NUMERICAL ANALYSIS

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Lecture 6

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NUMERECAL INTEGRATION

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Numerical Integration

Introduction

We know that $\int_a^b f(x)dx$ represents the area between y=*f*(x), x-axis and the ordinates x=a and x=b. This integration is possible only if the *f*(x) is explicitly and if it is integral. The problem of numerical integration can be stated as follows: Given a set of (n+1) paired values (x_i, y_i), *i*= 0,1,2,…….n of the function y=f(x). Where f(x) is not known explicitly, it is required to compute $\int_{x_0}^{x_n} y dx$.

1- Trapezoidal Rule

In Calculus, "**Trapezoidal Rule**" is one of the important integration rules. The name trapezoidal is because when the area under the curve is evaluated, then the total area is divided into small trapezoids instead of rectangles. This rule is used for approximating the definite integrals where it uses the linear approximations of the functions.
The trapezoidal rule is mostly used in the numerical analysis process. To evaluate the definite integrals, where we use small rectangles to evaluate the area under the curve

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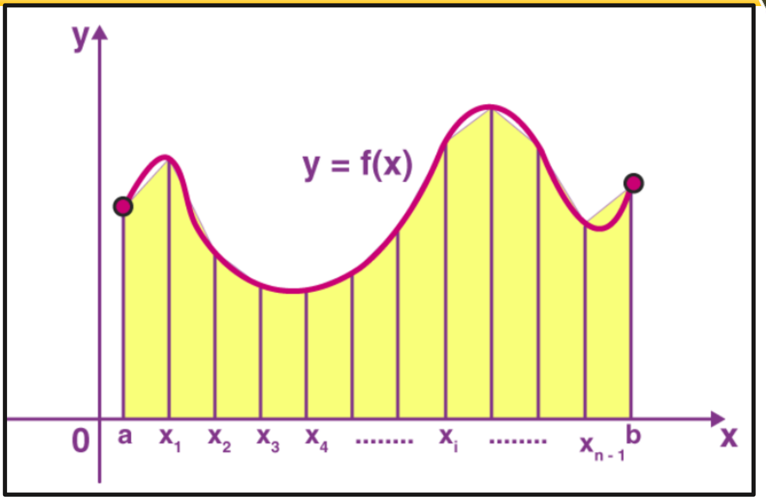
Trapezoidal Rule is a rule that evaluates the area under the curves by dividing the total area into smaller trapezoids rather than using rectangles. This integration works by approximating the region under the graph of a function as a trapezoid, and it calculates the area. This rule takes the average of the left and the right sum.



$$\int_{x_0}^{x_n} f(x)\ dx = \frac{h}{2}\left[(y_0+y_n) + 2\,(y_1 + y_2 + y_3 + \ldots\ldots + y_{n-1})\right]$$

$$\int_{x_0}^{x_n} f(x)\ dx = \frac{h}{2}\left[\mathit{Sum\ of\ the\ first\ and\ last\ ordinates}) + 2(\mathit{Sum\ of\ the\ remaining\ ordinates})\right]$$

Frequently Asked Questions – FAQs

What is Trapezoidal Rule?

Trapezoidal Rule is an integration rule, in Calculus, that evaluates the area under the curves by dividing the total area into smaller trapezoids rather than using rectangles.

Why the rule is named after trapezoid?

The name trapezoidal is because when the area under the curve is evaluated, then the total area is divided into small trapezoids instead of rectangles.
Then we find the area of these small trapezoids in a definite interval.

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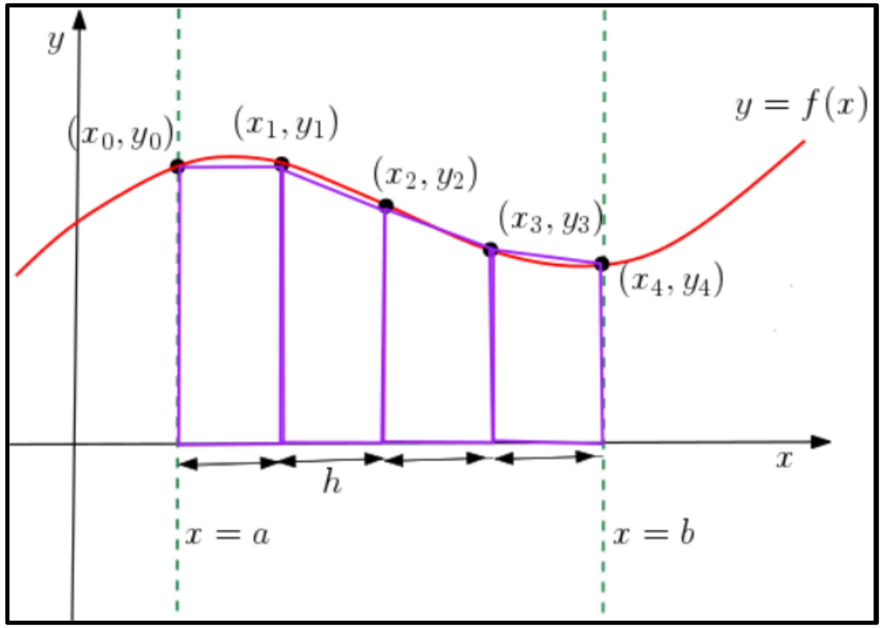
- Geometrical Interpolation

Geometrically, if the ordered pairs (x_i, y_i), i=0, 1, 2, …..n are plotted, and if any two consecutive points are joined by straight lines, we get the figure as shown.

The area between ***f(x)***, x-axis and ordinates $\boldsymbol{x=x_0}$ and $\boldsymbol{x=x_n}$ is approximated to the sum of the trapeziums as shown in figure.



Note:

Though this method is very simple for calculation purposes of numerical integration, the error in this case is significant. The accuracy of the results can be improved by increasing the number of intervals and decreasing the value of ***h***.

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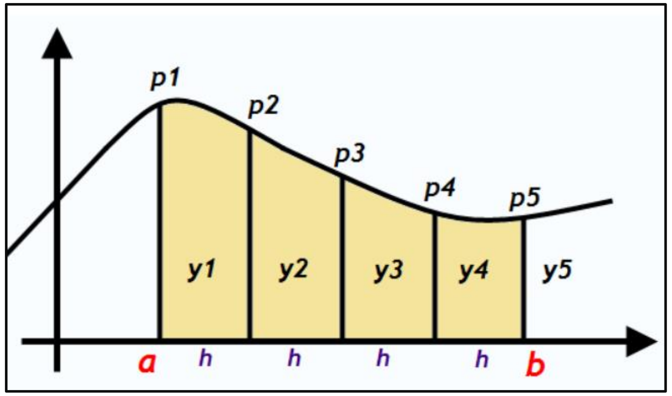
2- Simpsons Rule

Simpson’s rule methods are more accurate than the other numerical approximations and its formula for n+1 equally spaced subdivision

A- Simpsons one - third Rule



Simpson’s 1/3rd rule is an extension of the trapezoidal rule in which the integrand is approximated by a second-order polynomial. Simpson rule can be derived from the various way using Newton’s divided difference polynomial, Lagrange polynomial and the method of coefficients. Simpson’s 1/3 rule is defined by:

$$\int_{x_0}^{x_n} f(x)\ dx = \frac{h}{3}\left[(y_0+y_n) + 2\,(y_2 + y_4 + y_6 \ldots + y_{n-2}) + 4\,(y_1 + \ y_3 + y_5 \ldots + y_{n-1})\right]$$

$$\int_{x_0}^{x_n} f(x)\ dx = \frac{h}{3}[Sum\ of\ the\ first\ and\ last\ ordinates) + 2(Sum\ of\ the\ remaining\ odd\ ordinates) + 4(Sum\ of\ the\ even\ ordinates)]$$

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B - Simpsons three - eights Rule

Another method of numerical integration is called "Simpson's 3/8 rule". It is completely based on the cubic interpolation rather than the quadratic interpolation. Simpson's 3/8 or three-eight rule is given by:

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3\,(y_1 + y_2 + y_4 + y_5 + y_7 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots + y_n)\right]$$

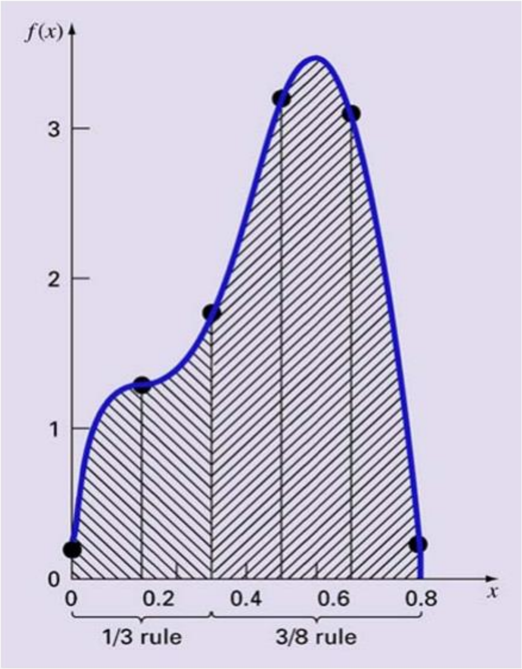

Illustration of how Simpson's 1/3 and 3/8 rules can be applied in tandem to handle multiple applications with odd numbers of intervals.

This rule is **more accurate** than the standard method, as it uses one more functional value. For 3/8 rule, the composite Simpson's 3/8 rule also exists which is similar to the generalized form. The 3/8 rule is known as Simpson's second rule of integration.

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Simpsons three - eights Rule applicable only when n is multiple of 3

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4- Weddle's rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[(y_0+5\,y_1 + \,y_2 + 6y_3 + y_4 + 5y_5) + \,(2y_6 + 5y_7 + \,y_8 + 6y_9 + y_{10} + 5y_{11}) + (2y_{n-6} + 5y_{n-5} + \,y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n)]$$

Note:

In above formula, the coefficient may remembered in groups of six.

First group: Coefficients : 1,5, 1,6, 1,5

All interior groups: Coefficients : 2,5, 1,6, 1,5

Last group: Coefficients :2,5, 1,6, 1,5, 1

Note: If there are only 7 ordinates, the coefficients are 1,5, 1,6, 1,5, 1.

Notes:

1- In Trapezoidal rule, y(x) is a linear function of x. The rule is the simplest one but it is least accurate.

2- In Simpson's one-third rule, y(x) is a polynomial of degree **two**. To apply this rule, ***n***, the number of **interval** must be ***even***. That is, the number of **ordinates** must be ***odd***.

3- In Simpson's third-eighths rule, y(x) is a polynomial of degree **three.** This rule is applicable if ***n***, number of **intervals** is multiple of 3.

4- In Weddle's rule, y(x) is a polynomial of degree ***six*** and this rule is applicable only if ***n***, number of **intervals,** is a multiple of six. A minimum number of 7 ordinates is necessary.

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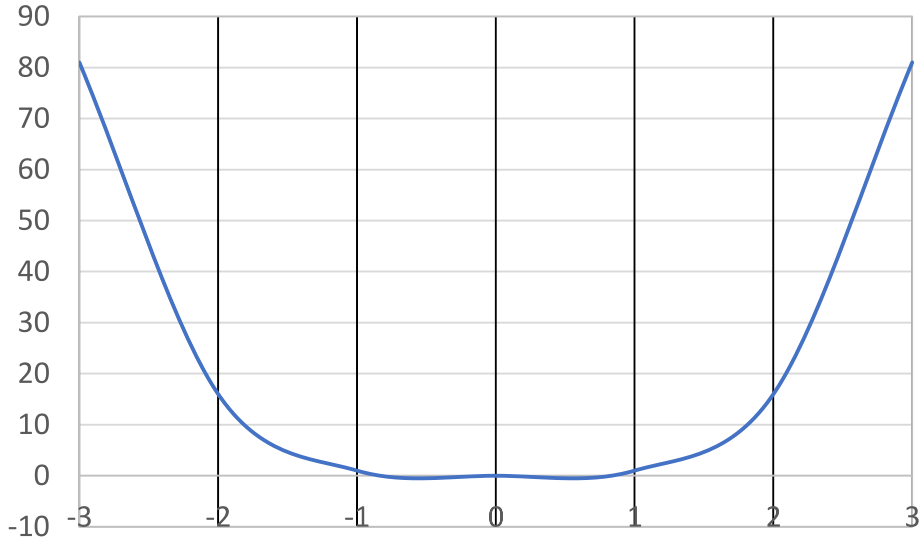
Example

$$Evaluate \int_{-3}^{3} x^4 \ dx \quad by\, using:$$

1) Trapezoidal Rule ,
2) Simpsons Rule , n=6
3) Verify the results by actual integration.

$$f(x) = x^4$$



$$h = \frac{b-a}{n} = \frac{3-(-3)}{6} = \frac{6}{6} = 1$$

	y_0	y_1	y_2	y_3	y_4	y_5	y_n
x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

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1) By Trapezoidal rule

$$\int_{-3}^{3} f(x)\ dx = \frac{h}{2}\left[(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \ \dots \dots . + y_{n-1})\right]$$

$$= \frac{1}{2}\left[(81 + 81) + 2\,(16 + 1 + 0 + 1 + 16)\right] = 115$$

2) By Simpson's one-third rule (since number of ordinates is odd)

n=6 , even number , one – third rule

$$\int_{-3}^{3} f(x)\ dx = \frac{h}{3}\left[(y_0 + y_n) + 2\,(y_2 + y_4 + \ \dots \dots) + 4\,(y_1 + y_3 + \ \dots \dots)\right]$$

$$= \frac{1}{3}\left[(81 + 81) + 2\,(1 + 1) + 4\,(16 + 0 + 16)\right] = 98$$

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3) Since n=6 (multiple of three), we can use also Simpson's three –eighths rule.

$$\int_{-3}^{3} f(x)\ dx = \frac{3h}{8}\left[y_0 + 3\left(y_1 + y_2 + y_4 + y_5 + y_7 + \ldots\ldots\right) + 2\left(y_3 + y_6 + y_9 + \ldots\ldots\ldots..\right) + y_n\right]$$

$$= \frac{3 \times 1}{8}\left[81 + 3\left(16 + 1 + 1 + 16\right) + 2\left(0\right) + 81\right] = 99$$

4) By actual integration,

$$\int_{-3}^{3} x^4 dx = 2 * \left(\frac{x^5}{5}\right)_0^3 = \frac{2 * 243}{5} = 97.2$$

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Example

$$Evaluate\ the\ integral\quad I=\int_{4}^{5.2} lnx\ dx\quad by\ using:$$

1) Trapezoidal Rule , 2) Simpsons Rule , n=6

$$f(x)=ln\ x$$

$$h=\frac{b-a}{n}=\frac{5.2-4}{6}=\frac{1.2}{6}=0.2$$

	y_0	y_1	y_2	y_3	y_4	y_5	y_n
x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
y	1.38629	1.43508	1.48160	1.52606	1.56862	1.60944	1.64866

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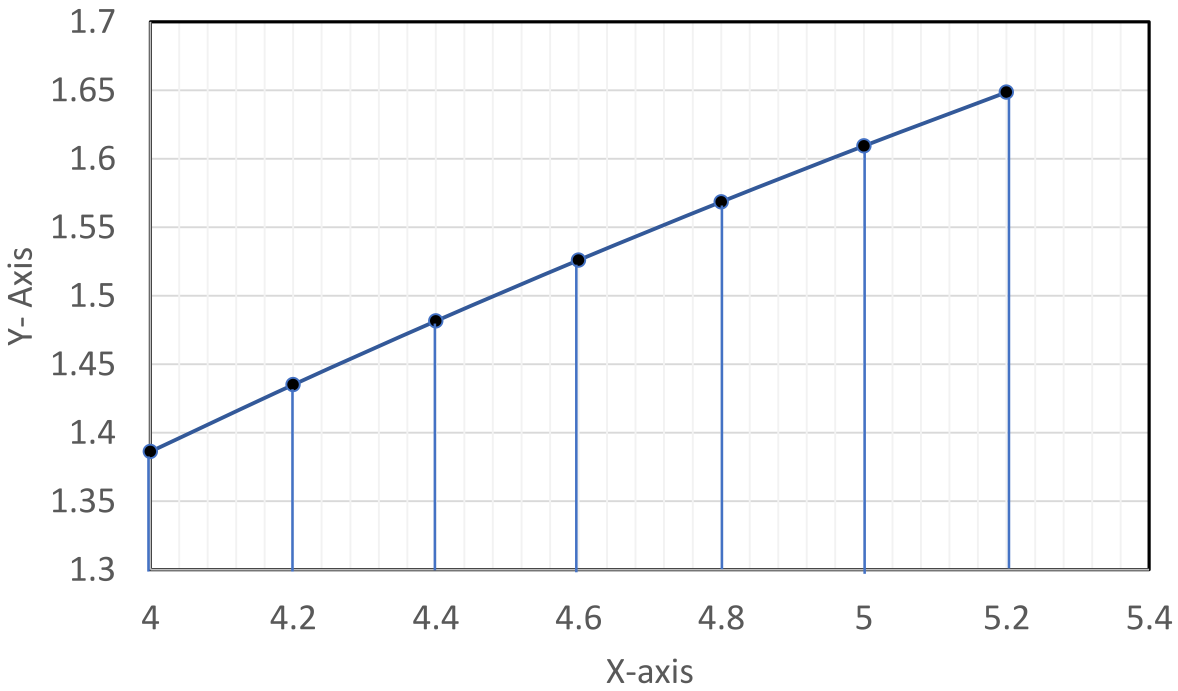
1) Trapezoidal Rule $\int_4^{5.2} lnx\ dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right) + y_6\right]$

$$= \frac{0.2}{2}\left[1.38629 + 2\left(1.43508 + 1.4816 + 1.52606 + 1.56862 + 1.60944\right) + 1.64866\right]$$

$$= 1.82784$$



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2) Simpsons Rule n=6 , even number , one – third rule

$$\int_4^{5.2} \ln x \; dx = \frac{h}{3}\left[y_0 + 2\,(y_2 + y_4) + 4\,(y_1 + y_3 + y_5) + y_6\right]$$

$$= \frac{0.2}{3}\left[1.38629 + 2\,(1.4816 + 1.56862) + 4\,(1.43508 + 1.52606 + 1.60944) + 1.64866\right]$$

$$= 1.82784$$

3) Simpsons Rule n=6 , multiple of three , three – eight rule

$$\int_4^{5.2} \ln x \, dx = \frac{3h}{8}\left[y_0 + 3\,(y_1 + y_2 + y_4 + y_5) + 2\,(y_3) + y_6\right]$$

$$= \frac{3 \times 0.2}{8}\left[1.38629 + 3\,(1.43508 + 1.4816 + 1.56862 + 1.60944) + 2(1.52606) + 1.64866\right]$$

$$= 1.82785$$

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Example

Find from the following table, the area bounded by the curve and x-axis from $x = 7.47 \;\; to \;\; x = 7.52$, using the Trapezoid Rule.

x	7.47	7.48	7.49	7.50	7.51	7.52
$f(x)$	1.93	1.95	1.98	2.01	2.03	2.06

Solution:

$$\text{The area is given by the integral} \int_{7.47}^{7.52} f(x)\,dx$$

$$\int_{7.47}^{7.52} f(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5\right]$$

$$= \frac{0.01}{2}\left[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)\right] = 0.09965$$

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Example

A river is 80 m wide the depth (d) in meter at a distance (x) meter from on tank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross section, using the Trapezoidal and Simpson rules

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1) Trapezoidal Rule

$$A = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$= \frac{10}{2}[0 + 2(4 + 7 + 9 + 12 + 15 + 14 + 8) + 3] = 705$$

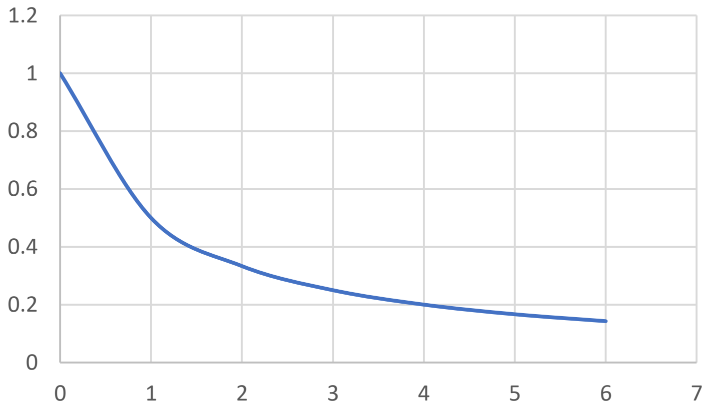
2) Simpsons Rule

$$A = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_7]$$

$$= \frac{10}{3}[0 + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14) + 8] = 710$$

Example

Evaluate $I = \int_0^6 \frac{1}{1+X}\, dx$ using (1) Trapezoidal rule (2) Simpson's rule (both) (3) Weddle's rule. Also, check up direct integration



Solution:

Take the number of interval as 6

$$\therefore h = \frac{6-0}{6} = 1$$

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$

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1) Trapezoidal Rule

$$\int_0^6 \frac{dx}{1+x} = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$= \frac{1}{2}\left[\left(1 + \frac{1}{7}\right) + 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)\right] = 2.02142857$$

2) Simpsons Rule one–third rule

$$I = \frac{1}{3}\left[\left(1 + \frac{1}{7}\right) + 2\left(\frac{1}{3} + \frac{1}{5}\right) + 4\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6}\right)\right] = \frac{1}{3}\left[\left(1 + \frac{1}{7}\right) + \frac{16}{15} + \frac{22}{6}\right] = 1.95873$$

3) Simpsons Rule three–eighth's rule

$$I = \frac{3 * 1}{8}\left[\left(1 + \frac{1}{7}\right) + 3\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6}\right) + 2\left(\frac{1}{4}\right)\right] = 1.96607$$

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4) Weddle's Rule

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}[(y_0+5\,y_1 + \ y_2 + 6y_3 + y_4 + 5y_5) + \ (2y_6 + 5y_7 + \ y_8 + 6y_9 + y_{10}$$

$$I = \frac{3*1}{10}[1 + 5 * (0.5) + \frac{1}{3} + 6\left(\frac{1}{4}\right) + \frac{1}{5} + 5\left(\frac{1}{6}\right) + \frac{1}{7}] = 1.952857$$

5) By actual integration

$$\int_0^6 \frac{1}{1+x}\,dx = [log(1+x)]_0^6 = log7 = 1.94591$$

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