $\lim\limits_{x\to\infty} \dfrac{3^{x}-3^{-x}}{3^{x}+3^{-x}}$

4) $\lim\limits_{x\to-\infty} \dfrac{3^{x}-3^{-x}}{3^{x}+3^{-x}}$

High Infinity Limit

Example 5 : Find the following limits :

1) $\lim\limits_{x\to\infty}\left(7x^5+3+\dfrac{1}{\sqrt{x}}\right)=\infty+3+0=\infty$

2) $\lim\limits_{x\to-\infty}(1-x)^5(5-2x^2)=(1+\infty)(5-\infty)=\infty.(-\infty)=-\infty$

Example 6 : if the function $f(x) = \frac{ax^2}{bx^2-3}$, and $\lim_{x\to\infty} f(x) = -2$

$\lim_{x\to 1} f(x) = 1$, Find the value of $a, b \in R$

Solution :

$$\lim_{x\to\infty} \frac{ax^2}{bx^2-3} = -2$$

$$\frac{a}{b} = 2 \Rrightarrow a = -2b$$

$$\lim_{x\to 1} \frac{ax^2}{bx^2-3} = \frac{a}{b-3} \Rrightarrow 1$$

$$a = b - 3 \quad , \quad -2b = b - 3$$

$$\therefore b = 1 \, , a = -2$$

Notes :

- When $x \to \infty$ then $|x| = x$ and $|x| = \sqrt{x^2}$
- When $x \to -\infty$ then $|x| = -x$ and $|x| = \sqrt{x^2}$
- In many function as the radical function or fractional function wew try to avoid the result that is indetriminate form (I F) by multiplying by the conjugate or by the common factor .

Example 7 : Find the limit of

$$\lim_{x\to\infty}\sqrt{3x^2-5x+4}$$

Solution :

$$\lim_{x\to\infty}\sqrt{3x^2-5x+4}=\lim_{x\to\infty}\sqrt{x^2\left(3-\frac{5}{x}+\frac{4}{x^2}\right)}$$

$$\lim_{x\to\infty}\sqrt{3x^2-5x+4}=\lim_{x\to\infty}x\sqrt{\left(3-\frac{5}{x}+\frac{4}{x^2}\right)}=\infty(3-0+0)=\infty$$

Example 8 : Find the limit of

$$\lim_{x\to\infty}\sqrt{x^2+x}-x$$

Solution :

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{\left[\sqrt{x^2+x}-x\right]\left[\sqrt{x^2+x}+x\right]}{\left[\sqrt{x^2+x}+x\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{x^2+x-x^2}{\left[\sqrt{x^2+x}+x\right]}=\lim_{x\to\infty}\frac{x}{\left[\sqrt{x^2+x}+x\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{x}{\left[\sqrt{x^2\left(1+\frac{1}{x}\right)}+x\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{x}{\left[x\sqrt{\left(1+\frac{1}{x}\right)}+x\right]}=\lim_{x\to\infty}\frac{x}{x\left[\sqrt{\left(1+\frac{1}{x}\right)}+1\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+x}-x=\lim_{x\to\infty}\frac{1}{\left[\sqrt{\left(1+\frac{1}{x}\right)}+1\right]}=\frac{1}{\sqrt{1+\frac{1}{\infty}}+1}=\frac{1}{2}$$

Example 9 : Find the limit of

$$\lim_{x\to\infty}\sqrt{x^2+x}+x$$

Solution :

$$\lim_{x\to\infty}\sqrt{x^2+x}+x=\sqrt{\infty^2.\infty}+\infty=\infty$$

Example 10 : Find the limit of

$$\lim_{x\to\infty}\sqrt{x^2+1}-x$$

Solution :

$$\lim_{x\to\infty}\sqrt{x^2+1}-x=\lim_{x\to\infty}\frac{\left[\sqrt{x^2+1}-x\right]\left[\sqrt{x^2+1}+x\right]}{\left[\sqrt{x^2+1}+x\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+1}-x=\lim_{x\to\infty}\frac{x^2+1-x^2}{\left[\sqrt{x^2+1}+x\right]}=\lim_{x\to\infty}\frac{1}{\left[\sqrt{x^2+1}+x\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+1}-x=\lim_{x\to\infty}\frac{1}{\left[\sqrt{x^2(1+\frac{1}{x^2})}+x\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+1}-x=\lim_{x\to\infty}\frac{1}{\left[x\sqrt{\left(1+\frac{1}{x^2}\right)}+x\right]}=\lim_{x\to\infty}\frac{1}{x\left[\sqrt{\left(1+\frac{1}{x^2}\right)}+1\right]}$$

$$\lim_{x\to\infty}\sqrt{x^2+x}-x==\frac{1}{\infty\sqrt{1+\frac{1}{\infty}}+1}=\frac{1}{\infty.2}=0$$

Example 11 : Find the limit of

$$\lim_{x\to\infty}2x+1-\sqrt{x^2+x-2}$$

Solution :

$$\lim_{x\to\infty}2x+1-\sqrt{x^2+x-2}$$

$$=\lim_{x\to\infty}\frac{\left[2x+1-\sqrt{x^2+x-2}\right]\left[2x+1+\sqrt{x^2+x-2}\right]}{\left[2x+1+\sqrt{x^2+x-2}\right]}$$

$$=\lim_{x\to\infty}\frac{[4x^2+4x+1-x^2-x+2]}{\left[2x+1+\sqrt{x^2+x-2}\right]}$$

$$=\lim_{x\to\infty}\frac{[3x^2+3x+3]}{\left[2x+1+\sqrt{x^2+x-2}\right]}$$

$$=\lim_{x\to\infty}\frac{x^2\left[3+\frac{3}{x}+\frac{3}{x^2}\right]}{\left[x(2+\frac{1}{x})+\sqrt{x^2(1+\frac{1}{x}-\frac{2}{x^2})}\right]}$$

$$=\lim_{x\to\infty}\frac{x^2\left[3+\frac{3}{x}+\frac{3}{x^2}\right]}{\left[x(2+\frac{1}{x})+\sqrt{x^2(1+\frac{1}{x}-\frac{2}{x^2})}\right]}$$

$$= \lim_{x\to\infty} \frac{x^2\left[3+\frac{3}{x}+\frac{3}{x^2}\right]}{\left[x(2+\frac{1}{x})+x\sqrt{(1+\frac{1}{x}-\frac{2}{x^2})}\right]}$$

$$= \lim_{x\to\infty} \frac{x^2\left[3+\frac{3}{x}+\frac{3}{x^2}\right]}{x\left[(2+\frac{1}{x})+\sqrt{(1+\frac{1}{x}-\frac{2}{x^2})}\right]}$$

$$= \lim_{x\to\infty} \frac{x\left[3+\frac{3}{x}+\frac{3}{x^2}\right]}{\left[(2+\frac{1}{x})+\sqrt{(1+\frac{1}{x}-\frac{2}{x^2})}\right]} = \frac{\infty.3}{3} = \infty$$

Another method

$$\lim_{x\to\infty} 2x+1-\sqrt{x^2+x-2} = \lim_{x\to\infty} 2x+1-\sqrt{x^2(1+\frac{1}{x}-\frac{2}{x^2})}$$

$$= \lim_{x\to\infty} 2x+1-x\sqrt{(1+\frac{1}{x}-\frac{2}{x^2})}$$

$$= \lim_{x\to\infty} x\left[2+\frac{1}{x}-\sqrt{(1+\frac{1}{x}-\frac{2}{x^2})}\right] = \infty(2-1) = \infty$$

Example 12 : Find the limit of

$$\lim_{x\to-\infty} x+2+\sqrt{x^2+x}$$

Solution :

$$\lim_{x\to-\infty} x+2+\sqrt{x^2+x}$$

$$=\lim_{x\to-\infty}\frac{\left[x+2+\sqrt{x^2+x}\right]\left[x+2-\sqrt{x^2+x}\right]}{\left[x+2-\sqrt{x^2+x}\right]}$$

$$=\lim_{x\to-\infty}\frac{\left[x^2+4x+4-x^2-x\right]}{\left[x+2-\sqrt{x^2+x}\right]}$$

$$=\lim_{x\to-\infty}\frac{[3x+4]}{\left[x+2-\sqrt{x^2+x}\right]}$$

$$=\lim_{x\to-\infty}\frac{x\left[3+\frac{4}{x}\right]}{\left[x\left(1+\frac{2}{x}\right)-\sqrt{x^2(1+\frac{1}{x})}\right]}$$

$$=\lim_{x\to-\infty}\frac{x\left[3+\frac{4}{x}\right]}{\left[x\left(1+\frac{2}{x}\right)+x\sqrt{\left(1+\frac{1}{x}\right)}\right]}$$

$$=\lim_{x\to-\infty}\frac{x\left[3+\frac{4}{x}\right]}{x\left[\left(1+\frac{2}{x}\right)+\sqrt{(1+\frac{1}{x})}\right]}$$

$$=\lim_{x\to-\infty}\frac{\left[3+\frac{4}{x}\right]}{\left[\left(1+\frac{2}{x}\right)+\sqrt{(1+\frac{1}{x})}\right]}=\frac{3}{1+1}=\frac{3}{2}$$

Example 13 : Find the limit of

$$\lim_{x\to-\infty} \sqrt{4x^2+x}+2x$$

Solution :

$$\lim_{x\to-\infty} \sqrt{4x^2+x}+2x$$

$$= \lim_{x\to-\infty} \frac{\left[\sqrt{4x^2+x}+2x\right]\left[\sqrt{4x^2+x}-2x\right]}{\left[\sqrt{4x^2+x}-2x\right]}$$

$$= \lim_{x\to-\infty} \frac{[4x^2+x-4x^2]}{\left[\sqrt{x^2(4+\frac{1}{x})}-2x\right]} = \lim_{x\to-\infty} \frac{[x]}{\left[-x\sqrt{(4+\frac{1}{x})}-2x\right]}$$

$$= \lim_{x\to-\infty} \frac{[3x+4]}{\left[x+2-\sqrt{x^2+x}\right]}$$

$$= \lim_{x\to-\infty} \frac{x}{-x\left[\sqrt{(4+\frac{1}{x})}+2\right]} = \lim_{x\to-\infty} \frac{1}{-\left[\sqrt{\left(4+\frac{1}{x}\right)}+2\right]} = -\frac{1}{2+2} = \frac{-1}{4}$$

Example 14 : Find the limit of

$$\lim_{x\to\infty} \sqrt{\frac{2x+3}{x-1}}$$

Solution :

$$\lim_{x\to\infty}\sqrt{\frac{2x+3}{x-1}}=\lim_{x\to\infty}\sqrt{\frac{x(2+\frac{3}{x})}{x(1-\frac{1}{x})}}=\lim_{x\to\infty}\sqrt{\frac{(2+\frac{3}{x})}{(1-\frac{1}{x})}}=\sqrt{\frac{2+0}{1-0}}=\sqrt{2}$$

Example 15 : *Find the limit of*

$$\lim_{x\to-\infty}\frac{\sqrt{5+x^2}}{x}$$

Solution :

$$\lim_{x\to-\infty}\frac{\sqrt{5+x^2}}{x}=\lim_{x\to-\infty}\frac{\sqrt{x^2(\frac{5}{x^2}+1)}}{x}=\lim_{x\to-\infty}\frac{-x\sqrt{(\frac{5}{x^2}+1)}}{x}$$

$$\lim_{x\to-\infty}\frac{\sqrt{5+x^2}}{x}=\lim_{x\to-\infty}-\sqrt{(\frac{5}{x^2}+1)}=-1$$

Example 15 : *Find the limit of*

$$\lim_{x\to\infty}\frac{|3-2x|+5}{4x-7}$$

Solution :

$$|3-2x|=\begin{cases}3-2x & x\geq\frac{3}{2}\\ \\ 2x-3 & x<\frac{3}{2}\end{cases}$$

$$\lim_{x\to\infty}\frac{|3-2x|+5}{4x-7}=\lim_{x\to\infty}\frac{3-2x+5}{4x-7}=\lim_{x\to\infty}\frac{8-2x}{4x-7}$$

$$\lim_{x\to\infty}\frac{|3-2x|+5}{4x-7}=\lim_{x\to\infty}\frac{x(\frac{8}{x}-2)}{x(4-\frac{7}{x})}=\lim_{x\to\infty}\frac{(\frac{8}{x}-2)}{(4-\frac{7}{x})}=-\frac{2}{4}=\frac{-1}{2}$$

Example 16 : *Find the limit of*

$$\lim_{x\to\infty}\sqrt{x^2+4x-1}-x$$

Solution :

$$\lim_{x\to\infty}\sqrt{x^2+4x-1}-x$$

$$=\lim_{x\to\infty}\frac{\left[\sqrt{x^2+4x-1}-x\right]\left[\sqrt{x^2+4x-1}+x\right]}{\left[\sqrt{x^2+4x-1}+x\right]}$$

$$=\lim_{x\to\infty}\frac{[x^2+4x-1-x^2]}{\left[\sqrt{x^2+4x-1}+x\right]}=\lim_{x\to\infty}\frac{[4x-1]}{\left[\sqrt{x^2+4x-1}+x\right]}$$

$$=\lim_{x\to\infty}\frac{x\left[4-\frac{1}{x}\right]}{\left[\sqrt{x^2(1+\frac{4}{x}-\frac{1}{x^2}}+x\right]}=\lim_{x\to\infty}\frac{x\left[4-\frac{1}{x}\right]}{\left[x\sqrt{(1+\frac{4}{x}-\frac{1}{x^2}}+x\right]}$$

$$=\lim_{x\to\infty}\frac{x\left[4-\frac{1}{x}\right]}{x\left[\sqrt{(1+\frac{4}{x}-\frac{1}{x^2}}+1\right]}=\lim_{x\to\infty}\frac{4-\frac{1}{x}}{\sqrt{(1+\frac{4}{x}-\frac{1}{x^2}}+1}=\frac{4-0}{1+1}=2$$

Example 17 : Find the limit of

$$\lim_{x\to\infty}\frac{\sqrt[3]{8x^3+5x-2}}{3x+2}$$

Solution :

$$\lim_{x\to\infty}\frac{\sqrt[3]{8x^3+5x-2}}{3x+2}=\lim_{x\to\infty}\frac{\sqrt[3]{x^3(8+\frac{5}{x^2}-\frac{2}{x^3})}}{3x+2}$$

$$=\lim_{x\to\infty}\frac{x\sqrt[3]{(8+\frac{5}{x^2}-\frac{2}{x^3})}}{x(3+\frac{2}{x})}=\lim_{x\to\infty}\frac{\sqrt[3]{(8+\frac{5}{x^2}-\frac{2}{x^3})}}{3+\frac{2}{x}}=\frac{\sqrt[3]{8}}{3}=\frac{2}{3}$$

Example 18 : Find the limit of

$$\lim_{x\to\infty}\frac{\sqrt[4]{x^6+2x^3-3}}{\sqrt{9x^3+x^2-1}}$$

Solution :

$$\lim_{x\to\infty}\frac{\sqrt[4]{x^6+2x^3-3}}{\sqrt{9x^3+x^2-1}}=\lim_{x\to\infty}\frac{\sqrt[4]{x^6(1+\frac{2}{x^3}-\frac{3}{x^6})}}{\sqrt{x^3(9+\frac{1}{x}-\frac{1}{x^3})}}$$

$$=\lim_{x\to\infty}\frac{x^{\frac{3}{2}}\sqrt[4]{1+\frac{2}{x^3}-\frac{3}{x^6}}}{x^{\frac{3}{2}}\sqrt{(9+\frac{1}{x}-\frac{1}{x^3})}}=\lim_{x\to\infty}\frac{\sqrt[4]{1+\frac{2}{x^3}-\frac{3}{x^6}}}{\sqrt{(9+\frac{1}{x}-\frac{1}{x^3})}}=\frac{\sqrt[4]{1}}{\sqrt{9}}=\frac{1}{3}$$

CHECK POINT

Example H·W :

Find the following limits :

1) $\lim\limits_{x\to\infty} \dfrac{x^3}{x^2+1}$

2) $\lim\limits_{x\to\infty} \dfrac{2x^{-2}-3x^{-3}}{x^{-4}+4x^{-2}}$

3) $\lim\limits_{x\to\infty} x\left[\left(2+\dfrac{1}{x}\right)^5-32\right]$

4) $\lim\limits_{x\to\infty} \dfrac{1+2+3+\cdots+x}{x^2}$

Indeterminate form :

Is the expression has no meaning and it is :

$$\frac{\mathbf{0}}{\mathbf{0}}, \frac{\infty}{\infty}, \infty - \infty, \mathbf{0}.\infty, \infty^{\mathbf{0}}, \mathbf{1}^{\infty}, \mathbf{0}^{\mathbf{0}}$$

Lohiptal's Rule

suppose that f and g are differentiable functions on an open interval containing x=a except possibly at x=a and that :

$\lim_{x\to a}\frac{f(x)}{g(x)}$ is one of the seven indeterminate forms then :

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

This statement is true in the case of limit as $x \to a^+, a^-, \infty, -\infty$

Note :

- In Lohiptal's Rule the numerator and denominator are differentiated separately which is not the same as differentiating of f(x)/g(x) .
- $\frac{0}{\infty}, \frac{\infty}{0}, 0^{\infty}, \infty.\infty, \infty + \infty, -\infty - \infty$ are not indeterminate forms .

To applying lohiptal rule you must apply the following three steps :

1) Check the limit of $\frac{f(x)}{g(x)}$ is indeterminate form , if it is not then lohiptal rule can not be used .

2) Differentiate f and g separately .

3) Find the limit of $\frac{f'(x)}{g'(x)}$, if the limit is finite ∞ or $-\infty$ then its equal the limit , if its not finite then apply lohiptal rule again .

1) Indeterminiate Form $\frac{0}{0}$

Example 1 : Find the following limits:

1) $\lim_{x\to 2} \frac{x^2-4}{x-2} \quad = \langle \frac{0}{0} \rangle$

$$\lim_{x\to 2} \frac{2x}{1} = 2(2) = 4$$

2) $\lim_{x\to 0} \frac{sin2x}{x} \quad = \langle \frac{0}{0} \rangle$

$$\lim_{x\to 0} \frac{2cos2x}{1} = 2(1) = 2$$

3) $\lim_{x\to \frac{\pi}{2}} \frac{1-sinx}{cosx} \quad = \langle \frac{0}{0} \rangle$

$$\lim_{x\to \frac{\pi}{2}} \frac{-cosx}{-sinx} = \lim_{x\to \frac{\pi}{2}} cotx = 0$$

4) $\lim\limits_{x\to 0} \frac{e^x-1}{x^3} \qquad = \langle\frac{0}{0}\rangle$

$$\lim_{x\to 0} \frac{e^x}{3x^2} = \infty$$

5) $\lim\limits_{x\to 0^-} \frac{tanx}{x^2} \qquad = \langle\frac{0}{0}\rangle$

$$\lim_{x\to 0^-} \frac{sec^2x}{2x} = -\infty$$

6) $\lim\limits_{x\to 0} \frac{1-cosx}{x^2} \qquad = \langle\frac{0}{0}\rangle$

$$\lim_{x\to 0} \frac{sinx}{2x} = \lim_{x\to 0} \frac{cosx}{2} = \frac{1}{2}$$

7) $\lim\limits_{x\to \infty} \frac{x^{\frac{-4}{3}}}{sin(\frac{1}{x})} \qquad = \langle\frac{0}{0}\rangle$

$$\lim_{x\to\infty} \frac{x^{-\frac{4}{3}}}{sin(\frac{1}{x})} = \lim_{x\to\infty} \frac{-\frac{4}{3}x^{-\frac{7}{3}}}{\frac{-1}{x^2}cos(\frac{1}{x})} = \lim_{x\to\infty} \frac{4}{3}\frac{x^{-\frac{1}{3}}}{cos(\frac{1}{x})} = 0$$

2) Indeterminiate Form $\frac{\infty}{\infty}$

Example 2 : Find the following limits:

1) $\lim\limits_{x\to\infty} \frac{x}{e^x} \qquad = \langle\frac{\infty}{\infty}\rangle$

$$\lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

2) $\lim\limits_{x\to 0^+} \frac{Lnx}{cscx} \qquad = \langle\frac{\infty}{\infty}\rangle$

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-cscx\,cotx} = \lim_{x\to 0^+} -\frac{sinx}{x}tanx = \lim_{x\to 0^+} -\frac{sinx}{x}.\lim_{x\to 0^+} tanx$$

$$\lim_{x\to 0^+} -\frac{cosx}{1}.\lim_{x\to 0^+} tanx = (-1)(0) = 0$$

CHECK POINT

Example H.W :

Find the following limit:

1) $\lim\limits_{x\to 1} \frac{x^3-7x+6}{x^2-3x+2}$

2) $\lim\limits_{x\to 0} \frac{sinx}{tan4x}$

3) $\lim\limits_{x\to 2} \frac{\sqrt{2-x}-x}{x-1}$

4) $\lim\limits_{x\to\infty} x^2e^{-x}$

5) $\lim\limits_{x\to 2^+} \frac{Ln(x-2)}{Ln(x^2-4)}$

3) Indeterminiate Form $(0.\infty)$

Example 3 : Find the following limits:

1) $\lim\limits_{x\to 0^+} xLnx \qquad = \langle 0.\infty\rangle$

$$\lim_{x\to 0^+}\frac{Lnx}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

2) $\lim\limits_{x\to\infty} x^2 sin\frac{\pi}{x} \qquad = \langle 0.\infty\rangle$

$$\lim_{x\to\infty}\frac{sin\frac{\pi}{x}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-\frac{\pi}{x^2}cos\frac{\pi}{x}}{\frac{-2}{x^3}} = \lim_{x\to\infty}\frac{\pi}{x^2}\frac{x^3}{2}cos\frac{\pi}{x}$$

$$\lim_{x\to\infty}\frac{\pi}{2}x\,cos\frac{\pi}{x} = (\infty)(1) = \infty$$

3) $\lim\limits_{x\to\infty} x^2 e^{-x} \qquad = \langle 0.\infty\rangle$

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

4) Indeterminiate Form $(\infty - \infty)$

Example 4 : Find the following limits:

1) $\lim\limits_{x\to 0} \frac{1}{x} - \frac{1}{e^x - 1} \qquad = \langle \infty - \infty \rangle$

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x\to 0} \frac{e^x - 1}{x(e^x) + (e^x - 1)} = \lim_{x\to 0} \frac{e^x}{xe^x + e^x + e^x}$$

$$\lim_{x\to 0} \frac{e^x}{e^x(x+2)} = \lim_{x\to 0} \frac{1}{(x+2)} = \frac{1}{2}$$

2) $\lim\limits_{x\to \infty} \sqrt{x^2 + 1} - 2x \qquad = \langle \infty - \infty \rangle$

$$\lim_{x\to \infty} \frac{(\sqrt{x^2 + 1} - 2x)(\sqrt{x^2 + 1} + 2x)}{\sqrt{x^2 + 1} + 2x}$$

$$\lim_{x\to \infty} \frac{(x^2 + 1 - 4x^2)}{\sqrt{x^2 + 1} + 2x} = \lim_{x\to \infty} \frac{(1 - 3x^2)}{\sqrt{x^2 + 1} + 2x}$$

$$\lim_{x\to \infty} \frac{-6x}{\frac{x}{\sqrt{x^2 + 1}} + 2} = \lim_{x\to \infty} \frac{-6x}{\frac{x}{x\sqrt{1 + \frac{1}{x^2}}} + 2} = \lim_{x\to \infty} \frac{-6x}{\frac{1}{\sqrt{1 + \frac{1}{x^2}}} + 2} = -\infty$$

CHECK POINT

Example H·W :

Find the following limit:

1) $\lim_{x\to\infty} xsin\frac{\pi}{x}$

2) $\lim_{x\to\pi^-} (x-\pi)tan\frac{x}{2}$

3) $\lim_{x\to 0^+} Lnx\,.\,tanx$

4) $\lim_{x\to\pi} (x-\pi)cotx$

5) $\lim_{x\to 0} (cscx-\frac{1}{x})$

6) $\lim_{x\to\infty} \left(x-Ln(x^2+1)\right)$

7) $\lim_{x\to\infty} (Ln(x)-x)$

8) $\lim_{x\to\infty} \frac{Lnx}{\sqrt[3]{x}}$

9) $\lim_{x\to\infty} (\sqrt{x^2-x}-x)$

5) Indeterminiate Form $(1^{\infty}\ ,\ \infty^{0}\ ,\ 0^{0})$

Example 5 : Find the following limits:

1) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} \qquad = \langle 1^{\infty}\rangle$

Let $y=(1+x)^{\frac{1}{x}}$ by taking Ln for the right and the left side

$$Lny=\frac{1}{x}Ln(1+x)$$

$$\lim_{x\to 0}Lny=\lim_{x\to 0}\frac{Ln(1+x)}{x}$$

$$Lny=\lim_{x\to 0}\frac{\frac{1}{(1+x)}}{1}\Longrightarrow Lny=\lim_{x\to 0}\frac{1}{(1+x)}$$

$$Lny=1\Longrightarrow y=e$$

$$\lim_{x\to 0}\frac{Ln(1+x)}{x}=e$$

2) $\lim_{x\to 0^+} (Lnx)^x \qquad = \langle \infty^0 \rangle$

Let $y = (Lnx)^x$ *by taking Ln for the right and the left side*

$$Lny = xLnLn(x)$$

$$\lim_{x\to 0^+} Lny = \lim_{x\to 0^+} xLnLn(x)$$

$$Lny = \lim_{x\to 0^+} \frac{LnLnx}{\frac{1}{x}} \Longrightarrow Lny = \lim_{x\to 0^+} \frac{\frac{1}{xLnx}}{\frac{-1}{x^2}} \Longrightarrow Lny = \lim_{x\to 0^+} \frac{-x}{Lnx}$$

$$Lny = \frac{0}{\infty} \Longrightarrow Lny = 0.\frac{1}{\infty} = 0.0 = 0 \Longrightarrow Lny = 0 \Longrightarrow y = 1$$

$$\lim_{x\to 0^+} (Lnx)^x = 1$$

3) $\lim_{x\to 1} (x-1)^{sin\pi x} \qquad = \langle 0^0 \rangle$

Let $y = (x-1)^{sin\pi x}$ *by taking Ln for the right and the left side*

$$Lny = sin\pi x\, Ln(x-1) \Longrightarrow Lny = \frac{Ln(x-1)}{csc\pi x}$$

$$\lim_{x\to 1} Lny = \lim_{x\to 1} \frac{Ln(x-1)}{csc\pi x}$$

$$Lny = \lim_{x\to 1} \frac{\frac{1}{x-1}}{-\pi csc\pi x cot\pi x} \Longrightarrow Lny = \lim_{x\to 1} \frac{sin\pi x\, tan\pi x}{-\pi(x-1)}$$

$$Lny = \lim_{x\to 1} \frac{\pi(sin\pi x\, sec^2\pi x + tan\pi x\, cos\pi x)}{-\pi(1)}$$

$$Lny = \lim_{x \to 1} \frac{0}{-1}$$

$$Lny = 0 \Rightarrow y = 1$$

$$\lim_{x \to 1} (x-1)^{sin\pi x} = 1$$

Example H.W :

Find the following limit:

1) $\lim_{x \to \infty} \left(1 - \frac{3}{x}\right)^{x}$

2) $\lim_{x \to \infty} (Lnx)^{\frac{1}{x}}$

3) $\lim_{x \to 1} (x^2 - 1)^{cos\frac{\pi}{2}x}$

The Sequence :

Is a function whose domain is a set of integers specifically , we will regard , the expression $\langle a_n \rangle$ to be an alternative notation for the function $f(n) = a_n \quad , n = 1, 2, 3, \dots$

A Sequence is an infinite list of numbers written in a definite order :

2,4,8,16,32,$\cdots$,

The numbers in the previous list are called the terms of the sequence , in the sequence above the 1st term is '2' , the 2nd term is '4' , the 3rd term is '8' and so , with each successive term being twice the previous term ·

So the general term can be writting as : 2^n , n=1,2,$\cdots$

Writting Sequence :

Case 1: (from the general term)

Example 1 : Consider the sequence whose general term is :

A) $f(n) = \frac{1}{n}$

B) $\langle an \rangle = \langle n^2 + n - 1 \rangle$

Solution :

A) $\langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}, \dots \rangle$

B) $\langle a_n \rangle = \langle 1, 5, 11, 19, \dots, n^2 + n - 1, \dots \rangle$

Case 2: (formulas for sequence)

The trick to finding the formula for a sequence is to recognize the pattern and figure out how to describe it in terms of n .

Example 2 : Find the formulas for the following sequences :

A) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

B) $4, 5, 6, 7, 8, \dots$

C) $1, 4, 9, 16, 25, \dots$

Solution :

A)

n	1	2	3	4	5	…
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	…

The general term of this sequence is $\langle \frac{1}{n} \rangle$

B)

n	1	2	3	4	5	⋯
a_n	4	5	6	7	8	⋯

The general term of this sequence is $\langle n+3 \rangle$

C)

n	1	2	3	4	5	⋯
a_n	1	4	9	16	25	⋯

The general term of this sequence is $\langle n^2 \rangle$.

There are certain sequences that you should know on sight :

$\langle 2^n \rangle$	$2,4,8,16,\cdots$
$\langle 3^n \rangle$	$3,9,27,81,\cdots$
$\langle n^2 \rangle$	$1,4,9,16,25,\cdots$
$\langle n! \rangle$	$1,2,6,24,120,\cdots$

Example 3 : Find the formulas for the following sequences :

A) $1, \frac{9}{2}, \frac{27}{6}, \frac{81}{24}, \frac{243}{120} \ldots$

B) $\sqrt{3}, 4, 3\sqrt{5}, 4\sqrt{6}, 5\sqrt{7}, \ldots$

C) $16, 25, 36, 49, 64, \ldots$

Solution :

A)

n	1	2	3	4	5	…
a_n	$3 = \frac{3^0}{1!}$	$\frac{9}{2} = \frac{3^2}{2!}$	$\frac{27}{6} = \frac{3^3}{3!}$	$\frac{81}{24} = \frac{3^4}{4!}$	$\frac{243}{120} = \frac{3^5}{5!}$	…

The general term of this sequence is $\langle \frac{3^n}{n!} \rangle$.

B)

n	1	2	3	4	5	…
a_n	$\sqrt{3} = 1\sqrt{3}$	$4 = 2\sqrt{4}$	$3\sqrt{5}$	$4\sqrt{6}$	$5\sqrt{7}$	…

The general term of this sequence is $\langle n\sqrt{n+2} \rangle$

C)

n	a_n
1	$16 = 4^2 = (1+3)^2$
2	$25 = 5^2 = (2+3)^2$
3	$36 = 6^2 = (3+3)^2$
4	$49 = 7^2 = (4+3)^2$
5	$64 = 8^2 = (5+3)^2$
⋮	⋮

The general term of this sequence is $\langle (n+3)^2 \rangle$.

Example H.W :

CHECK POINT

Find the formula of the general term an of the sequence:

- $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$
- $\frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}, ..$
- $1, \frac{2}{4}, \frac{6}{9}, \frac{24}{16}, \frac{120}{25}, \ldots$
- $3, 2.5, 2, 1.5, \ldots$
- $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$

Special Sequences

Two types of sequences that we will encounter repeatedly are arithmetic sequence and geometric sequence .

1) Arthemitic Sequence

is a sequence for which each term is a constant plus the previous term . for example the sequence :

$$5, 8, 11, 14, \cdots$$

each term is obtained from the previous term by adding 3 , this number 3 is called the common difference , since it can be obtained from subtracting any two consecutive terms .

the formula for an arthemitic sequence is always a linear function .

Arthemitic Sequence :

if $\langle a_n \rangle$ is an arthemitic sequence with common difference 'd' then :

$$a_n = k + nd$$

for some value of k ·

Example 4 : Find the formulas for the following sequences :

A) 5 , 8 , 11 , 14 , 17 , ⋯

B) 9 , 5 , 1 , -3 , -7 , ⋯

Solution :

A) the common difference d =8-5 =3

when n=1 any apply the arthemitic formula

$$a_1 = k + 1(3) \to 5 = k + 3 \ \to k = 2$$
$$\therefore \ \langle a_n \rangle = \langle 2 + 3n \rangle$$

B) the common difference d =5-9 =-4

when n=1 any apply the arthemitic formula

$$a_1 = k + 1(-4) \to 9 = k - 4 \ \to k = 13$$
$$\therefore \ \langle a_n \rangle = \langle 13 - 4n \rangle$$

2) Geometric Sequence

is a sequence for which each term is a constant multiplied by the previous term . for example the sequence :

$$6,12,24,48,\cdots$$

each term is exactly 2 times the previous term , this number 2 is called the common ratio , since it can be obtained by taking the ratio of any two consecutive terms .

the formula for a geometric sequence is always an exponetial function

Geometric Sequence :

if $\langle a_n \rangle$ is a geometric sequence with common ratio 'r' then :

$$a_n = k\, r^n$$

for some value of k .

Example 5 : Find the formulas for the following sequences :

A) $6 , 12 , 24 , 48 , 96 , \cdots$

B) $12 , 6 , 3 , 1.5 , 0.75 , \cdots$

Solution :

A) the common ratio $r = 12 \div 6 = 2$

when n=1 any apply the geometric formula

$$a_1 = k\,(2)^1 \to 6 = 2k \;\to k = 3$$
$$\therefore\ \langle a_n \rangle = \langle 3.\ 2^n \rangle$$

B) the common ratio $r = 6 \div 12 = 0.5$

when n=1 any apply the geometric formula

$$a_1 = k\,(0.5)^1 \to 12 = 0.5k \;\to k = 24$$
$$\therefore\ \langle a_n \rangle = \langle 24.\ 0.5^n \rangle = \langle \frac{24}{2^n} \rangle$$

Example H.W :

CHECK POINT

Find the formula of the general term an of the sequence:

- $13 , 10 , 7 , 4 , \ldots$
- $\frac{1}{2} , 1 , \frac{3}{2} , 2 , \ldots$
- $10 , 50 , 250 , 1250 , ..$
- $\frac{2}{3} , \frac{1}{3} , \frac{1}{6} , \frac{1}{12} , \ldots$

The Limit of the Sequence

you can take the limit of a sequence as $\langle a_n \rangle$ as $n \to \infty$ in the same way that you take the limit of the function f(x) as x involving to infinity . the only difference is that there is one term a_n for every positive integer n , while there is one value of f(x) for every real number .

Convergence And Divergence

We say that the sequence $\langle a_n \rangle$ convergence if $\lim_{n\to\infty} a_n$ is real number , if $\lim_{n\to\infty} a_n$ is infinite or doesnot exists , then the sequence is diverges

Example 6 : Does the sequence $\langle n^2 \rangle$ converges or diverges ?

Solution :

$$\lim_{n\to\infty} n^2 = \infty$$

Since the limit is infinite then the sequence diverges .

Example 7 : Does the sequence $\langle \frac{n^2+1}{3n^2+4n+2} \rangle$ converges or diverges ?

Solution :

$$\lim_{n\to\infty} \frac{n^2+1}{3n^2+4n+2} = \frac{\infty}{\infty}$$

By using lopital rule we have :

$$\lim_{n\to\infty} \frac{n^2+1}{3n^2+4n+2} = \frac{1}{3}$$

Since the limit is real finite then the sequence converges to $\frac{1}{3}$.

Example 8 : Does the sequence $\langle(-1)^n\rangle$ *converges or diverges ?*

Solution :

$\langle(-1)^n\rangle = -1\,,1-1\,,1-1\,,1\,,\ldots$

As n approaches to infinity the sequence will continue to oscillate between -1 and 1 , and therefore have no limit (it approaches neither -1 nor 1) so the limit does not exist , then the sequence is diverges .

Example 9 : Does the sequence $\langle\frac{n+1}{n}\rangle$ *converges or diverges ?*

Solution :

$$\lim_{n\to\infty} \frac{n+1}{n} = 1$$

Since the limit is 1 , then the sequence is converges .

Example 10 : Does the sequence $\langle 1+(\frac{-1}{2})^n\rangle$ converges or diverges ?

Solution :

$$\lim_{n\to\infty} 1+(\frac{-1}{2})^n = 1+0 = 1$$

Since the limit is 1 , then the sequence is converges .

Example 11 : Does the sequence $\langle (-1)^n \frac{n}{2n+1}\rangle$ converges or diverges ?

Solution :

$$\lim_{n\to\infty} (-1)^n \frac{n}{2n+1} = \mp\frac{1}{2}$$

Since the limit is alternating between $\frac{1}{2}$ and $\frac{-1}{2}$, then the sequence is diverges .

Example 12 : Does the sequence $\langle \frac{\pi^n}{4^n}\rangle$ converges or diverges ?

Solution :

$$\lim_{n\to\infty} \frac{\pi^n}{4^n} = \lim_{n\to\infty} \left(\frac{\pi}{4}\right)^n$$

Since the expression $\frac{\pi}{4} < 1$, then the sequence is converges to zero.

Example H.W :

CHECK POINT

Does the sequences converges or diverges ?

- $\langle \frac{n}{e^n} \rangle$, $\langle \sqrt[n]{n} \rangle$, $\langle n^2 e^{-n} \rangle$
- $\langle n \sin \frac{\pi}{n} \rangle$, $\langle \cos \frac{3}{n} \rangle$, $\langle \cos \frac{\pi n}{2} \rangle$
- $\langle \frac{Ln(n)}{n} \rangle$, $\langle Ln(\frac{1}{n}) \rangle$,
- $\langle \left(\frac{n+3}{n+1}\right)^n \rangle$, $\langle \left(1 - \frac{2}{n}\right)^n \rangle$, $\langle (1 + (-1)^n \rangle$, $\langle (-1)^n \frac{2n^3}{n^3+1} \rangle$

Monotone Sequence

a sequence $\langle a_n \rangle$ is called increasing if : $a_1 \le a_2 \le a_3 \le \cdots \le a_n$

and the sequence called decreasing if : $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$

- $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \frac{n}{n+1}$ is increasing sequence .
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}$ is decreasing sequence .
- $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots (-1)^{n+1}\frac{1}{n}$ Neither increasing nor decreasing.

Testing for monotonicity

1) Difference Technique :

In order for a sequence to be increasing all pairs of successive terms a_n and a_{n+1} must satisfy $a_n < a_{n+1}$ or $a_{n+1} - a_n > 0$.

monotone sequence can be classifed according this technique as follows :

$a_{n+1} - a_n > 0$ sequence is increasing

$a_{n+1} - a_n < 0$ sequence is decreasing

Example 13 : show that the sequence $\langle\frac{n}{n+1}\rangle$ increasing ?

Solution :

$$\langle a_n\rangle = \langle\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots\rangle$$

$$a_n = \frac{n}{n+1} \quad , \qquad a_{n+1} = \frac{n+1}{n+2}$$

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1}$$

$$a_{n+1} - a_n = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)}$$

$$a_{n+1} - a_n = \frac{n^2 + 2n + 1 - n^2 - 2n}{n^2 + 3n + 2}$$

$$a_{n+1} - a_n = \frac{1}{n^2 + 3n + 2} > 0$$

The sequence is increasing .

Example 14 : show that the sequence $\langle \frac{1}{n} \rangle$ decreasing ?

Solution :

$$\langle a_n \rangle = \langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots \rangle$$

$$a_n = \frac{1}{n} \quad , \qquad a_{n+1} = \frac{1}{n+1}$$

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n}$$

$$a_{n+1} - a_n = \frac{n - n - 1}{n(n+1)}$$

$$a_{n+1} - a_n = \frac{-1}{n^2 + n} < 0$$

The sequence is decreasing .

Testing for monotonicity

2) Ratio Technique :

If a_n and a_{n+1} are any successive terms in increasing sequence then $a_n < a_{n+1}$, if the terms in the sequence are all positive then we divide both sides of this inequality by a_n to obtain $\frac{a_{n+1}}{a_n} > 1$

More generally monotone sequence with positive terms can be classifed according this technique as follows :

$\frac{a_{n+1}}{a_n} > 1$ sequence is increasing

$\frac{a_{n+1}}{a_n} < 1$ sequence is decreasing

Example 15 : show that the sequence $\langle\frac{n}{n+1}\rangle$ increasing ?

Solution :

$$\langle a_n\rangle = \langle\frac{1}{2},\frac{2}{3},\frac{3}{4},\dots,\frac{n}{n+1},\dots\rangle$$

$$a_n = \frac{n}{n+1} \quad , \qquad a_{n+1} = \frac{n+1}{n+2}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2}\cdot\frac{n+1}{n}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^2+2n+1}{n^2+2n} > 1$$

The sequence is increasing .

Example 16 : show that the sequence $\langle 3+\frac{1}{n}\rangle$ decreasing ?

Solution :

$$\langle a_n\rangle = \langle 4, \frac{7}{2}, \frac{10}{3}, \dots, 3+\frac{1}{n}, \dots\rangle$$

$$a_n = 3+\frac{1}{n} \quad , \qquad a_{n+1} = 3+\frac{1}{n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{3+\frac{1}{n+1}}{3+\frac{1}{n}} = \frac{\frac{3n+4}{n+1}}{\frac{3n+1}{n}}$$

$$\frac{a_{n+1}}{a_n} = \frac{3n+4}{n+1} \cdot \frac{n}{3n+1}$$

$$\frac{a_{n+1}}{a_n} = \frac{3n^2+4n}{3n^2+4n+1} < 1$$

The sequence is decreasing .

Testing for monotonicity

3) Differentiation Technique :

If a_n is the termin increasing or decreasing sequence , then can you test the increasing or decreasing by transform the general term a_n to a function f(x) by instead every n in general term to x , and calculate the $f'(x)$ to the function f .

monotone sequence can be classifed according this technique as follows :

$f'(x) > 0$ sequence is increasing

$f'(x) < 0$ sequence is decreasing

Example 17 : show that the sequence $\langle \frac{n}{n+1} \rangle$ increasing ?

Solution :

$$\langle a_n \rangle = \langle \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \rangle$$

$$f(x) = \frac{x}{x+1}$$

$$f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$$

The sequence is increasing .

Example 18 : show that the sequence $\langle \frac{n}{2n-1} \rangle$ decreasing ?

Solution :

$$\langle a_n \rangle = \langle 1, \frac{2}{3}, \frac{3}{5}, \dots, \frac{n}{2n-1}, \dots \rangle$$

$$f(x) = \frac{x}{2x-1}$$

$$f'(x) = \frac{2x-1-2x}{(2x-1)^2} = \frac{-1}{(2x-1)^2} < 0$$

The sequence is decreasing .

Example H.W :

CHECK POINT

Does the sequences increasing or decreasing ?

- $\langle 1-\frac{1}{n} \rangle$, $\langle \frac{n}{4n-1} \rangle$, $\langle n-2^n \rangle$, $\langle n-n^2 \rangle$
- $\langle \frac{2n}{1+2^n} \rangle$, $\langle \frac{5^n}{2^{n^2}} \rangle$, $\langle ne^{-n} \rangle$, $\langle \frac{10^n}{(2n)!} \rangle$, $\langle \frac{n^n}{n!} \rangle$
- $\langle \frac{Ln(n+2)}{n+2} \rangle$, $\langle \frac{1}{n+Ln(n)} \rangle$, $\langle 3-\frac{1}{n} \rangle$, $\langle ne^{-2n} \rangle$, $\langle tan^{-1}x \rangle$

Eventually sequence

If discarding finitely many terms from the begining of a sequence produces a sequence with a certain property then the original sequence is said to have the property : eventually

Example 19 : show that the sequence $\langle \frac{10^n}{n!} \rangle$ Is eventually decreasing ?

Solution :

$$a_n = \frac{10^n}{n!} \quad , \qquad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{10^n}{n!}} = \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{10}{n+1} < 1 \ , for\ all\ n \geq 10$$

The sequence is eventually decreasing .

Example H.W :

Does the sequences eventually increasing or decreasing ?

CHECK POINT

- $\langle 2n^2 - 7n \rangle$, $\langle n^3 - 4n^2 \rangle$, $\langle \frac{n}{n^2+10} \rangle$, $\langle n + \frac{17}{n} \rangle$, $\langle \frac{n!}{3^n} \rangle$

The Series :

an infinite series is an exppression can be written in the form :

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

the numbers $u_1, u_2, u_3, \ldots, u_k$ are called the terms of the series .

let $\langle a_n \rangle$ be a sequence with partial sum of the series $u_1+u_2+\cdots+u_k$

if the sequence $\langle a_n \rangle$ convergent to a limit S then the series is said to be convergence to S and S is called the sum of the series , we denote by :

$$S = \sum_{k=1}^{\infty} u_k$$

Note :

the sequence of partial sum is diverges then the series is said to be diverges , a divergent series has no sum .

Geometric series

If the initial term of the series is 'a' and each term is obtained by multiplying the preceding term by r then the series has the form :

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots, a \neq 0$$

Such series are called geometrical series and the number r is called the ratioof the series , for examples :

- $1+2+4+8+\cdots+2^k+\cdots \quad ,a=1,r=2$
- $\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\cdots+\frac{3}{10^k}+\cdots \quad ,a=\frac{3}{10} \quad ,r=\frac{1}{10}$
- $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}+\cdots+(-1)^{k+1}\frac{1}{2^k}+\cdots ,a=\frac{1}{2} \ ,r=-\frac{1}{2}$

Tests of Convergence & Divergence

1) The Convergence of Geometrical series :

the geometry series :

$$\sum_{k=0}^{\infty} ar^k = a+ar+ar^2+ar^3+\cdots+ar^k+\cdots ,a\neq 0$$

is convergence if $|r|<1$ *and divergence if* $|r|\geq 1$ *,*

if the series is convergence then the sum of the series is :

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

Example 1 : Test the convergence or divergence of the series , and find sum if it is convergence :

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5+\frac{5}{4}+\frac{5}{4^2}+\frac{5}{4^3}+\cdots+\frac{5}{4^k}+\cdots$$

Solution :

The geometrical series with a=5 and $r = \frac{1}{4} < 1$,

then the series it is convergence and its sum is

$$S = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}$$

Example2 : Test the convergence or divergence of the series , and find sum if it is convergence :

$$\sum_{k=0}^{\infty} 3^{2k}\, 5^{1-k}$$

Solution :

The geometrical series can be written as :

$$\sum_{k=0}^{\infty} \frac{9^k}{5^{k-1}} = \sum_{k=1}^{\infty} 9\,\frac{9^{k-1}}{5^{k-1}} = \sum_{k=1}^{\infty} 9\left(\frac{9}{5}\right)^{k-1}$$

The geometrical series with a=9 and $r = \frac{9}{5} > 1$,

then the series it is divergence and its no sum

Example 3 : Find the rational numberrepresent by repeating decimal for :

$$0.784784784784\ldots$$

Solution :

The number $0{\cdot}784784784784\cdots$ can written as a series

$0{\cdot}784784784784\cdots = 0{\cdot}784 + 0{\cdot}000784 + 0{\cdot}000000784 + \cdots$

previous expression is geometrical series with $a=0{\cdot}784$ and $r = 0.001 < 1$,

then the series it is convergence and its sum is

$$S = \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{0.784}{0.999} = \frac{784}{999}$$

Example H·W :

CHECK POINT

1) Determine the convergence or divergence of the following series , and find sum if it is convergence :

- $\sum_{k=1}^{\infty}\left(\frac{-3}{4}\right)^{k-1}$, $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^{k+2}$, $\sum_{k=1}^{\infty}\left(\frac{-3}{2}\right)^{k+1}$, $\sum_{k=5}^{\infty}\left(\frac{e}{\pi}\right)^{k-1}$
- $\sum_{k=1}^{\infty}\frac{4^{k+2}}{7^{k-1}}$, $\sum_{k=1}^{\infty}\frac{3}{4^k}+\frac{2}{5^{k-1}}$, $\sum_{k=1}^{\infty}5^{3k}+7^{1-k}$

2) express the given reapeating decimal as a fractional :

$0{\cdot}9999\cdots$, $0{\cdot}159159159\cdots$, $0{\cdot}782178217821\cdots$, $0{\cdot}45114141414\cdots$

Tests of Convergence & Divergence

2) The Integral Test

Let $\sum u_k$ be a series with positive terms and let f(x) be a function that result when k is replaced by x in the general term of the series ,

If f is decreasing and continous on the interval $[a, \infty)$ then :

$$\sum_{k=a}^{\infty} u_k \text{ , and } \int_a^{\infty} f(x)dx$$

Both convergence or both divergence

Example 4 : use the integral test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

Solution :

$$u_k = \frac{1}{k} \rightarrow f(x) = \frac{1}{x}$$

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x} dx = \lim_{b\to\infty} Ln(x)|_1^b$$

$$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} Ln(b) - Ln(1) = Ln(\infty) = \infty$$

The integral is divergent and consequently so does the series .

Example 5 : use the integral test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

Solution :

$$u_k = \frac{1}{k^2} \to f(x) = \frac{1}{x^2}$$

$$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2} dx = \lim_{b \to \infty} \left. \frac{-1}{x} \right|_{1}^{b}$$

$$\int_{1}^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} 1 - \frac{1}{b} = 1 - 0 = 1$$

The integral is convergent and consequently so does the series .

CHECK POINT

Example H.W :

confirm that the integral test is applicable and use it to

Determine whether the series convergence or divergence :

❖ $\sum_{k=1}^{\infty}\frac{1}{5^{k+2}}$, $\sum_{k=1}^{\infty}\frac{1}{1+9k^2}$, $\sum_{k=1}^{\infty}\frac{k}{k^2+1}$, $\sum_{k=1}^{\infty}\frac{1}{(4+2k)^{\frac{3}{2}}}$

Tests of Convergence & Divergence

3) P - Series convergence

The series :

$$\sum_{k=1}^{\infty}\frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{k^p} + \cdots$$

Is convergence if $p > 1$ and divergent if $p \leq 1$

Example 6 : use the P series test to determine the convergence or divergenc of the series :

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

Solution :

This series is p-series with $p = \frac{2}{3} < 1$

So it is divergence .

Example 7 : *use the P series test to determine the convergence or divergenc of the series :*

$$1 + \frac{1}{8} + \frac{1}{64} + \cdots + \frac{1}{k^3} + \cdots$$

Solution :

This series is p-series with $p = 3 > 1$

So it is convergence .

Example H.W :

CHECK POINT

Use P-series toDetermine whether the series convergence or divergence :

- $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, $\sum_{k=1}^{\infty} k^{\frac{-2}{5}}$, $\sum_{k=1}^{\infty} k^{\frac{-4}{3}}$

Tests of Convergence & Divergence

4) Comparsion Test

Let $\sum a_k$ and $\sum b_k$ be a series with non-negative terms and suppose that :

$$a_1 \leq b_1 , a_2 \leq b_2 , a_3 \leq b_3 , \dots , a_k \leq b_k , \dots$$

1) if the bigger series $\sum b_k$ convergence , then the smaller series $\sum a_k$ also convergence .

2) if the smaller series $\sum a_k$ divergence , then the bigger series $\sum b_k$ also divergence .

Example 8 : use the comparsion test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} - \frac{1}{2}}$$

Solution :

$$\sqrt{k} > \sqrt{k} - \frac{1}{2}$$

$$\underbrace{\frac{1}{\sqrt{k}}}_{\sum a_k} < \underbrace{\frac{1}{\sqrt{k} - \frac{1}{2}}}_{\sum b_k}$$

The series $\frac{1}{\sqrt{k}}$ is diverges by P-series

So the series $\frac{1}{\sqrt{k}-\frac{1}{2}}$ is also divergent .

Example 9 : use the comparsion test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty}\frac{1}{2k^2+k}$$

Solution :

$$2k^2+k>2k^2$$

$$\underbrace{\frac{1}{2k^2+k}}_{\sum a_k}<\underbrace{\frac{1}{2k^2}}_{\sum b_k}$$

The series $\frac{1}{2k^2}$ is converges by P-series

So the series $\frac{1}{2k^2+k}$ is also convergent .

Example H.W :

CHECK POINT

Test the convergence or divergence of the series by comparsion

$\sum_{k=1}^{\infty}\frac{1}{5k^2-k}$, $\sum_{k=1}^{\infty}\frac{3}{k-4}$, $\sum_{k=1}^{\infty}\frac{k+1}{k^2-k}$, $\sum_{k=1}^{\infty}\frac{2}{k^4+k}$,

$\sum_{k=1}^{\infty}\frac{5\sin^2 k}{k!}$, $\sum_{k=1}^{\infty}\frac{Lnk}{k}$, $\sum_{k=1}^{\infty}\frac{k}{k^{\frac{3}{2}}-1}$

Tests of Convergence & Divergence

5) Limit Comparsion Test

Let $\sum a_k$ and $\sum b_k$ be a series with positive terms and suppose that :

$$H = \lim_{k\to\infty} \frac{a_k}{b_k}$$

if H is finite and $H > 0$ then the series both convergence or both divergence .

Example 10 : use the limit comparsion test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{1}{3k^3 + 2k^2 + k}$$

Solution :

$3k^3 + 2k^2 + k > 3k^3$

$$\underbrace{\frac{1}{3k^3 + 2k^2 + k}}_{\sum a_k} < \underbrace{\frac{1}{3k^3}}_{\sum b_k}$$

$$H = \lim_{k\to\infty} \frac{\frac{1}{3k^3 + 2k^2 + k}}{\frac{1}{3k^3}} = \lim_{k\to\infty} \frac{3k^3}{3k^3 + 2k^2 + k} = 1 > 0$$

Since $H = 1 > 0$ and finite ,

The series $\sum_{k=1}^{\infty} \frac{1}{3k^3}$ is convergence (By P-series)

$\therefore \sum_{k=1}^{\infty} \frac{1}{3k^3+2k^2+k}$ is also convergent .

Example 11 : use the Limit comparsion test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}$$

Solution :

Let $a_k = \sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$, $b_k = \sum_{k=1}^{\infty} \frac{3k^3}{k^7} = \frac{3}{k^4}$

$$H = \lim_{k\to\infty} \frac{\frac{3k^3 - 2k^2 + 4}{k^7 - k^3 + 2}}{\frac{3}{k^4}} = \lim_{k\to\infty} \frac{3k^7 - 2k^6 + 4k^4}{3k^7 - 3k^3 + 6} = 1 > 0$$

Since $H = 1 > 0$ and finite ,

The series $\sum_{k=1}^{\infty} \frac{3}{k^4}$ is convergence (By P-series)

$\therefore \sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$ is also convergent .

CHECK POINT

Example H.W :

Test the convergence or divergence of the series by limit comparsion

$\sum_{k=1}^{\infty} \frac{4k^2-2k+6}{8k^7+k-8}$, $\sum_{k=1}^{\infty} \frac{5}{3^k+1}$, $\sum_{k=1}^{\infty} \frac{1}{\sqrt[3]{8k^2-3k}}$, $\sum_{k=1}^{\infty} \frac{1}{(2k+3)^{17}}$, $\sum_{k=1}^{\infty} \frac{k(k+3)}{(k+1)(k+2)(k+5)}$

Tests of Convergence & Divergence

6) The Ratio Test

Let $\sum u_k$ be a series with positive terms and suppose that :

$$R = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

1) if $R < 1$ then the series convergence.

2) if $R > 1$ then the series divergence.

3) if $R = 1$ then the series may be convergence or divergence , so that another test must be tried .

Example 12 : use the ratio test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{k}{2^k}$$

Solution :

$$R = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{k+1}{2^{k+1}} \frac{2^k}{k} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2} < 1$$

Since $R = \frac{1}{2} < 1$, then the series is convergence .

Example 13 : use the ratio test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

Solution :

$$R = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}} = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{(k+1)!} \frac{k!}{k^k}$$

$$R = \lim_{k\to\infty} \frac{(k+1)^k (k+1) k!}{k^k\,(k+1)k!} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = e > 1$$

Since $R = e > 1$, then the series is divergence .

Example 14 : use the ratio test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$$

Solution :

$$R = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{(2(k+1))!}{4^{k+1}}}{\frac{(2k)!}{4^k}} = \lim_{k\to\infty} \frac{(2k+2)!}{4^{k+1}} \frac{4^k}{(2k)!} =$$

$$R = \frac{1}{4}\lim_{k\to\infty}(2k+2)(2k+1) = \infty$$

Since R is infinite , then the series is divergence .

Example 15 : use the ratio test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} \frac{1}{2k-1}$$

Solution :

$$R = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{\frac{1}{2(k+1)-1}}{\frac{1}{2k-1}} = \lim_{k\to\infty} \frac{2k-1}{2k+1} = 1$$

Since R =1 , then the test is fail .

We try another test such as (Integral Test)

$$I = \int_{1}^{\infty} \frac{1}{2x-1}dx = \lim_{b\to\infty} \int_{1}^{b} \frac{1}{2x-1}dx = \frac{1}{2}\lim_{b\to\infty} Ln|2x-1| = \infty$$

The integral is divergent ,

So the series is divergent .

CHECK
POINT

Example H.W :

Test the convergence or divergence of the series by Ratio Test

$\sum_{k=1}^{\infty}\frac{3^k}{k!}$, $\sum_{k=1}^{\infty}\frac{4^k}{k^2}$, $\sum_{k=1}^{\infty}\frac{1}{5x}$, $\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^k$,

$\sum_{k=1}^{\infty}\frac{k}{k^3}$, $\sum_{k=1}^{\infty}\frac{k}{k^2+1}$

Tests of Convergence & Divergence

7) The Root Test

Let $\sum u_k$ be a series with positive terms and suppose that :

$$Q=\lim_{k\to\infty}\sqrt[k]{u_k}=\lim_{k\to\infty}(u_k)^{\frac{1}{k}}$$

1) if $Q<1$ then the series convergence.

2) if $Q>1$ then the series divergence.

3) if $Q=1$ then the series may be convergence or divergence , so that another test must be tried .

Example 16 : use the Root Test to determine the convergence or divergenc of the series :

$$\sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^{k}$$

Solution :

$$Q=\lim_{k\to\infty}\left[\left(\frac{4k-5}{2k+1}\right)^{k}\right]^{\frac{1}{k}}=\lim_{k\to\infty}\frac{4k-5}{2k+1}=2>1$$

Since $Q=2>1$, then the series is divergence ∙

Example 17 : use the Root Test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty}\frac{1}{(Ln(k+1))^{k}}$$

Solution :

$$Q=\lim_{k\to\infty}\left[\frac{1}{(Ln(k+1))^{k}}\right]^{\frac{1}{k}}=\lim_{k\to\infty}\frac{1}{Ln(k+1)}=\frac{1}{\infty}=0<1$$

Since $Q=0<1$, then the series is convergence ∙

CHECK POINT

Example H.W :

Test the convergence or divergence of the series by Root Test

$\sum_{k=1}^{\infty}\left(\frac{3k+2}{2k-1}\right)^k$, $\sum_{k=1}^{\infty}\left(\frac{k}{100}\right)^k$, $\sum_{k=1}^{\infty}\frac{k}{5^k}$, $\sum_{k=1}^{\infty}(1-e^{-k})^k$

Tests of Convergence & Divergence

8) Alternating series Test

Series whose term is alternate between positive and negative value is called alternating series in general an alternating series has one of the following forms :

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

Or

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

Alternating series of the two forms above convergent if the following conditions are satisfied :

1) $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots \geq a_k \geq \cdots$

2) $\lim_{k\to\infty} a_k = 0$

Example 18 : use the alternating series Test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

Solution :

$$a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1}$$

$$\lim_{k\to\infty} \frac{1}{k} = 0$$

the series is convergence .

Example 19 : use the alternating series Test to determine the convergence or divergenc of the series :

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$$

Solution :

$$\frac{a_{k+1}}{a_k} = \frac{k+4}{(k+1)(k+2)} \; \frac{k(k+1)}{k+3}$$

$$\frac{a_{k+1}}{a_k} = \frac{k^2+4k}{k^2+5k+6} = \frac{k^2+4k}{k^2+4k+k+6} < 1$$

$$\therefore \; a_k > a_{k+1}$$

$$\lim_{k\to\infty} \frac{k+3}{k(k+1)} = \lim_{k\to\infty} \frac{k+3}{k^2+k} = 0$$

the series is convergence .

CHECK POINT

Example H.W :

Test the convergence or divergence of the series by Alternating Test

$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{3k+1}$, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+1}{\sqrt{k}+1}$

$\sum_{k=1}^{\infty}(-1)^{k+1}e^{-k}$, $\sum_{k=3}^{\infty}(-1)^{k}\frac{Lnk}{k}$

Tests of Convergence & Divergence

9) Absolute Alternating series Test

A Series $\sum u_k$ is said to be convergent absolutely if the series of the absolute values

$$\sum_{k=1}^{\infty}|u_k| = |u_1| + |u_2| + |u_3| + \cdots + |u_k| + \cdots$$

Convergent , and is said to be divergent absolutely if the series of the absolute values divergent .

Example 20 :Determine whethere the following series convergent absolutely

$$1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \cdots$$

Solution :

The series of the absolute value is : $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots$

This series is convergent (geometry series with a=1 , $r = \frac{1}{2}$)

Example 21 : Determine whethere the following series convergent absolutely

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\cdots$$

Solution :

The series of the absolute value is : $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$

This series is divergent (P- series with p=1)

Note :

if $\sum a_n$ and $\sum b_n$ are convergent series with sum A and B respectivly then :

1) $\sum(a_n \mp b_n)$ is convergent and its sum $A \mp B$

2) $\sum C\, a_n$ is convergent and its sum is CA

CHECK POINT

Example H.W :

Test the convergence or divergence of the series by Absolutely Alternating Test

$$\sum_{k=1}^{\infty}(-1)^k\frac{2^k}{k!} \qquad , \qquad \sum_{k=3}^{\infty}(-1)^k\frac{(2k-1)!}{3^k}$$

Iterates of function :

Suppose that we press key 2 into a calculater and then repeatedly depress the x^2 buttom , the calculater would display the numbers : $2,4,16,256,\cdots$ one after another , these numbers called iterates of x_o for f and represented by : $x_o\ , f(x_o), f\big(f(x_o)\big), f\Big(f\big(f(x_o)\big)\Big), \dots$

x_o is the initial point

$f(x_o)$ is the 1^{st} iterate for f

$f\big(f(x_o)\big)$ is the 2^{nd} iterate for f and can write as $f^2(x_o)$

$f(f\big(f(x_o)\big))$ is the 3^{rd} iterate for f and can write as $f^3(x_o)$

And etc$\cdots$

Orbit of function :

The sequence $\langle f^n(x_o)\rangle = \langle x_o\ , f(x_o)\ , f^2(x_o), f^3(x_o), \dots , f^n(x_o), \dots\ \rangle$ of all iterates is called the orbit of x_o .

Example 1 : Find the orbit of the following function at the point x_o

Solution :

F(x)	x_0	Orbit of f at x_0
$f(x) = x^2$	1	1,1,1,1,1,⋯
$f(x) = x^2 - 1$	-1	-1,0,-1,0,-1,0,-1,⋯
$f(x) = x^2 + 1$	-2	-2,5,26,677, ⋯
$f(x) = x^2 + 0.25$	0	0 , 0.25 , 0.3125 , 01347656 ,⋯
$f(x) = 4x - 4x^2$	1/3	0.333⋯ , 0.888⋯ , 0.395061⋯, ⋯

Note

The orbits seems to have very different behavior , indeed the orbit of 1 in the 1st function is constantly 1 , next the orbit of -1 for the 2nd function is alternating between -1 and 0 , the orbit of -2 in the 3rd function unbounded (increasing) , and its no clear how the iterates of 0 and 1/3 in the last functions behaves .

Fixed Points

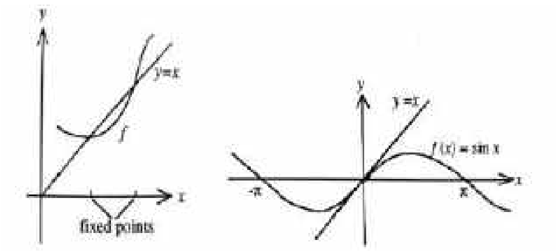
A point whose iterates are the same point is called a fixed point , fixed points are very important in the study of the dynamics of functions .

Def :

Let p be in the domain of f , then p is called a fixed point of f if

$$f(p) = p$$

Graphically a point p in the domain of f is a fixed point of f iff the graph of f touches or crosses the line y=x at (p,p)



Example 2 : Find the fixed point to the following functions :

1) $f(x) = 3x - 6$

Solution :

$$f(p) = p$$

$$3p - 6 = p$$

$$2p = 6 \rightarrow p = 3$$

$$(3,3)\ fixed\ point$$

2) $f(x) = (x - 2)^2$

Solution :

$$f(p) = p$$

$$(p - 2)^2 = p$$

$$p^2 - 4p + 4 = p$$

$$p^2 - 5p + 4 = 0$$

$$(p-1)(p-4) = 0 \,,\;\; p = 1\,,4$$

$(1,1)\,,(4,4)$ *are fixed points*

3) $f(x) = 6x^2 - 15$

Solution :

$$f(p) = p$$

$$6p^2 - 15 = p$$

$$6p^2 - p - 15 = 0$$

$$(3p-5)(2p+3) = 0 \,,\;\; p = \frac{5}{3}\,,\frac{-3}{2}$$

$\left(\frac{5}{3},\frac{5}{3}\right)\,,\left(\frac{-3}{2},\frac{-3}{2}\right)$ *are fixed points*

4) $f(x) = x^3 + x^2 - 9x + 8$

Solution :

$$f(p) = p$$

$$p^3 + p^2 - 9p + 8 = p$$

$$p^3 + p^2 - 10p + 8 = 0$$

$$(p-1)(p^2+2p-8)=0$$

$$(p-1)(p-2)(p+4)=0$$

$$p=1\,,2\,,4$$

$$(1,1)\,,(2,2)\,,(3,3)\ are\ \ fixed\ \ points$$

Example 3 : if the function $f(x)=Ax^2+11x-7$ touch the line y=x at x=0·5 , find the value of A ?

Solution :

Since the function touch line y=x , then the function having fixed point at x=0·5 , so the point satisfy the equation f(x) :

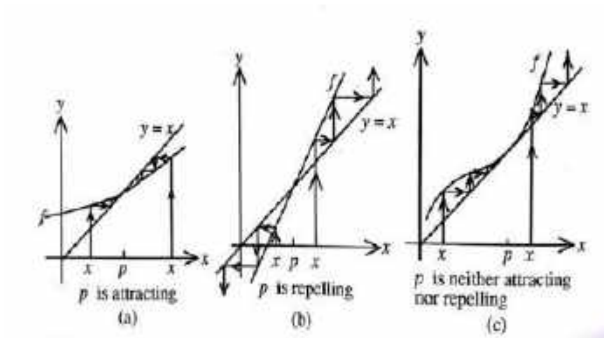
$$(0.5\,,0.5)\in f(x)$$

$$\frac{1}{2}=\frac{1}{4}A+\frac{11}{2}-7$$

$$2=A+22-28\ \rightarrow A=8$$

Attracting & Repelling Fixed Points

By applying graphical analysis , we can see diverse behavior for the iterates of various points . Indeed in figure A the iterates of x approaches to fixed point p , where as in figure B tends towards infinity , the iterates of x in figure C have each of these characteristics depending on the x value .



p is attracting (a) p is repelling (b) p is neither attracting nor repelling (c)

It follows from the figure above that the fixed point in figure A is attracting and that the one in figure B is repelling , that not every fixed point is attracting or repelling is demonstarted in figure C where points to the left of p are attracted to p and point to the right of p are repelled .

Theorem :

Suppose that f is differentiable at a fixed point p then :

I. If $|f'(p)| < 1$, then p is attracting

II. If $|f'(p)| > 1$, then p is repelling

III. If $|f'(p)| = 1$, then p is neither attracting nor repelling

Example 4 : Find the fixed point and determine its type (Attracting or Repelling)for the following functions ?

1) $f(x) = 3x - 3x^2$

Solution :

$$f(p) = p$$

$$3p - 3p^2 = p$$

$$2p - 3p^2 = 0$$

$$p(2 - 3p) = 0 \,, \quad p = 0 \;, \frac{2}{3}$$

$(0,0)\,,\left(\frac{2}{3},\frac{2}{3}\right)$ *are fixed points*

To know type of this points we find f'(x)

$$f'(x) = 3 - 6x$$

$|f'(0)| = |3 - 6(0)| = 3 > 1$, the point is repelling at x=0

$\left|f'\left(\frac{2}{3}\right)\right| = \left|3 - 6\left(\frac{2}{3}\right)\right| = |-1| = 1$, the point is neither attracting nor repelling at x=2/3

2) $f(x) = x^2$

Solution :

$$f(p) = p$$

$$p^2 = p$$

$$p - p^2 = 0$$

$$p(1-p) = 0\,,\ \ p = 0\ ,1$$

$$(0,0)\,,(1,1)\ are\ fixed\ points$$

To know type of this points we find f'(x)

$$f'(x) = 2x$$

$|f'(0)| = |2(0)| = 0 < 1$, the point is attracting at x=0

$|f'(1)| = |2(1)| = |2| = 2 > 1$, the point repelling at x=1

Eventually Fixed Points

Let x be in the domain of f , then x is an eventually fixed point of f if there is a positive integer n such that $f^{(n)}(x)$ is a fixed point of f .

i.e :

$$f(a) = b\,, f(b) = c\,, f(c) = d\,, \ldots, f(z) = z$$

Example 5 : Show that π is an eventually fixed point to $f(x) = sinx$

Solution :

$$f(\pi) = sin\pi = 0$$

$$f(0) = sin0 = 0$$

Then π is eventually fixed point .

Example 6 : Show that $\frac{1}{8}$ is an eventually fixed point to

$$f(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} < x \le 1 \end{cases}$$

Solution :

$$f\left(\frac{1}{8}\right) = 2(\frac{1}{8}) = \frac{1}{4}$$

$$f\left(\frac{1}{4}\right) = 2(\frac{1}{4}) = \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = 2(\frac{1}{2}) = 1$$

$$f(1) = 2 - 2(1) = 0$$

$$f(0) = 2(0) = 0$$

Then $\frac{1}{8}$ is eventually fixed point .

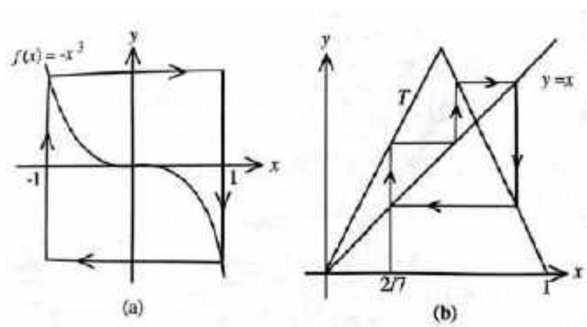
The Periodic points

Let x_o be in the domain of f , then x_o has a period-n if : $f^{(n)}(x_o) = x_o$

And if in addition : $x_o , f(x_o), f^{(2)}(x_o), f^{(3)}(x_o), \dots, f^{(n-1)}(x_o)$ is a periodic orbit and is called an n-cycle . i.e :

$$f(a) = b , f(b) = c , f(c) = d , \dots , f(z) = a$$

Note : Every fixed point is period-1 or 1-cycle



Example 7 : Show that $\{-1, 1\}$ is 2-cycle to $f(x) = -x^3$

Solution :

$$f(-1) = 1$$

$$f(1) = -1$$

Then $\{-1, 1\}$ is 2-cycle .

Example 8 : *Show that* $\left\{\frac{2}{7},\frac{4}{7},\frac{6}{7}\right\}$ *is 3-cycle to* :

$$f(x)=\begin{cases} 2x & 0\le x\le \frac{1}{2} \\ 2-2x & \frac{1}{2}< x\le 1 \end{cases}$$

Solution :

$$f\left(\frac{2}{7}\right)=2\left(\frac{2}{7}\right)=\frac{4}{7}$$

$$f\left(\frac{4}{7}\right)=2-\frac{8}{7}=\frac{6}{7}$$

$$f\left(\frac{6}{7}\right)=2-\frac{12}{7}=\frac{2}{7}$$

Then $\left\{\frac{2}{7},\frac{4}{7},\frac{6}{7}\right\}$ is 3-cycle .

Example 9 : *classify the periodic cycle containing 0 for the function* :

$$f(x)=1-\frac{x}{3}+2x^2-\frac{2x^3}{3}$$

Solution :

$$f(0)=1$$

$$f(1)=1-\frac{1}{3}+2-\frac{2}{3}=2$$

$$f(2)=1-\frac{2}{3}+8-\frac{16}{3}=3$$

$$f(3) = 1 - 1 + 18 - 18 = 0$$

Then $\{0\,,1\,,2\,,3\}$ is 4-cycle .

CHECK POINT

Example H.W :

1) Find the orbit of the function :

$$f(x) = x^2 - 2x + 3 \quad \text{at } x_o=2$$

2) Find the fixed point of the functions :

A. $f(x) = x^2 + 5x + 3$
B. $f(x) = x^3 - 4x^2 + 6x - 2$
C. $f(x) = 2x^4 + x^3 - 3x^2 + 1$

3) Classify the type of the fixed points of the function :

$$f(x) = x^3$$

4) Show that $\frac{1}{8}$ and $\frac{4}{5}$ are eventually fixed points for the function :

$$f(x) = \begin{cases} 4x & 0 \le x < \frac{1}{4} \\ 2 - 4x & \frac{1}{4} \le x < \frac{1}{2} \\ 4x - 2 & \frac{1}{2} \le x < \frac{3}{4} \\ 4 - 4x & \frac{3}{4} \le x \le 1 \end{cases}$$

5) prove that $\frac{1}{3}, \frac{1}{5}, \frac{1}{9}$ is a periodic point and find the cycle for every point in the function :

$$f(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

6) Find the value of c such that $\{0, c\}$ is 2-cycle such that c be a constant and let the function be :

$$f(x) = x^3 - 3x + c$$

Gauss Method for solving Linear Equation System

This method uses to solve linear equation system , to solve system we must follow the steps :

1) Transform the system to the formula $[A : B]$ such that A is cofactor Matix , B is the result vector .
2) Equations must be arranged so that the first element of the first row and the first column a_{11} are non-zero value .
3) Convert the matrix to upper or lower triangular matrix , and this converting needs to :
   a) Use a_{11} to zeros the elements below it (a_{21}, a_{31})
   b) Use a_{22} to zeros the elements below it (a_{32})
   c) The row to be zeroed is called the secondary row .
   d) The row using to zeros the secondary row called the primary row .
   e) A mechanism to zeros any element in particular row that is by the rule :

$$\boxed{-\frac{\text{secondary row element cofactor}}{\text{primary row element cofactor}} * R_{primary} + R_{secondary}}$$

   f) multiply the last row by the number remaining inverted .
4) solve the produced in a reverse gradient method .

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \begin{array}{l} cR_p + R_s \\ dR_p + R_s \end{array}$$

Example 1 :

Solve the system :

$$x_1 + x_2 - x_3 = 3$$

$$-2x_1 + x_2 - 4x_3 = 3$$

$$x_1 - 2x_3 = 3$$

Solution :

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ -2 & 1 & -4 & 3 \\ 1 & 0 & -2 & 3 \end{array}\right] \begin{array}{l} 2R_1 + R_2 \\ -R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 3 & -6 & 9 \\ 0 & -1 & -1 & 0 \end{array}\right] \frac{1}{3}R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 3 \\ 0 & 3 & -6 & 9 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

$$-3x_3 = 3 \quad \Longrightarrow \quad \boxed{x_3 = -1}$$

$$3x_2 - 6x_3 = 9 \quad \Longrightarrow \quad \boxed{x_2 = 1}$$

$$x_1 + x_2 - x_3 = 3 \quad \Longrightarrow \quad \boxed{x_1 = 1}$$

Note :

If it is possible to simplify the numbers of any row , we divide or multiply this row by a certain number for the purpose of simplification .

Example 2 :

Solve the system :

$$x_1 + 2x_2 + x_3 = 5$$

$$2x_1 - 4x_2 + 7x_3 = -16$$

$$3x_1 + x_2 - x_3 = 13$$

Solution :

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & -4 & 7 & -16 \\ 3 & 1 & -1 & 13 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \\ -3R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -8 & 5 & -26 \\ 0 & -5 & -4 & -2 \end{array}\right] \frac{-5}{8} R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -8 & 5 & -26 \\ 0 & 0 & \frac{-57}{8} & \frac{114}{8} \end{array}\right] \frac{-8}{57} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -8 & 5 & -26 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\boxed{x_3 = -2}$

$-8x_2 + 5x_3 = -26 \quad \Longrightarrow \quad \boxed{x_2 = 2}$

$x_1 + 2x_2 + x_3 = 3 \quad \Longrightarrow \quad \boxed{x_1 = 3}$

Example 3 :

Solve the system :

$$2x_1 + 3x_2 - 4x_3 = -9$$

$$x_1 + x_2 - x_3 = -1$$

$$5x_1 - 2x_2 - x_3 = 4$$

Solution :

$$\left[\begin{array}{ccc|c} 2 & 3 & -4 & -9 \\ 1 & 1 & -1 & -1 \\ 5 & -2 & -1 & 4 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 2 & 3 & -4 & -9 \\ 5 & -2 & -1 & 4 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \\ -5R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & -2 & -7 \\ 0 & -7 & 4 & 9 \end{array}\right] 7R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & -10 & -40 \end{array}\right]$$

$$-10x_3 = -40 \implies \boxed{x_3 = 4}$$

$$x_2 - 2x_3 = -7 \implies \boxed{x_2 = 1}$$

$$x_1 + x_2 - x_3 = -1 \implies \boxed{x_1 = 2}$$

Example 4 :

Solve the system :

$$2x_1 + 3x_2 - 5x_3 = -7$$

$$3x_1 - x_2 + x_3 = 9$$

$$6x_1 - 2x_2 + 2x_3 = 10$$

Solution :

$$\left[\begin{array}{ccc|c} 2 & 3 & -5 & -7 \\ 3 & -1 & 1 & 9 \\ 6 & -2 & 2 & 10 \end{array}\right] \begin{array}{l} \frac{-3}{2}R_1 + R_2 \\ -3R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -5 & -7 \\ 0 & \frac{-11}{2} & \frac{17}{2} & \frac{39}{2} \\ 0 & -11 & 17 & 31 \end{array}\right] 2R_2$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -5 & -7 \\ 0 & -11 & 17 & 39 \\ 0 & -11 & 17 & 31 \end{array}\right] -R_2+R_3$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -5 & -7 \\ 0 & -11 & 17 & 39 \\ 0 & 0 & 0 & -8 \end{array}\right] -R_2+R_3$$

$$0=-8$$

The system is wrong , then the system dont have any solution .

Note :

- *If the number of variables = the number of equations then there is only one solution for the system .*
- *If the number of variables smaller the number of equations then there is many solution for the system .*
- *If the number of variables greater the number of equations then the solution for the system is dependence .*

Example 5 :

Solve the system :

$$x_1 + 2x_2 - x_3 = 7$$

$$x_1 + x_2 - 2x_3 = 3$$

$$3x_1 - 4x_2 + x_3 = -5$$

$$5x_1 + x_2 - 3x_3 = 10$$

Solution :

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 1 & 1 & -2 & 3 \\ 3 & -4 & 1 & -5 \\ 5 & 1 & -3 & 10 \end{array}\right] \begin{array}{l} -R_1 + R_2 \\ -3R_1 + R_3 \\ -5R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -1 & -1 & -4 \\ 0 & -10 & 4 & -26 \\ 0 & -9 & 2 & -25 \end{array}\right] \begin{array}{l} -10R_2 + R_3 \\ -9R_2 + R_4 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 14 & 14 \\ 0 & 0 & 11 & 11 \end{array}\right] \frac{-11}{14}R_3 + R_4$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -1 & -1 & -4 \\ 0 & 0 & 14 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$14x_3 = 14 \implies \boxed{x_3 = 1}$$

$$-x_2 - x_3 = -4 \implies \boxed{x_2 = 3}$$

$$x_1 + 2x_2 - x_3 = 7 \implies \boxed{x_1 = 2}$$

Example 6 :

Solve the system :

$$x_1 + 2x_2 - x_3 = 2$$

$$2x_1 + x_2 + 4x_3 = 16$$

Solution :

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & 4 & 16 \end{array}\right] -2R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 6 & 12 \end{array}\right] \frac{-1}{3}R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & -2 & -4 \end{array}\right]$$

$$x_2 - 2x_3 = -4 \implies \boxed{x_2 = 2x_3 - 4}$$

$$x_1 + 2x_2 - x_3 = 2 \implies \boxed{x_1 = 10 - 3x_3}$$

Let $x_3 = 3 \implies x_2 = 2 \implies x_1 = 1$

Gauss – Jordan Method

This method transforms the matrix A to a Diagonal matrix using two techniques :

First : is the Gauss Method techniques whish we explained earlier .

Second : is Jordan techniques , which zeros the remaining elements in the upper triangle (above the main diagonal)

a) Use a_{33} to zeros the elements below it (a_{13} , a_{23})

b) Use a_{22} to zeros the elements below it (a_{12})

c) The produced matrix is Diagonal and solved by bijective .

Example 7 :

Solve the system :

$$x_1 + x_2 + x_3 = 1$$

$$2x_1 - 2x_2 - 3x_3 = -3$$

$$5x_1 + x_2 - x_3 = 7$$

Solution :

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -2 & -3 & -3 \\ 5 & 1 & -1 & 7 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \\ -5R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -5 & -5 \\ 0 & -4 & -6 & 2 \end{array}\right] -R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -5 & -5 \\ 0 & 0 & -1 & 7 \end{array}\right] \begin{array}{c} -5R_3 + R_2 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & -4 & 0 & -40 \\ 0 & 0 & -1 & 7 \end{array}\right] \frac{-1}{4} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & -1 & 7 \end{array}\right] -R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & -1 & 7 \end{array}\right]$$

$$x_1 = -2 \implies \boxed{x_1 = -2}$$

$$x_2 = 10 \implies \boxed{x_2 = 10}$$

$$-x_3 = 7 \implies \boxed{x_3 = -7}$$

Example 8 :

Solve the system :

$$3x_1 - x_2 + x_3 = 5$$

$$2x_1 + 3x_2 + 5x_3 = 28$$

$$x_1 - x_2 - x_3 = -5$$

Solution :

$$\left[\begin{array}{ccc|c} 3 & -1 & 1 & 5 \\ 2 & 3 & 5 & 28 \\ 1 & -1 & -1 & -5 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -5 \\ 2 & 3 & 5 & 28 \\ 3 & -1 & 1 & 5 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \\ -3R_1 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -5 \\ 0 & 5 & 7 & 38 \\ 0 & 2 & 4 & 20 \end{array}\right] \frac{-2}{5} R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -5 \\ 0 & 5 & 7 & 38 \\ 0 & 0 & \frac{6}{5} & \frac{24}{5} \end{array}\right] \frac{5}{6} R_3$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -5 \\ 0 & 5 & 7 & 38 \\ 0 & 0 & 1 & 4 \end{array}\right] \begin{array}{c} -7R_3 + R_2 \\ R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right] \frac{1}{5}R_2 + R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 5 & 0 & 10 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$x_1 = 1 \implies \boxed{x_1 = 1}$

$5x_2 = 10 \implies \boxed{x_2 = 2}$

$x_3 = 4 \implies \boxed{x_3 = 4}$

Example 9 :

Solve the system :

$$2x_1 + 3x_2 + 2x_3 = 2$$

$$3x_1 + x_2 + x_3 = 5$$

$$x_1 - 2x_2 - x_3 = 1$$

Solution :

$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 2 \\ 3 & 1 & 1 & 5 \\ 1 & -2 & -1 & 1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 3 & 1 & 1 & 5 \\ 2 & 3 & 2 & 2 \end{array}\right] \begin{matrix} -3R_1 + R_2 \\ -2R_1 + R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 7 & 4 & 2 \\ 0 & 7 & 4 & 0 \end{array}\right] - R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 7 & 4 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The system dont have any solution .

Inverse Matrix Using Gauss – Jordan

The summary of this method is to transform the extended matrix $(A|I)$ to the another extended matrix $(I|B)$ by performing the gauss and jordan steps for rowsas explained earlier , such that A is the cofactor matrix , I is the identity element and B is the inverse of the matrix A and denoted by A^{-1} .

Example 10 :

Find inverse matrix if available :

$$A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$

Solution :

$$[A \vdots I] = \left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \frac{-2}{3}R_1 + R_2$$

$$\left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 0 & \frac{1}{3} & \frac{-2}{3} & 1 \end{array}\right] 3R_2$$

$$\left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] -7R_2 + R_1$$

$$\left[\begin{array}{cc:cc} 3 & 0 & 15 & -21 \\ 0 & 1 & -2 & 1 \end{array}\right] \frac{1}{3}R_1$$

$$\left[\begin{array}{cc:cc} 1 & 0 & 5 & -7 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 1 \end{bmatrix}$$

To check the result :

$$AA^{-1} = I$$

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 11 :

Find inverse matrix if available :

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix}$$

Solution :

$$[A \vdots I] = \left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] -2R_1 + R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 0 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right] \frac{3}{5}R_2 + R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 0 & 1 & 0 \\ 0 & 0 & \frac{-3}{5} & -2 & \frac{3}{5} & 1 \end{array}\right] 5R_3$$

$$\left[\begin{array}{ccc:ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 5 & -1 & 0 & 1 & 0 \\ 0 & 0 & -3 & -10 & 3 & 5 \end{array}\right] \begin{array}{l} \frac{-1}{3}R_3 + R_2 \\ \frac{2}{3}R_3 + R_1 \end{array}$$

$$\left[\begin{array}{ccc:ccc} 1 & 3 & 0 & \frac{-17}{3} & 2 & \frac{10}{3} \\ 0 & 5 & 0 & \frac{10}{3} & 0 & \frac{-5}{3} \\ 0 & 0 & -3 & -10 & 3 & 5 \end{array}\right] \frac{-3}{5}R_2 + R_1$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{-23}{3} & 2 & \frac{13}{3} \\ 0 & 5 & 0 & \frac{10}{3} & 0 & \frac{-5}{3} \\ 0 & 0 & -3 & -10 & 3 & 5 \end{array}\right] \begin{array}{l} \frac{1}{5}R_2 \\ \frac{-1}{3}R_3 \end{array}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{-23}{3} & 2 & \frac{13}{3} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{10}{3} & -1 & \frac{-5}{3} \end{array}\right] \begin{array}{l} \frac{1}{5}R_2 \\ \frac{-1}{3}R_3 \end{array}$$

$$A^{-1} = \begin{bmatrix} \frac{-23}{3} & 2 & \frac{13}{3} \\ \frac{2}{3} & 0 & \frac{-1}{3} \\ \frac{10}{3} & -1 & \frac{-5}{3} \end{bmatrix}$$

To check the result :

$$AA^{-1} = I$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \frac{-23}{3} & 2 & \frac{13}{3} \\ \frac{2}{3} & 0 & \frac{-1}{3} \\ \frac{10}{3} & -1 & \frac{-5}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

CHECK POINT

Example H.W :

1) Solve the following systems using gauss Method and using Gauss - Jordan Method

1)

$$2x_1 + 3x_2 - x_3 = 1$$

$$3x_1 + x_2 - x_3 = 0$$

$$x_1 + x_2 + 4x_3 = 18$$

2)

$$3x_2 - x_1 + x_3 = -1$$

$$4x_1 + x_3 + x_2 = 1$$

$$5x_1 + 2x_2 - x_3 = -4$$

3)

$$4x_1 + x_3 = 8$$

$$x_2 + 2x_3 = 11$$

$$x_1 + 3x_2 = 10$$

4)

$$x_1 + 2x_2 + x_3 = 12$$

$$x_1 - 2x_2 + x_3 = 0$$

$$x_1 + x_2 - x_3 = -1$$

2) Find inverse matrix using Gauss - Jordan Method if available

1)

$$A=\begin{bmatrix}1 & 3 & 4\\ 1 & -5 & 0\\ 3 & -2 & 4\end{bmatrix}$$

2)

$$A=\begin{bmatrix}3 & 2 & -1\\ 6 & 2 & 0\\ 5 & 0 & 1\end{bmatrix}$$

3)

$$A=\begin{bmatrix}1 & -2 & 4\\ 2 & 0 & 8\\ -1 & -3 & -4\end{bmatrix}$$

Linear Differential Equation (1^{st} order 1^{st} degree) L·D·E

The general form of L·D·E from 1^{st} order and 1^{st} degree can be written as :

First case (x is independent variables)

$$\frac{dy}{dx} + p(x)y = Q(x)$$

And this equation can solved by finding integral cofactor I such that :

$$I = e^{\int p(x)dx}$$

And the solution of L·D·E is :

$$y.I = \int Q(x).I\,dx$$

second case (y is independent variables)

$$\frac{dx}{dy} + g(y)x = h(y)$$

And this equation can solved by finding integral cofactor I such that :

$$I = e^{\int g(y)dy}$$

And the solution of L·D·E is :

$$x.I = \int h(y).I\,dy$$

Example 1 :

Solve the D.E : $\frac{dy}{dx} + \frac{y}{x} = x^2$

Solution :

$$p(x) = \frac{1}{x} \quad , \quad Q(x) = x^2$$

$$I = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{Lnx} = x$$

$$yI = \int Q(x)I dx$$

$$yx = \int x^2 x \, dx$$

$$yx = \frac{x^4}{4} + C$$

$$y = \frac{x^3}{4} + \frac{C}{x}$$

Example 2 :

Solve the D.E : $xdy + ydx = xsinx^2 \, dx$

Solution :

By dividing on dx

$$x\frac{dy}{dx} + y = xsinx^2$$

$$\frac{dy}{dx} + \frac{y}{x} = sinx^2$$

$$p(x) = \frac{1}{x} \quad , \quad Q(x) = sinx^2$$

$$I = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{Lnx} = x$$

$$yI = \int Q(x)Idx$$

$$yx = \int xsinx^2\, dx$$

$$yx = \frac{-1}{2}cosx^2 + C$$

$$y = \frac{-1}{2x}cosx^2 + \frac{C}{x}$$

Example 3 :

Solve the D.E : $dy + 2xydx = xe^{-x^2}\, dx$

Solution :

By dividing on dx

$$\frac{dy}{dx} + 2xy = xe^{-x^2}$$

$$p(x) = 2x \quad , \quad Q(x) = xe^{-x^2}$$

$$I = e^{\int p(x)dx} = e^{\int 2xdx} = e^{x^2}$$

$$yI = \int Q(x) I dx$$

$$ye^{x^2} = \int e^{x^2} x e^{-x^2}\, dx$$

$$ye^{x^2} = \int x dx$$

$$ye^{x^2} = \frac{x^2}{2} + C \rightarrow y = \frac{x^2 e^{-x^2}}{2} + C\, e^{-x^2}$$

Example 4 :

Solve the D.E : $y\frac{dx}{dy} + 2x = y^3$

Solution :

By dividing on y

$$\frac{dx}{dy} + \frac{2x}{y} = y^2$$

$$g(y) = \frac{2}{y} \quad , \quad h(y) = y^2$$

$$I = e^{\int g(y)dy} = e^{\int \frac{2}{y} dy} = e^{2Lny} = e^{Lny^2} = y^2$$

$$xI = \int h(y) I dy$$

$$xy^2 = \int y^2 y^2\, dy$$

$$xy^2 = \frac{y^5}{5} + C$$

$$x = \frac{y^3}{5} + \frac{C}{y^2}$$

Example 5 :

Solve the D·E : $\frac{dx}{dy} + 2xy = 4y$

Solution :

$$g(y) = 2y \quad , \quad h(y) = 4y$$

$$I = e^{\int g(y)dy} = e^{\int 2ydy} = e^{y^2}$$

$$xI = \int h(y)Idy$$

$$xe^{y^2} = \int 4ye^{y^2}\, dy$$

$$xe^{y^2} = 2e^{y^2} + C$$

$$x = 2 + Ce^{-y^2}$$

CHECK POINT

Example H.W :

Solve the following D.E

1) $y' + y = sinx$

2) $x\frac{dy}{dx} - 2y = x^3 cos4x$

3) $x\frac{dy}{dx} = y + x^3 + 3x^2 - 2x$

4) $\frac{dx}{dy} + x = 4cos2y$

5) $\frac{dy}{dx} + ycotx = 5e^{cosx}$

Bernoli Equation

The general form to bernoli equation is :

$$\frac{dy}{dx} + p(x)y = Q(x)\, y^n \quad ,n \neq 1 \qquad \cdots\cdots(1)$$

Such that p and Q functions for x only .

method of solution :

transform eq(1) to L.D.E by multiplying by y^{-n}

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = Q(x) \qquad \cdots\cdots(2)$$

Let $z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \;\; \Rightarrow \frac{1}{1-n}\,\frac{dz}{dx} = y^{-n}\frac{dy}{dx} \quad \cdots\cdots(3)$

Subisitituting eq(3) in eq(2) we get :

$$\frac{1}{1-n}\frac{dz}{dx}+p(x)z=Q(x)$$

The last equation is L·D·E and its solution is :

$$z.I=\int Q(x).I\,dx$$

By the same method ,The general form to bernoli equation is :

$$\frac{dx}{dy}+g(y)x=h(y)\,x^{n}\quad ,n\neq 1 \qquad \cdots\cdots(1)$$

Such that g and h functions for y only .

method of solution :

transform eq(1) to L·D·E by multiplying by x^{-n}

$$x^{-n}\frac{dx}{dy}+g(y)x^{1-n}=h(y) \qquad \cdots\cdots(2)$$

Let $z=x^{1-n}\Rightarrow \frac{dz}{dy}=(1-n)x^{-n}\frac{dx}{dy}\ \Rightarrow \frac{1}{1-n}\frac{dz}{dy}=x^{-n}\frac{dx}{dy}$ ····(3)

Subisitituting eq(3) in eq(2) we get :

$$\frac{1}{1-n}\frac{dz}{dy}+g(y)z=h(y)$$

The last equation is L·D·E and its solution is :

$$z.I=\int h(y).I\,dy$$

Example 6 :

Solve the D·E : $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Solution :

multiplying by (-1)

$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$ .....(1)

Eq(1) is bernoli equation , multiply (1) by y^{-3}

$y^{-3}\frac{dy}{dx} - xy^{-2} = -e^{-x^2}$ .....(2)

Let $z = y^{-2} \Longrightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \Longrightarrow \frac{-1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx}$ .....(3)

Subistitute eq(3) in eq(2) we get :

$$\frac{-1}{2}\frac{dz}{dx} - xz = -e^{-x^2}$$

Multiply by (-2)

$\frac{dz}{dx} + 2xz = 2e^{-x^2}$ is L·D·E

$$p(x) = 2x \quad , \quad Q(x) = 2e^{-x^2}$$

$$I = e^{\int p(x)dx} = e^{\int 2x dx} = e^{x^2}$$

$$zI = \int Q(x) I dx$$

$$ze^{x^2} = \int 2e^{-x^2}e^{x^2}\,dx$$

$$ze^{x^2} = \int 2dx$$

$$ze^{x^2} = 2x + C \implies z = 2xe^{-x^2} + C\,e^{-x^2}$$

Example 7 :

Solve the D.E : $\frac{dy}{dx} - y = -xy^5$

Solution :

$\frac{dy}{dx} - y = -xy^5$ .....(1)

Eq(1) is bernoli equation , multiply (1) by y^{-5}

$y^{-5}\frac{dy}{dx} - y^{-4} = -x$ .....(2)

Let $z = y^{-4} \implies \frac{dz}{dx} = -4y^{-5}\frac{dy}{dx} \implies \frac{-1}{4}\frac{dz}{dx} = y^{-5}\frac{dy}{dx}$ .....(3)

Subistitute eq(3) in eq(2) we get :

$$\frac{-1}{4}\frac{dz}{dx} - z = -x$$

Multiply by (-4)

$\frac{dz}{dx} + 4z = 4x$ is L.D.E

$$p(x) = 4 \;,\; Q(x) = 4x$$

$$I = e^{\int p(x)dx} = e^{\int 4dx} = e^{4x}$$

$$zI = \int Q(x)Idx$$

$$ze^{4x} = \int 4x\, e^{4x}\, dx$$

$$ze^{4x} = xe^{4x} - \frac{1}{4}e^{4x} + C$$

$$\frac{e^{4x}}{y^4} = xe^{4x} - \frac{1}{4}e^{4x} + C$$

Example 8 :

Solve the D.E : $dx - xdy = yx^2dy$

Solution :

Dividing by dy

$\frac{dx}{dy} - x = yx^2$ .....(1)

Eq(1) is bernoli equation , multiply (1) by x^{-2}

$x^{-2}\frac{dx}{dy} - x^{-1} = y$ .....(2)

Let $z = x^{-1} \Rightarrow \frac{dz}{dy} = -x^{-2}\frac{dx}{dy} \Rightarrow -\frac{dz}{dy} = x^{-2}\frac{dx}{dy}$ .....(3)

Subisititute eq(3) in eq(2) we get :

$$-\frac{dz}{dy}-z=y$$

Multiply by (-1)

$\frac{dz}{dy}+z=-y$ *is L·D·E*

$$g(y)=1 \;\;,\;\; h(y)=-y$$

$$I=e^{\int g(y)dy}=e^{\int dy}=e^{y}$$

$$zI=\int h(y)I dy$$

$$ze^{y}=\int -y\,e^{y}\,dy$$

$$ze^{y}=-(y\,e^{y}-\;e^{y})+C$$

$$\frac{e^{y}}{x}=-y\,e^{y}+\;e^{y}+C$$

Example 9 :

Solve the D·E : $dx-2xydy=6x^3y^2e^{-2y^2}dy$

Solution :

Dividing by dy

$\frac{dx}{dy}-2xy=6x^3y^2e^{-2y^2}$ ·····(1)

Eq(1) is bernoli equation , multiply (1) by x^{-3}

$$x^{-3}\frac{dx}{dy} - 2x^{-2} = 6y^2e^{-2y^2} \quad \cdots\cdots(2)$$

Let $z = x^{-2} \Longrightarrow \frac{dz}{dy} = -2x^{-3}\frac{dx}{dy} \Longrightarrow \frac{-1}{2}\frac{dz}{dy} = x^{-3}\frac{dx}{dy}$ $\cdots\cdots(3)$

Subistitute eq(3) in eq(2) we get :

$$\frac{-1}{2}\frac{dz}{dy} - 2zy = 6y^2e^{-2y^2}$$

Multiply by (-2)

$\frac{dz}{dy} + 4zy = -12y^2e^{-2y^2}$ is L·D·E

$$g(y) = 4y \quad , \quad h(y) = -12y^2e^{-2y^2}$$

$$I = e^{\int 4y dy} = e^{2y^2}$$

$$zI = \int h(y)I dy$$

$$ze^{2y^2} = \int -12y^2e^{-2y^2}e^{2y^2}\,dy$$

$$ze^{2y^2} = \int -12y^2\,dy$$

$$ze^{2y^2} = -4y^3 + C$$

$$x^{-2}e^{2y^2} = -4y^3 + C$$

CHECK POINT

Example H·W :

Solve the following D·E

1) $xdy + ydx = x^3 y^6\, dx$

2) $\frac{dy}{dx} + xy = 6x\sqrt{y}$

3) $\frac{dy}{dx} + y = y^3$

4) $dx + xdy = x^2 e^y\, dy$

5) $\frac{dx}{dy} - \frac{x}{2y} = \frac{-1}{2}(cosy)x^3$

2nd Order Differential Equation (1st order 1st degree) L·D·E

The general form of D·E from 2nd order can be written as :

$$f(x, y, y', y'') = C$$

Reduction to 1st order Method

The type of D·E can be solved by transform it to 1st order by missing variables x or y :

First case (y is missing variables)

$$y' = p \Rightarrow y'' = \frac{dp}{dx}$$

And this equation will transform to the 1st order D·E and solving it for x , p , then back y'=p and solve it for x and y ·

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

second case (x is missing variables)

$$y' = p \Rightarrow y'' = p\frac{dp}{dy}$$

And this equation will transform to the 1st order D·E and solving it for y , p , then back y'=p and solve it for x and y ·

Example 1 :

Solve the D·E : $xy'' = y'$

Solution :

Let $y' = p \Rightarrow y'' = \frac{dp}{dx}$

$$x\frac{dp}{dx} = p$$

$$\frac{dp}{p} = \frac{dx}{x} \quad , by \int$$

$$Ln(p) = Ln(x) + Ln(c_1)$$

$$Ln(p) = Ln(xc_1)$$

$$p = xc_1$$

$$\frac{dy}{dx} = xc_1$$

$$dy = xc_1 dx \quad , \quad by \int$$

$$y = \frac{x^2}{2}c_1 + c_2$$

Example 2 :

Solve the D·E : $yy'' + (y')^3 = 0$

Solution :

Let $y' = p \Longrightarrow y'' = p\frac{dp}{dy}$

$$yp\frac{dp}{dy} + p^3 = 0$$

$$p(y\frac{dp}{dy} + p^2) = 0$$

$$p = 0 \Longrightarrow \frac{dy}{dx} = 0 \Longrightarrow y = c$$

$$y\frac{dp}{dy}+p^2=0$$

$$y\frac{dp}{dy}=-p^2$$

$$\frac{dy}{y}=-\frac{dp}{p^2} \quad ,by \int$$

$$Ln(y)=\frac{1}{p}+c_1$$

$$\frac{1}{p}=Ln(y)+c_1$$

$$\frac{dx}{dy}=Ln(y)+c_1$$

$$dx=(Ln(y)+c_1\,)dy \quad , \quad by \int$$

$$x=\int(Ln(y)+c_1\,)dy$$

$$x=y\,Ln(y)-y+c_1y\;+\;\;c_2\;y=\frac{x2}{2}c_1+c_2$$

Example 3 :

Solve the D·E : $x^2y''-(y')^2-2xy'=0$

Solution :

Let $y'=p\;\Rightarrow y''=\frac{dp}{dx}$

$$x^2\frac{dp}{dx} - p^2 - 2xp = 0$$

$$x^2\frac{dp}{dx} - 2xp = p^2 \qquad , \quad \% x^2$$

$$\frac{dp}{dx} - \frac{2p}{x} = \frac{p^2}{x^2} \qquad \textit{is Bernoli eq. ,} \qquad \textit{multiply by } p^{-2}$$

$$p^{-2}\frac{dp}{dx} - \frac{2}{x}p^{-1} = \frac{1}{x^2}$$

$$\text{Let} \quad z = p^{-1} \Rightarrow \frac{dz}{dx} = -p^{-2}\frac{dp}{dx} \quad \Rightarrow -\frac{dz}{dx} = p^{-2}\frac{dp}{dx}$$

$$-\frac{dz}{dx} - \frac{2}{x}z = \frac{1}{x^2}$$

$$\frac{dz}{dx} + \frac{2}{x}z = -\frac{1}{x^2}$$

$$p(x) = \frac{2}{x} \quad , \quad Q(x) = -\frac{1}{x^2}$$

$$I = e^{\int \frac{2}{x}dx} = e^{2Lnx} = x^2$$

$$zI = \int Q(x) I dx$$

$$z\,x^2 = \int -\frac{1}{x^2}x^2\,dx$$

$$z\,x^2 = -x + c_1$$

$$\frac{dx}{dy}x^2 = -x + c_1$$

$$x^2dx = (-x + c_1)dy$$

$$\frac{x^2}{-x+c_1}dx = dy$$

$$\boxed{\frac{x^2}{-x+c_1} = -x - c_1 + \frac{c_1^2}{-x+c_1}}$$

$$\left(-x - c_1 + \frac{c_1^2}{-x+c_1}\right)dx = dy \ , \ \ by \int$$

$$-\frac{x^2}{2} - c_1x + c_1^2 Ln|-x+c_1| = y + c_2$$

Example 4 :

Solve the D.E : $yy'' + 2y' - 2(y')^2 = 0$

Solution :

Let $y' = p \Rightarrow y'' = p\frac{dp}{dy}$

$$yp\frac{dp}{dy} + 2p - 2p^2 = 0$$

$$p(y\frac{dp}{dy} + 2 - 2p) = 0$$

$$p = 0 \ \Rightarrow \frac{dy}{dx} = 0 \ \Rightarrow y = c$$

$$y\frac{dp}{dy} + 2 - 2p = 0$$

$$y\frac{dp}{dy} = 2(p-1)$$

$$ydp = 2(p-1)dy$$

$$\frac{dp}{p-1} = 2\frac{dy}{y} \ \ ,by \int$$

$$Ln(p-1) = Ln(y^2 c_1)$$

$$p - 1 = y^2 c_1$$

$$p = 1 + y^2 c_1$$

$$\frac{dy}{dx} = 1 + y^2 c_1$$

$$dx = \frac{1}{1 + y^2 c_1} dy \quad , \ by \int$$

$$x = \frac{1}{\sqrt{c_1}} tan^{-1}(\sqrt{c_1} y) + c_2$$

Note :

In some untegrals the solve being hard then re-write the function as a series to integrate it easily

Example 5 :

Solve the D·E : $y'' - 2xy' + x = 0$

Solution :

Let $y' = p \Rightarrow y'' = \frac{dp}{dx}$

$$\frac{dp}{dx} - 2xp + x = 0$$

$$\frac{dp}{dx} - 2px = -x \qquad is\ L.D.E\ eq.$$

$$p(x) = -2x \quad , \quad Q(x) = -x$$

$$I = e^{\int -2x dx} = e^{-x^2}$$

$$pI = \int Q(x) I dx$$

$$p\, e^{-x^2} = \int -x e^{-x^2}\, dx$$

$$p\, e^{-x^2} = \frac{1}{2} e^{-x^2} + c_1$$

$$p = \frac{1}{2} + c_1 e^{x^2}$$

$$\frac{dy}{dx} = \frac{1}{2} + c_1 \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \right)$$

$$dy = \frac{1}{2} dx + c_1 \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \right) dx \quad , \qquad by \int$$

$$y = \frac{x}{2} + c_1 \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \right) + c_2$$

Example 6 :

Solve the D·E : $y'' - 3x^2y' = x^2$

Solution :

Let $y' = p \Longrightarrow y'' = \frac{dp}{dx}$

$$\frac{dp}{dx} - 3x^2p = x^2$$

$$p(x) = -3x^2 \quad , \quad Q(x) = x^2$$

$$I = e^{\int -3x^2dx} = e^{-x^3}$$

$$pI = \int Q(x)Idx$$

$$p\,e^{-x^3} = \int x^2e^{-x^3}\,dx$$

$$p\,e^{-x^2} = -\frac{1}{3}e^{-x^2} + c_1$$

$$p = -\frac{1}{3} + c_1e^{-x^3}$$

$$\frac{dy}{dx} = -\frac{1}{3} + c_1\left(1 - \frac{x^3}{1!} + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\right)$$

$$dy = -\frac{1}{3}dx + c_1\left(1 - \frac{x^3}{1!} + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\right)dx \quad , \quad by \int$$

$$y = \frac{x}{2} + c_1\left(x - \frac{x^4}{4} + \frac{x^7}{14} - \frac{x^{10}}{60} + \cdots\right) + c_2$$

Note :

If the equation from the 3^{rd} order then :

$$y'' = q \quad , \quad y''' = \frac{dq}{dx}$$

Example 7 :

Solve the D·E : $xy''' - 2y'' = 0$

Solution :

Let $y'' = q \Rightarrow y''' = \frac{dq}{dx}$

$x\frac{dq}{dx} - 2q = 0$

$$x\frac{dq}{dx} = 2q$$

$$2\frac{dx}{x} = \frac{dq}{q} \quad , \quad by \int$$

$$Ln|x^2c_1| = Ln|q|$$

$$q = x^2c_1$$

$$y'' = x^2c_1 \Rightarrow y' = \frac{x^3}{3}c_1 + c_2 \Rightarrow y = \frac{x^4}{12}c_1 + c_2x + c_3$$

Example 8 :

Solve the D·E : $x^2y''' - 2xy'' = 0$

Solution :

Let $y'' = q \Rightarrow y''' = \frac{dq}{dx}$

$x^2\frac{dq}{dx} - 2xq = 0$

$$x^2\frac{dq}{dx} = 2xq$$

$$\frac{dq}{q} = \frac{2}{x}dx \quad , \qquad by \int$$

$$Ln|q| = L|x^2c_1|$$

$$q = x^2c_1$$

$$y'' = x^2c_1 \Rightarrow y' = \frac{x^3}{3}c_1 + c_2 \Rightarrow y = \frac{x^4}{12}c_1 + c_2x + c_3$$

CHECK POINT

Example H·W :

Solve the following D·E

1) $xy'' - y'^2 - 2y' = 0$

2) $y^3y'' - 3y^4y' = 0$

3) $yy'' - 3y = 0$

4) $y^2y'' - 2y' = 0$

5) $xy'' + 2xy' = x^3$

6) $y'' + 2y' = 4x$

7) $1 + yy'' + (y')^2 = 0$