NUMERICAL ANALYSIS

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Lecture 5

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NUMERECAL DIFFERENTIATION AND INTEGRATION

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Introduction

We were finding the polynomial curve ***y = f(x)***, passing through the (***n+1***) ordered pairs (x_i, y_i), i= 0, 1, 2, …n. Now we are trying to find the derivative value of such curves at given $x=x_e$. Where $x_0 < x_e < x_n$ (or even outside the range but closer to starting or end values). To get derivative, we first find the curve ***y = f(x)*** through the points and then differentiate and get its value at the required point.

If the values of x are equally spaced, we get the interpolating polynomial due to Newton-Gregory. If the derivative is required at **point near the starting** value in the table we use **Newton's forward** interpolation formula. If the require the derivative at the **end of the table** we use **Newton's backward** interpolation formula. If the value of derivative is required **near the middle** of the table value we use on of the central difference interpolation formulae.

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Newton's forward difference formula to get the derivative

We are given (n+1) ordered pairs (x_i, y_i) $i = 0,1,2,\ldots\ldots, n$. we want to find the derivative of $y= f(x)$ passing through the n+1 points, at a point near to the starting value $x=x_o$.
Newton's forward difference interpolation formula

$$y(x_0 + uh) = y_u = y_0 + u\,\Delta y_0 + \frac{u\,(u-1)}{2!}\Delta^2 y_0 + \frac{u\,(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots, \qquad \ldots\ldots.(1)$$

Where ***y(x)*** is a polynomial of degree ***n*** in ***x*** and $u = \frac{x-x_0}{h}$

Differentiating ***y(x)*** with respect to ***x***

$$\frac{d_y}{d_x} = \frac{d_y}{d_u}\cdot\frac{d_u}{d_x} = \frac{1}{h}\cdot\frac{d_y}{d_u}$$

$$\frac{d_y}{d_x} = \frac{1}{h} + \left[\Delta y_0 + \frac{2u-1}{2}\,\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\,\Delta^3 y_0 + \frac{(4\,u^3 - 18\,u^2 + 22u - 6}{24}\,\Delta^4 y_0 + \cdots\right] \qquad \ldots\ldots.(2)$$

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Equation (2) gives the value of $\frac{d_y}{d_x}$ at general $\boldsymbol{x}$ which may be anywhere in the interval.

In special case like $\boldsymbol{x=x_0}$ i.e., u = 0, equation (2) reduces to

$$\boxed{\left(\frac{dy}{dx}\right)_{x=x_0} = \left(\frac{dy}{dx}\right)_{u=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\dots\dots..\right]} \quad \dots\dots\dots..(3)$$

Differentiating (2) again with respect to $\boldsymbol{x}$,

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right).\frac{du}{dx} = \frac{d}{du}\left(\frac{dy}{dx}\right).\frac{1}{h}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{(6u^2-18u+11)}{12}\Delta^4 y_0 + \cdots\right] \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..(4)$$

Equation (4) give the second derivative value at x=x. Setting x = x_0 i.e. u = 0 in (4)

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\dots\dots..\right]} \quad \dots\dots\dots\dots\dots..(5)$$

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Newton's backward difference formula to compute the derivative

Now consider Newton's backward difference interpolation formula

$$y(x) = y(x_n + vh) = yn + v\,\nabla y_n + \frac{v\,(v+1)}{2!}\,\nabla^2 y_n + \frac{v\,(v+1)(v+2)}{3!}\,\nabla^3 y_n + \ldots\ldots\ldots\ldots \qquad \text{.............(6)}$$

Where $\quad v = \frac{x - x_n}{h}$

Differentiate (8) with respect to x

$$\frac{d_y}{d_x} = \frac{d_y}{d_v}\cdot\frac{d_v}{d_x} = \frac{1}{h}\cdot\frac{d_y}{d_v}$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\nabla y_n + \frac{2v+1}{2}\nabla^2 y_n + \frac{3v^2+6v+2}{6}\,\nabla^3 y_n + \frac{4v^3+18v^2+22v+6}{24}\,\nabla^4 y_n + \ldots\ldots...\right] \text{.............(7)}$$

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\,\nabla^4 y_n + \ldots\ldots\ldots\right] \qquad \text{..........................(8)}$$

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Equations (7 and 8) give the first and second derivative at any general x.
Setting x = x_n or v=0 in (7 and 8) we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \ldots\ldots\ldots..\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \frac{5}{6}\nabla^5 y_0 + \frac{137}{180}\nabla^6 y_0\right]$$

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Example

Find the first and two derivatives of $(x)^{1/3}$ at x = 50 and x = 56 given in the table below:

x	50	51	52	53	54	55	56
y=x^1/3	3.684	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution:

Sine we require $\left(\frac{dy}{dx}\right)$ at x = 50 we use Newton's forward formula and to get $\left(\frac{dy}{dx}\right)$ at x = 56 we use Newton's backward formula.

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x	y	$1st\ diff.$	$2nd\ diff.$	$3rd\ diff.$	$4th\ diff.$
50	3.6840				
		0.0244			
51	3.7084		-0.0003		
		0.0241		0	
52	3.7325		-0.0003		
		0.0238		0	
53	3.7563		-0.0003		
		0.0235		0	
54	3.7798		-0.0003		
		0.0232		0	
55	3.8030		-0.0003		
		0.0229			
56	3.8259				

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By Newton's forward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\ldots..\right]$$

$$\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)_{\mathrm{x}=50} = \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}\,(0)\right] = 0.02455$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\ldots\ldots..\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = 1\,[-0.0003] = -0.0003$$

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By Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \dots\dots\dots..\right]$$

$$\left(\frac{dy}{dx}\right)_{x=56} = \frac{1}{1}\left[0.0229 + \frac{1}{2}(-0.0003) + 0\right] = 0.02275$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \frac{5}{6}\nabla^5 y_0 + \frac{137}{180}\nabla^6 y_0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=56} = \frac{1}{1}[-0.0003] = -0.0003$$

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Example

From the given table of values of x and y obtain:

$$(i) \quad \frac{dy}{dx} \; and \; \frac{d^2y}{dx^2} \; at \; x = 1.2$$

$$(ii) \quad \frac{dy}{dx} \; and \; \frac{d^2y}{dx^2} \; at \; x = 2.2$$

x	1	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

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x	y	$1st\ diff.$	$2nd\ diff.$	$3rd\ diff.$	$4th\ diff.$	$5th\ diff.$	$6th\ diff.$
1	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

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$$(i) \quad \left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}\times 0.1627 + \frac{1}{3}\times 0.0361\right.$$

$$\left. - \frac{1}{4}\times 0.008 + \frac{1}{5}\times 0.0014\right] = 3.3203$$

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$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \dots\dots\dots..\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = \frac{1}{(0.2)^2}\left[0.1627 - 0.0361 + \frac{11}{12}\times 0.008 - \frac{5}{6}\times 0.0014\right]$$

$$= 3.319$$

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$$(ii)\quad \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \frac{1}{5}\nabla^5 y_0 + \frac{1}{6}\nabla^6 y_0\right]$$

$$= \frac{1}{0.2}\left[1.6359 + \frac{1}{2}\times 0.2964 + \frac{1}{3}\times 0.0535 + \frac{1}{4}\times 0.0094\right.$$

$$\left. + \frac{1}{5}\times 0.0014 + \frac{1}{6}\times 0.0001\right] = 9.0229$$

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$$\left(\frac{d^2y}{dx^2}\right)_{x=2.2} = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \frac{5}{6}\nabla^5 y_0 + \frac{137}{180}\nabla^6 y_0\right]$$

$$= \frac{1}{(0.2)^2}\left[0.2964 + 0.0535 + \frac{11}{12}\times 0.0094 + \frac{5}{6}\times 0.0014\right.$$

$$\left. + \frac{137}{180}\times 0.0001\right] = 8.994$$

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Example

The distance moved by a particle along curve are recorded at intervals of 0.5 sec

$t\ (sec)$	4.5	5.0	5.5	6.0	6.5	7.0	7.5
$x\ (m)$	9.69	12.9	16.71	21.18	26.37	32.34	39.15

Find the velocity and acceleration at $t = 6\ sec$

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t	x	$1st\ diff.$	$2nd\ diff.$	$3rd\ diff.$	$4th\ diff.$	$5th\ diff.$	$6th\ diff.$
4.4	9.69						
		3.21					
5.0	12.9		0.6				
		3.81		0.06			
5.5	16.71		0.66		0		
		4.47		0.06		0	
6.0	21.18		0.72		0		0
		5.19		0.06		0	
6.5	26.37		0.78		0		
		5.97		0.06			
7.0	32.34		0.84				
		6.81					
7.5	39.15						

Newton's forward difference formula

$$V=\left(\frac{dy}{dx}\right)_{x=6}=\frac{1}{h}\left[\Delta y_0-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0\right]$$

$$=\frac{1}{0.5}\left[5.19-\frac{1}{2}\times 0.78+\frac{1}{3}\times 0.06\right]=9.64$$

$$A=\left(\frac{d^2y}{dx^2}\right)_{x=6}=\frac{1}{h^2}\left[\Delta^2 y_0-\Delta^3 y_0\right]$$

$$=\frac{1}{(0.5)^2}[0.78-0.06]=2.88$$

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Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \dots\dots\dots..\right]$$

$$= \frac{1}{0.5}\left[4.47 + \frac{1}{2}\times 0.66 + \frac{1}{3}\times 0.06\right] = 9.64$$

$$A = \left(\frac{d^2y}{dx^2}\right)_{x=6} = \frac{1}{h^2}\left[\Delta^2 y_0 + \Delta^3 y_0\right]$$

$$= \frac{1}{(0.5)^2}[0.66 + 0.06] = 2.88$$

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