FLUID MECHANICS

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LECTURE 9

Examples of Use of the Bernoulli Equation
a- Free Jets
b- Confined Flows
c- Flowrate Measurement
The Energy Line and the Hydraulic Grade Line

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Examples of Use of the Bernoulli Equation

In this section we illustrate various additional applications of the **Bernoulli equation**. Between any two points, (1) and (2), on a streamline in ***steady, inviscid, incompressible*** flow the Bernoulli equation can be applied in the form,

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \quad \text{.........(1)}$$

Obviously, if five of the six variables are known, the remaining one can be determined.

a- Free Jets

One of the oldest equations in fluid mechanics deals with the flow of a liquid from a large reservoir. The basic principles of this type of flow are shown in Fig. A. Where a jet of liquid of diameter d flows from the nozzle with velocity ***V***.

(A nozzle is a device shaped to accelerate a fluid.) Application of Eq. 1 between ***points (1)*** and ***(2)*** on the streamline shown gives

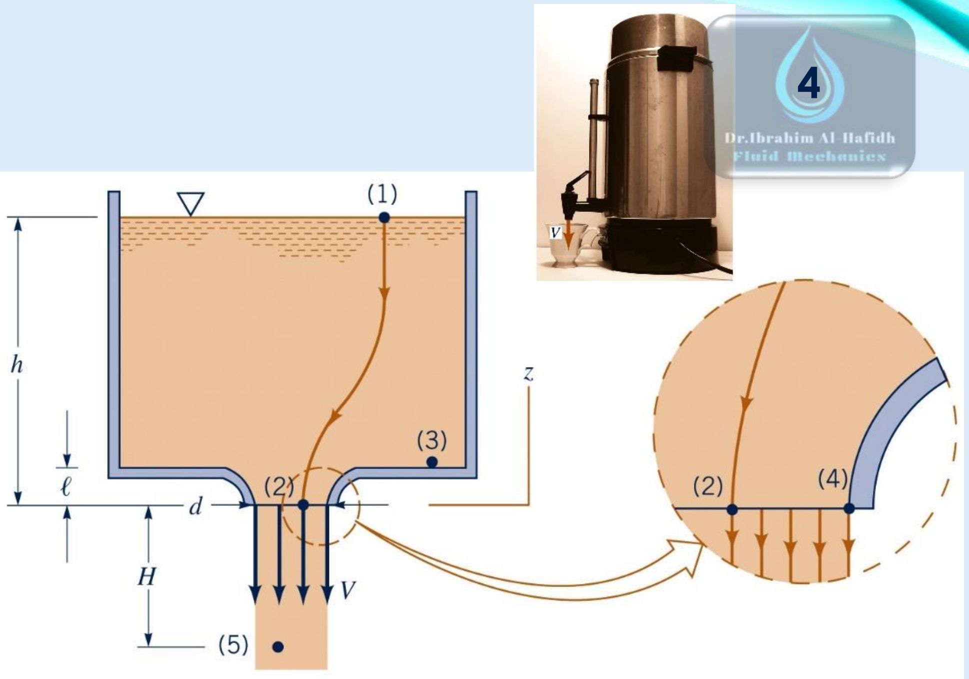


Figure (A)

$$\cancel{p_1} + \frac{1}{2}\rho \cancel{V_1^2} + \gamma z_1 = \cancel{p_2} + \frac{1}{2}\rho V_2^2 + \cancel{\gamma z_2}$$

$$\gamma h = \frac{1}{2}\rho V^2 \quad ....(2)$$

We have used the facts that $\boldsymbol{z_1 = h}$, $\boldsymbol{z_2 = 0}$, the reservoir is large (***V1≈ 0***) and open to the atmosphere (***p_1=0 gage***), and the fluid leaves as a "***free jet***" (***p_2=0***). Thus, we obtain

$$V = \sqrt{2\frac{\gamma h}{\rho}} = \sqrt{2gh} \quad ....(3)$$

Once outside the nozzle, the stream continues to fall as a free jet with zero pressure throughout (***p_5=0***) and as seen by applying Eq. 1 between ***points (1)*** and ***(5)***, the speed increases according to

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_5 + \frac{1}{2}\rho V_5^2 + \gamma z_5$$

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma h = p_5 + \frac{1}{2}\rho V_5^2 - \gamma H$$

$$\gamma h + \gamma H = \frac{1}{2}\rho V_5^2$$

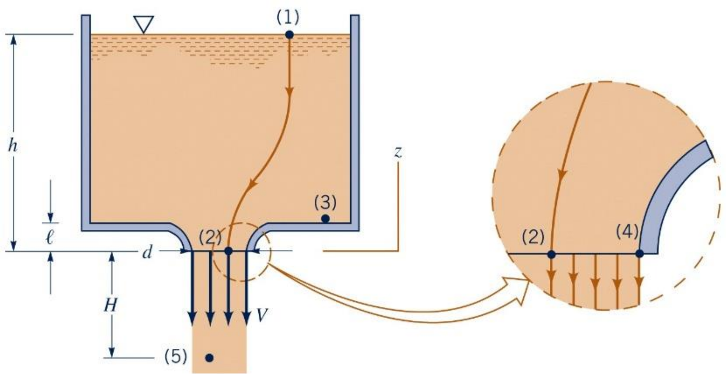


Figure (A)

$$V = \sqrt{2g\,(h + H)} \quad \text{.........(4)}$$

where, as shown in Fig. A, ***H*** is the distance the fluid has fallen outside the nozzle.

If the streamlines at the tip of the nozzle are straight ($R = \infty$) it follows that $\boldsymbol{p_2 = p_4}$. Since (4) is on the surface of the jet, in contact with the atmosphere, we have $\boldsymbol{p_4 = 0}$.

Equation 3 could also be obtained by writing the Bernoulli equation between ***points (3)*** and ***(4)*** using the fact that $\boldsymbol{z_4 = 0}$, $\boldsymbol{z_3 = l}$. Also, $\boldsymbol{V_3 = 0}$ since it is far from the nozzle,
and from hydrostatics, $\boldsymbol{p_3 = \gamma(h - l)}$.

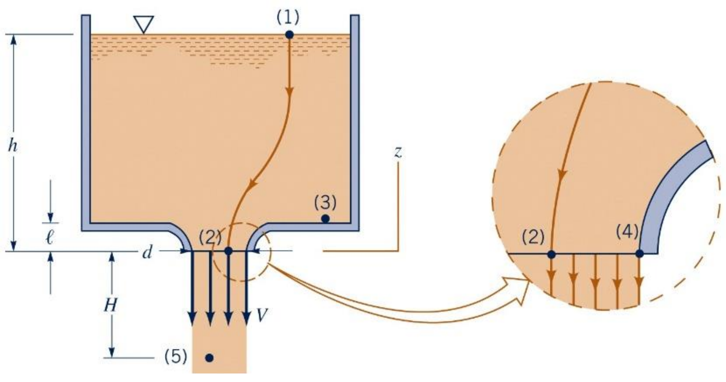


Figure (A)

$$p_3 + \frac{1}{2}\rho V_3^2 + \gamma z_3 = p_4 + \frac{1}{2}\rho V_4^2 + \gamma z_4$$

$$\gamma (h - l) + \gamma l = \frac{1}{2}\rho V_4^2 - \gamma z_4$$

$$\gamma h = \frac{1}{2}\rho V_4^2$$

$$V = \sqrt{2\frac{\gamma h}{\rho}} = \sqrt{2gh}$$

As learned in physics or dynamics and illustrated in the figure (B), any object dropped from rest that falls through a distance ***h*** in a vacuum will obtain the speed $V = \sqrt{2gh}$, the same as the water leaving the spout of the watering can shown in the figure (C).

This is consistent with the fact that all of the particle's ***potential energy*** is converted to ***kinetic energy***, provided viscous (friction) effects are negligible.
In terms of heads, the elevation head at point (1) is converted into the velocity head at point (2).
The pressure is the same (atmospheric) at points (1) and (2) in Figure (A).

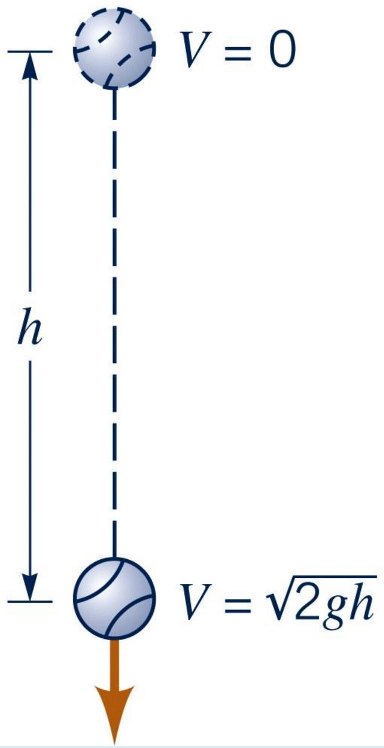


Figure (B)

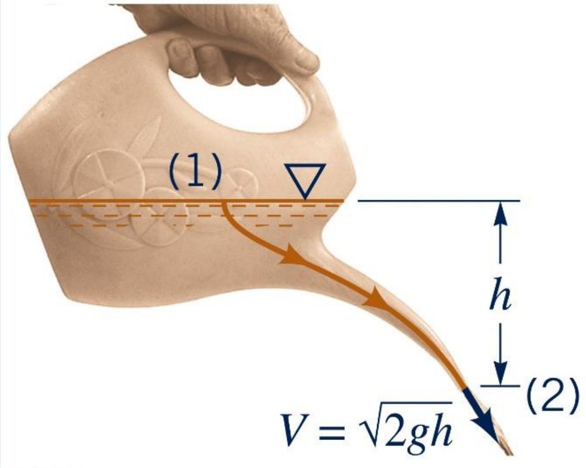


Figure (C)

For the horizontal nozzle of **Figure (D)**, the velocity of the fluid at the centerline, V_2, will be slightly greater than that at the top, V_1, and slightly less than that at the bottom, V_3, due to the differences in elevation. In general, $d \ll h$ as shown in **Figure (E)** and we can safely use the centerline velocity as a reasonable "average velocity".

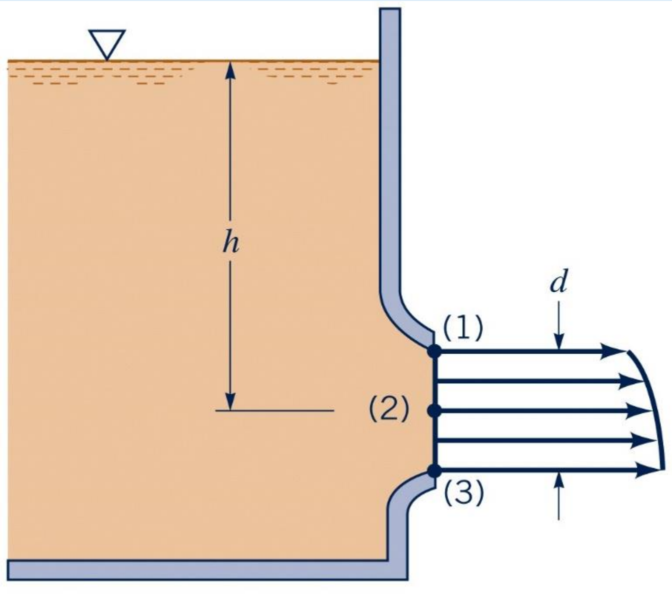


Figure (D)

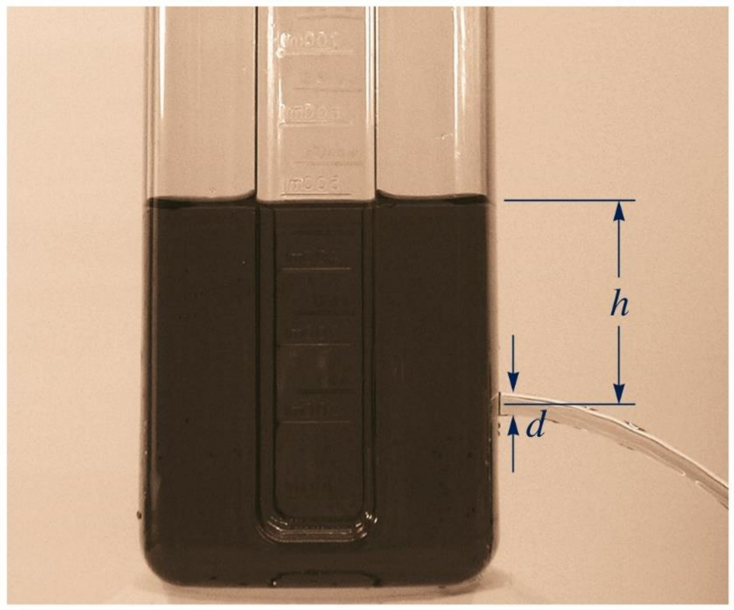


Figure (E)

Horizontal flow from a tank

If the exit is not a smooth, well-contoured nozzle, but rather a flat plate as shown in Figure (F), the diameter of the jet, d_j will be less than the diameter of the hole, d_h. This phenomenon, called a ***vena contracta*** effect, is a result of the inability of the fluid to turn the sharp 90° corner indicated by the dotted lines in the figure.

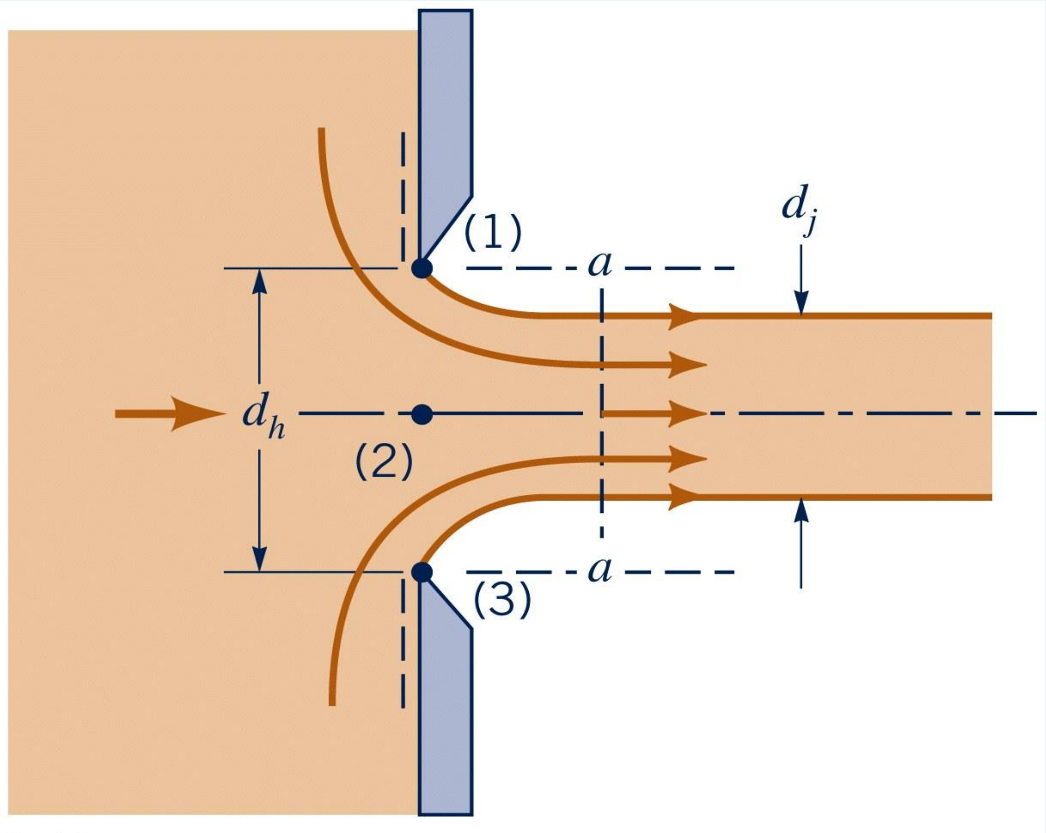


Figure (F)
Vena contracta effect for a sharp-edged orifice

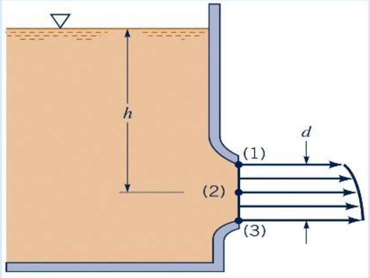


Figure (D)

Since the streamlines in the exit plane are curved **($R < \infty$)**, the pressure across them is not constant.

It would take an infinite pressure gradient across the streamlines to cause the fluid to turn a "sharp" corner (***R=0***).
The highest pressure occurs along the centerline at (2) and the lowest pressure, ***p_1=p_3=0***, is at the edge of the jet.
Thus, the assumption of uniform velocity with straight streamlines and constant pressure is not valid at the exit plane. It is valid, however, in the plane of the ***vena contracta***, section ***a–a***. The uniform velocity assumption is valid at this section provided $d_j \ll h$ as is discussed for the flow from the nozzle shown in Figure D.

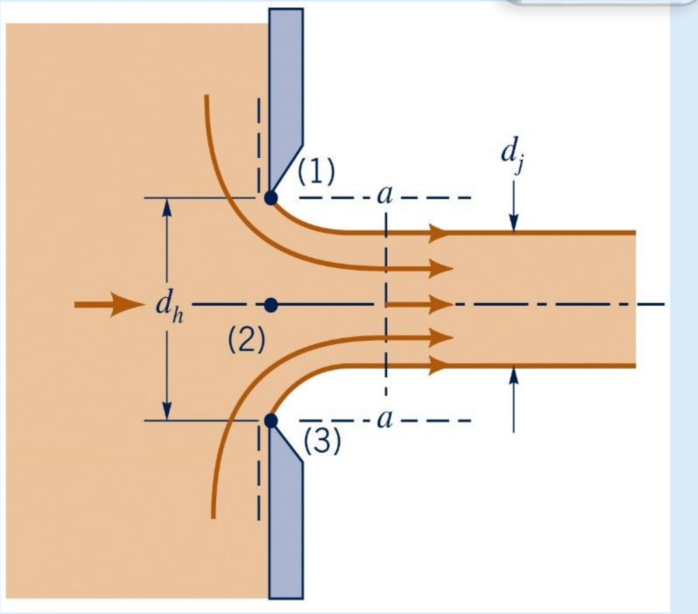


Figure (F)
Vena contracta effect for a sharp-edged orifice

The **vena contracta** effect is a function of the geometry of the outlet. Some typical configurations are shown in Figure (G) along with typical values of the experimentally obtained ***contraction coefficient***, $C_c = A_j/A_h$, where A_j and A_h are the areas of the jet at the vena contracta and the area of the hole, respectively

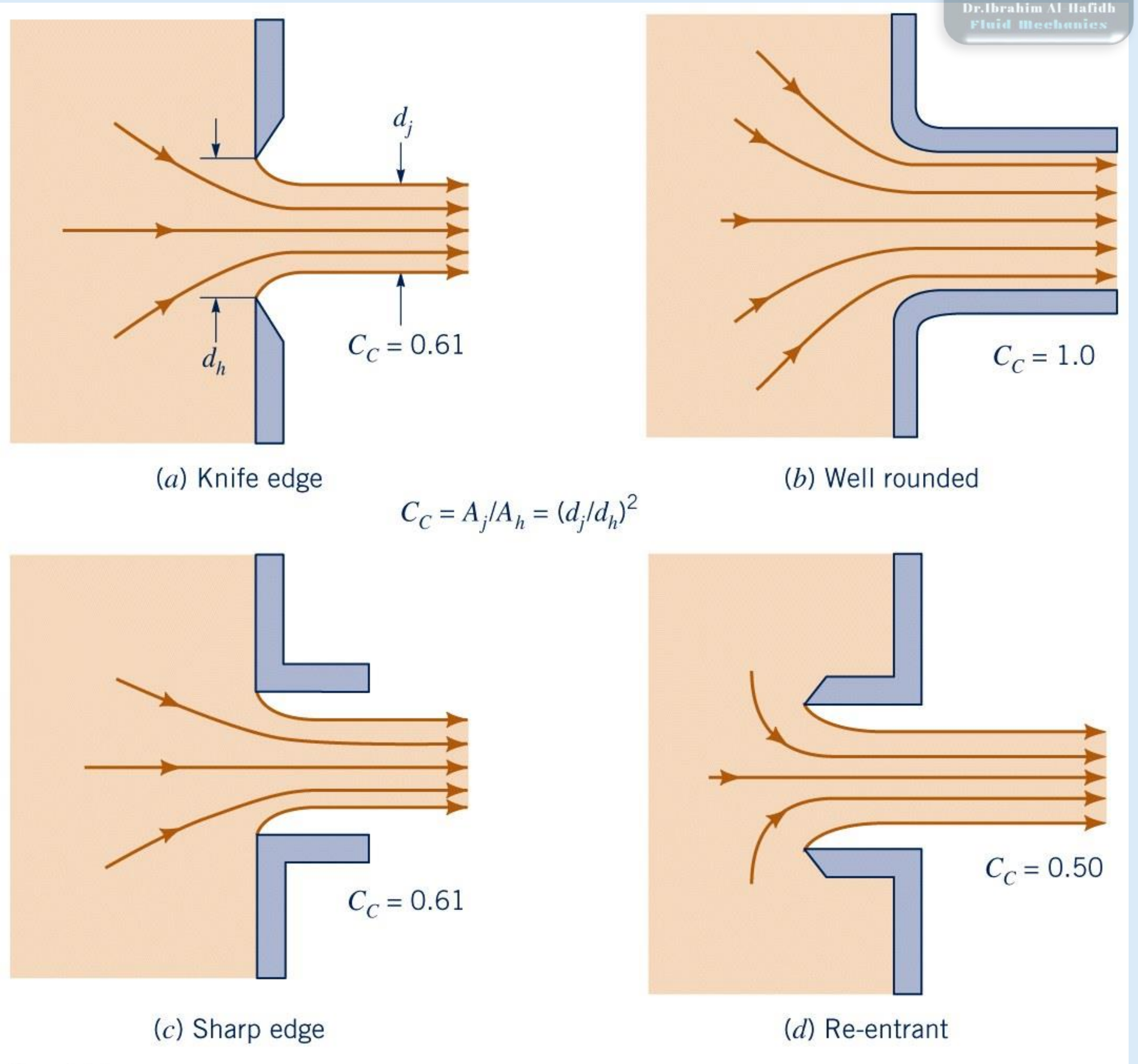


Figure (G)
Typical flow patterns and contraction coefficients for various round exit configurations.
(a) Knife edge, (b) Well rounded,
(c) Sharp edge, (d) Re-entrant.

b- Confined Flows

In many cases the fluid is physically constrained within a device so that its pressure cannot be prescribed a priori as was done for the free jet examples above. Such cases include nozzles and pipes of variable diameter for which the fluid velocity changes because the flow area is different from one section to another.
For this situations It is necessary to use the concept of conservation of mass (the continuity equation) along with the Bernoulli equation.

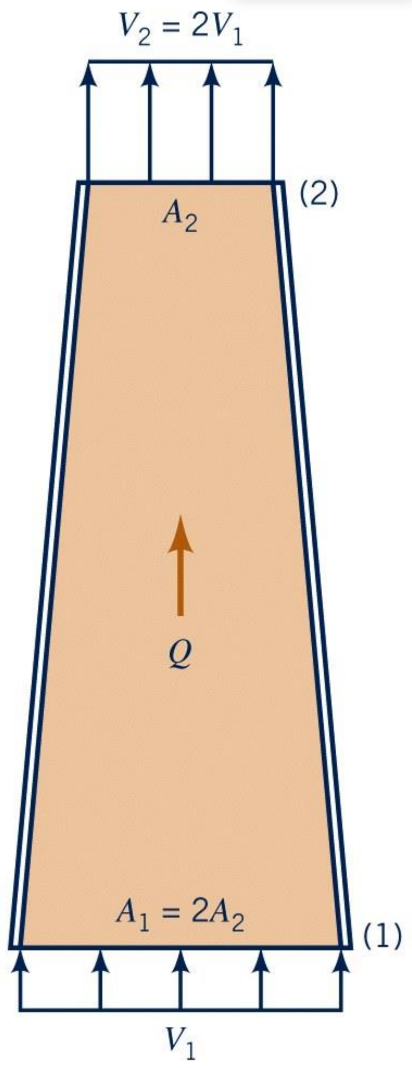


Figure (H)

Consider a fluid flowing through a fixed volume (such as a syringe) that has one inlet and one outlet as shown in Figure (I).
If the flow is steady so that there is no additional accumulation of fluid within the volume, *the rate at which the fluid flows into the volume must equal the rate at which it flows out of the volume* (otherwise, **mass** would not be conserved).

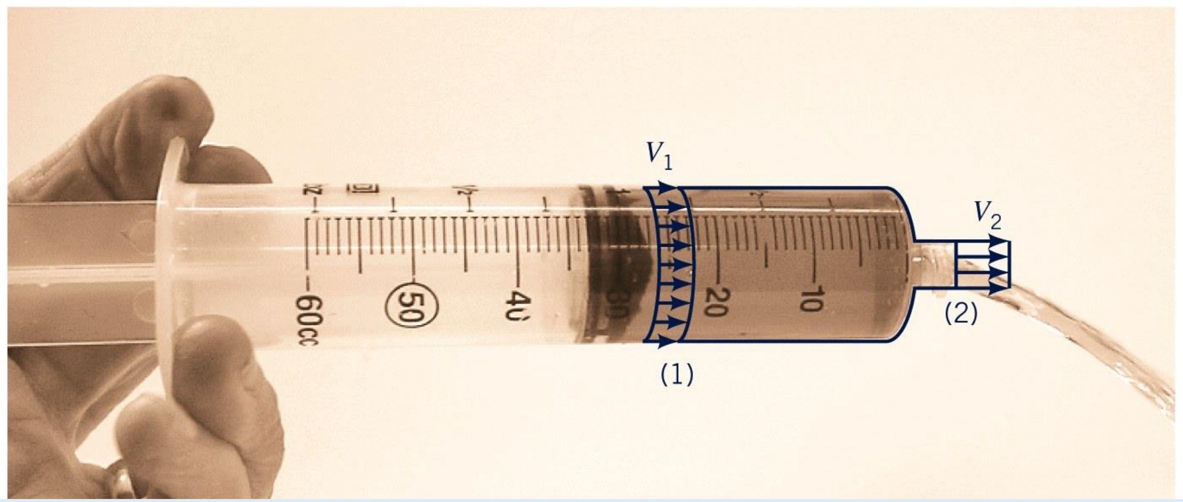


Figure (I)

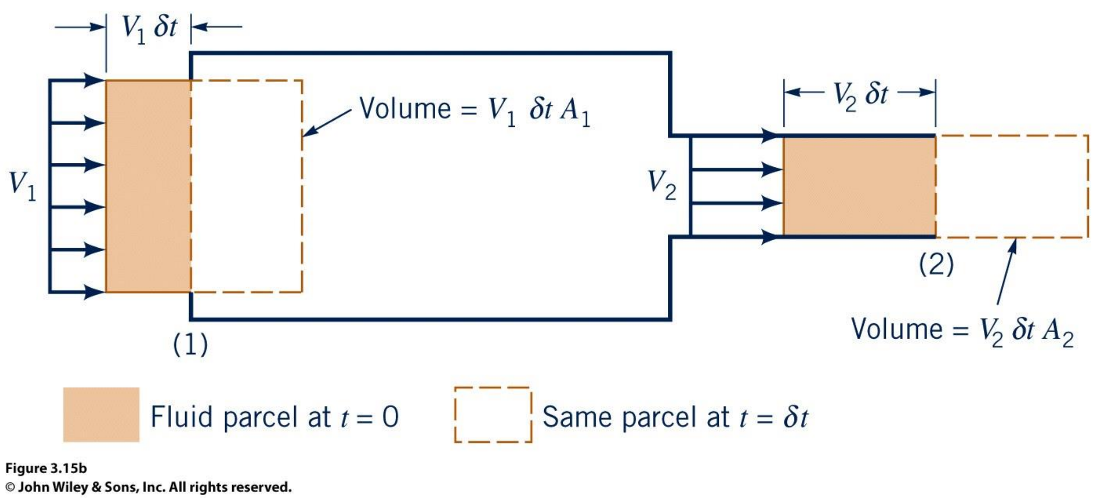


Figure (J)

The mass flowrate from an outlet, $\dot{m}$ (slugs/s or kg/s), is given by $\dot{m} = \rho Q$, where **Q (ft^3/s or m^3/s)** is the ***volume flowrate*** and ***ρ density* (slugs/ft3 or kg/m3)**. If the outlet area is ***A*** and the fluid flows across this area (normal to the area) with an average velocity ***V***, then the volume of the fluid crossing this area in a time interval ***δt*** is ***VA δt***, equal to that in a volume of length ***Vδt*** and cross-sectional area ***A*** (see Figure J). Hence, the volume flowrate (***volume per unit time***) is **Q=VA**. Thus, $\dot{m} = \rho VA$.

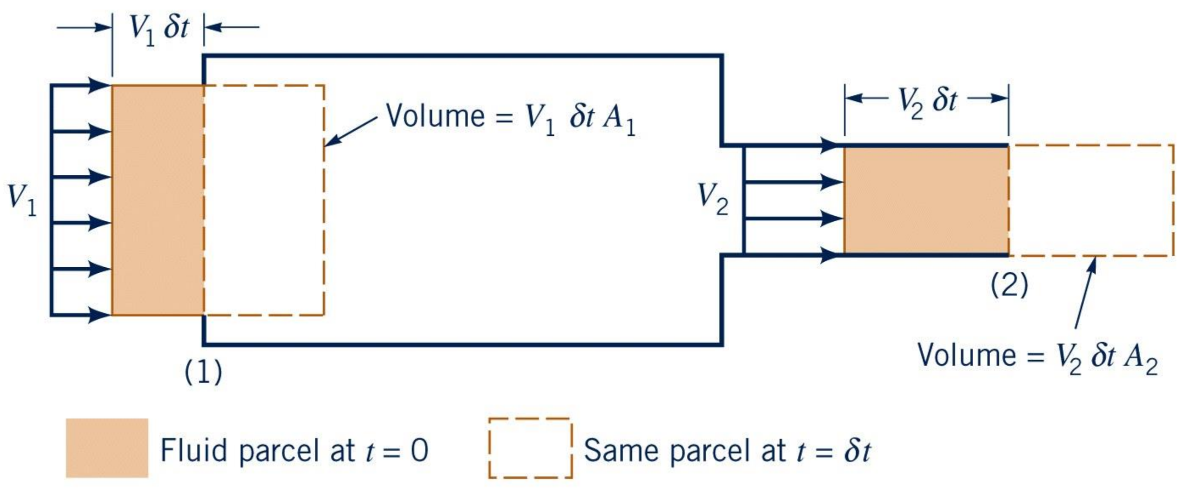


Figure 3.15b

Figure (J)

To conserve mass, the inflow rate must equal the outflow rate. If the inlet is designated as (1) and the outlet as (2), it follows that $\dot{m}_1 = \dot{m}_2$. Thus, ***conservation of mass requires,***

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

If the density remains constant, then $\rho_1 = \rho_2$ and the above becomes the ***continuity equation*** for incompressible flow

$$A_1 V_1 = A_2 V_2, \text{ or } Q_1 = Q_2 \qquad \text{.........(5)}$$

For example, if as shown by the figure (H) the outlet flow area is one-half the size of the inlet flow area, it follows that the outlet velocity is twice that of the inlet velocity, since $\boldsymbol{V_2 = A_1 V_1 / A_2 = 2V_1}$

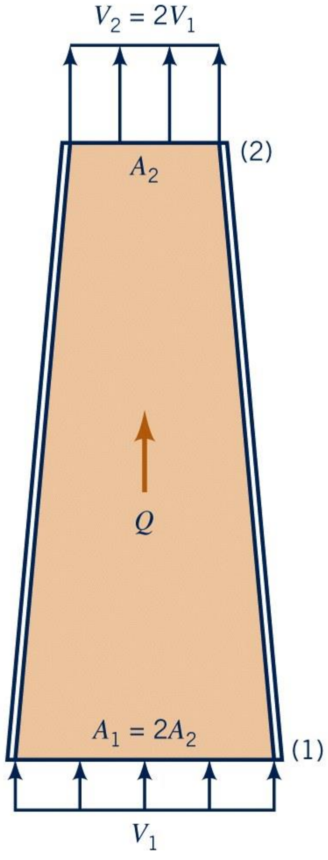
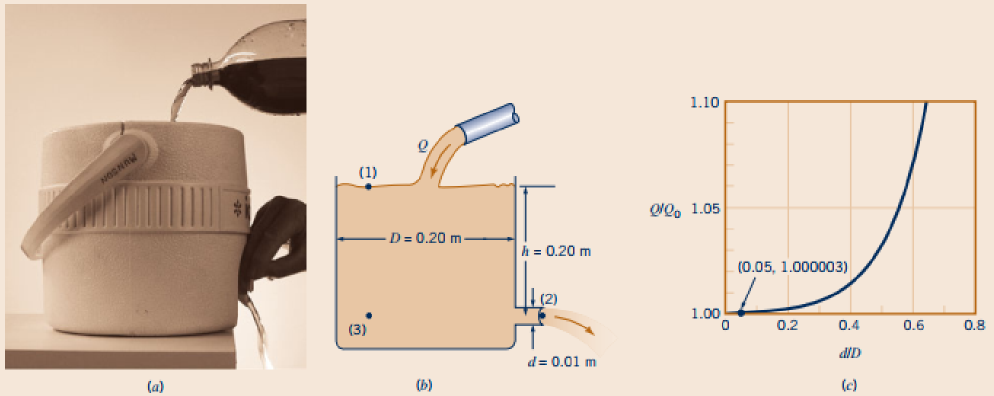


Figure (H)

EXAMPLE 3.7 Flow from a Tank—Gravity Driven

GIVEN A stream of refreshing beverage of diameter $d = 0.01$ m flows steadily from the cooler of diameter $D = 0.20$ m as shown in Figs. E3.7*a* and *b*.

FIND Determine the flowrate, Q, from the bottle into the cooler if the depth of beverage in the cooler is to remain constant at $h = 0.20$ m.



■ Figure E3.7

SOLUTION

For steady, inviscid, incompressible flow, the Bernoulli equation applied between points (1) and (2) is

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2 \qquad (1)$$

With the assumptions that $p_1 = p_2 = 0$, $z_1 = h$, and $z_2 = 0$, Eq. 1 becomes

$$\tfrac{1}{2}V_1^2 + gh = \tfrac{1}{2}V_2^2 \qquad (2)$$

Although the liquid level remains constant (h = constant), there is an average velocity, V_1, across section (1) because of the flow from the tank. From Eq. 5 for steady incompressible flow, conservation of mass requires $Q_1 = Q_2$, where $Q = AV$. Thus, $A_1V_1 = A_2V_2$, or

$$\frac{\pi}{4}D^2V_1 = \frac{\pi}{4}d^2V_2$$

Hence,

$$V_1 = \left(\frac{d}{D}\right)^2 V_2 \qquad (3)$$

Equations 1 and 3 can be combined to give

$$V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}} = \sqrt{\frac{2(9.81\ \text{m/s}^2)(0.20\ \text{m})}{1-(0.01\ \text{m}/0.20\ \text{m})^4}} = 1.98\ \text{m/s}$$

Thus,

$$Q = A_1V_1 = A_2V_2 = \frac{\pi}{4}(0.01\ \text{m})^2(1.98\ \text{m/s})$$
$$= 1.56 \times 10^{-4}\ \text{m}^3/\text{s} \qquad \textbf{(Ans)}$$

COMMENTS In this example we have not neglected the kinetic energy of the water in the tank ($V_1 \neq 0$). If the tank diameter is large compared to the jet diameter ($D \gg d$), Eq. 3 indicates that $V_1 \ll V_2$ and the assumption that $V_1 \approx 0$ would be reasonable. The error associated with this assumption can be seen by calculating the ratio of the flowrate assuming $V_1 \neq 0$, denoted Q, to that assuming $V_1 = 0$, denoted Q_0. This ratio, written as

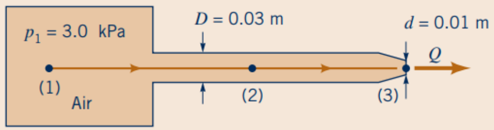
$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{D=\infty}} = \frac{\sqrt{2gh/[1-(d/D)^4]}}{\sqrt{2gh}} = \frac{1}{\sqrt{1-(d/D)^4}}$$

is plotted in Fig. E3.7*c*. With $0 < d/D < 0.4$ it follows that $1 < Q/Q_0 \lesssim 1.01$, and the error in assuming $V_1 = 0$ is less than 1%. For this example with $d/D = 0.01\ \text{m}/0.20\ \text{m} = 0.05$, it follows that $Q/Q_0 = 1.000003$. Thus, it is often reasonable to assume $V_1 = 0$.

Note that this problem was solved using points (1) and (2) located at the free surface and the exit of the pipe, respectively. Although this was convenient (because most of the variables are known at those points), other points could be selected and the same result would be obtained. For example, consider points (1) and (3) as indicated in Fig. E3.7*b*. At (3), located sufficiently far from the tank exit, $V_3 = 0$ and $z_3 = z_2 = 0$. Also, $p_3 = \gamma h$ since the pressure is hydrostatic sufficiently far from the exit. Use of this information in the Bernoulli equation applied between (3) and (2) gives the exact same result as obtained using it between (1) and (2). The only difference is that the elevation head, $z_1 = h$, has been interchanged with the pressure head at (3), $p_3/\gamma = h$.

EXAMPLE 3.8 Flow from a Tank—Pressure Driven

GIVEN Air flows steadily from a tank, through a hose of diameter $D = 0.03$ m, and exits to the atmosphere from a nozzle of diameter $d = 0.01$ m as shown in Fig. E3.8. The pressure in the tank remains constant at 3.0 kPa (gage) and the atmospheric conditions are standard temperature and pressure.



■ Figure E3.8*a*

FIND Determine

(a) the flowrate and

(b) the pressure in the hose.

SOLUTION

(a) If the flow is assumed steady, inviscid, and incompressible, we can apply the Bernoulli equation along the streamline from (1) to (2) to (3) as

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$
$$= p_3 + \tfrac{1}{2}\rho V_3^2 + \gamma z_3$$

With the assumption that $z_1 = z_2 = z_3$ (horizontal hose), $V_1 = 0$ (large tank), and $p_3 = 0$ (free jet), this becomes

$$V_3 = \sqrt{\frac{2p_1}{\rho}}$$

and

$$p_2 = p_1 - \tfrac{1}{2}\rho V_2^2 \qquad (1)$$

The density of the air in the tank is obtained from the perfect gas law, using standard absolute pressure and temperature, as

$$\rho = \frac{p_1}{RT_1}$$
$$= \frac{(3.0 + 101)\ \text{kN/m}^2 \times 10^3\ \text{N/kN}}{(286.9\ \text{N}\cdot\text{m/kg}\cdot\text{K})(15 + 273)\text{K}}$$
$$= 1.26\ \text{kg/m}^3$$

Thus, we find that

$$V_3 = \sqrt{\frac{2(3.0 \times 10^3\ \text{N/m}^2)}{1.26\ \text{kg/m}^3}} = 69.0\ \text{m/s}$$

or

$$Q = A_3 V_3 = \frac{\pi}{4} d^2 V_3 = \frac{\pi}{4}(0.01\ \text{m})^2(69.0\ \text{m/s})$$
$$= 0.00542\ \text{m}^3/\text{s} \qquad \textbf{(Ans)}$$

COMMENT Note that the value of V_3 is determined strictly by the value of p_1 (and the assumptions involved in the Bernoulli equation), independent of the "shape" of the nozzle. The pressure head within the tank, $p_1/\gamma = (3.0\ \text{kPa})/(9.81\ \text{m/s}^2)(1.26\ \text{kg/m}^3) = 243$ m, is converted to the velocity head at the exit, $V_2^2/2g = (69.0\ \text{m/s})^2/(2 \times 9.81\ \text{m/s}^2) = 243$ m. Although we used gage pressure in the Bernoulli equation ($p_3 = 0$), we had to use absolute pressure in the perfect gas law when calculating the density.

(b) The pressure within the hose can be obtained from Eq. 1 and the continuity equation (Eq. 5)

$$A_2 V_2 = A_3 V_3$$

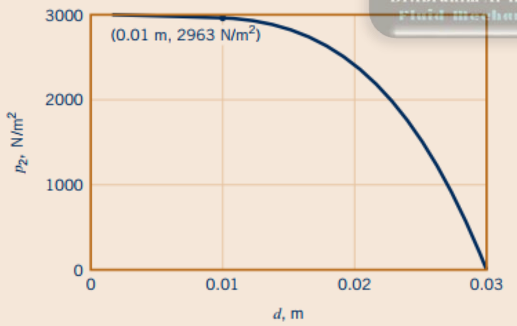
Hence,

$$V_2 = A_3 V_3 / A_2 = \left(\frac{d}{D}\right)^2 V_3$$
$$= \left(\frac{0.01\ \text{m}}{0.03\ \text{m}}\right)^2 (69.0\ \text{m/s}) = 7.67\ \text{m/s}$$

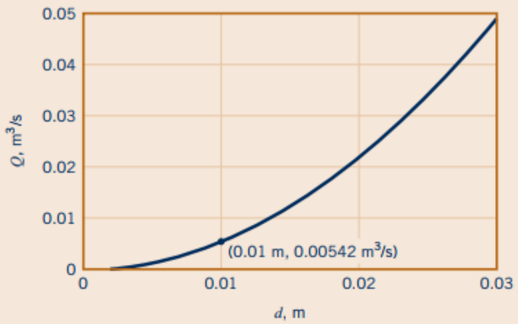
and from Eq. 1

$$p_2 = 3.0 \times 10^3\ \text{N/m}^2 - \tfrac{1}{2}(1.26\ \text{kg/m}^3)(7.67\ \text{m/s})^2$$
$$= (3000 - 37.1)\text{N/m}^2 = 2963\ \text{N/m}^2 \qquad \textbf{(Ans)}$$

COMMENTS In the absence of viscous effects, the pressure throughout the hose is constant and equal to p_2. Physically, the decreases in pressure from p_1 to p_2 to p_3 accelerate the air and increase its kinetic energy from zero in the tank to an intermediate value in the hose and finally to its maximum value at the nozzle exit. Since the air velocity in the nozzle exit is nine times that in the hose, most of the pressure drop occurs across the nozzle ($p_1 = 3000\ \text{N/m}^2$, $p_2 = 2963\ \text{N/m}^2$, and $p_3 = 0$).



■ Figure E3.8*b*



■ Figure E3.8*c*

Since the pressure change from (1) to (3) is not too great [i.e., in terms of absolute pressure $(p_1 - p_3)/p_1 = 3.0/101 = 0.03$], it follows from the perfect gas law that the density change is also not significant. Hence, the incompressibility assumption is reasonable for this problem. If the tank pressure were considerably larger or if viscous effects were important, application of the Bernoulli equation to this situation would be incorrect.

By repeating the calculations for various nozzle diameters, d, the results shown in Figs. E3.8*b,c* are obtained. The flowrate increases as the nozzle is opened (i.e., larger d). Note that if the nozzle diameter is the same as that of the hose ($d = 0.03$ m), the pressure throughout the hose is atmospheric (zero gage).

In general, an increase in velocity is accompanied by a decrease in pressure. For example, the velocity of the air flowing over the top surface of an airplane wing is, on the average, faster than that flowing under the bottom surface. Thus, the net pressure force is greater on the bottom than on the top—the wing generates a lift.