NUMERICAL ANALYSIS

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Lecture 1 & 2

FINITE DIFFERENCE

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Engineering Analysis
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FINITE DIFFERENCE

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Interpolation

Let $y = f(x)$ be a function of x

The corresponding values of y for a set $a, a+h, a+2h, a+3h, \dots\dots, a+nh$

are given as:

$y_0 = f(a)$, $y_1 = f(a+h)$, $y_2 = f(a+2h), \dots\dots\dots\dots\dots, y_n = f(a+nh)$

اذن هي عملية ايجاد قيمة y عند اي قيمة ل x محصورة بين a و $a+nh$

Extrapolation

هي عملية ايجاد قيمة y عند اي قيمة ل x خارج الفترة المحصورة بين a و $a+nh$

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Finite Differences

1- First difference

Let $y=f(x)$ be a given function of x and let $y_0, y_1, y_2, y_3, \ldots\ldots, y_n$ be the value of y corresponding to x_0, $x_1, x_2, x_3, \ldots\ldots, x_n$, the values of x. the independent variables x is called the ***argument*** and the corresponding dependent value y is called, the ***entry***. In general, the difference between any two consecutive values of x need not be same or equal.

We can write the arguments and entries as below.

x	X_0	x_1	x_2	x_3	……	Y_{n-1}	y_n
Y	y_0	y_1	Y_2	y_3	……	Y_{n-1}	y_n

if we subtract from each value of y (except y_0) the preceding value of y, we get

$$y_1 - y_0 , y_2 - y_1 , y_3 - y_2 , \ldots , y_n - y_{n-1}$$

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These results are called the first difference of ***y***. The first difference of ***y*** are denoted by **Δy**.

That is, $\Delta y_0 = y_1 - y_0$

$$\Delta y_1 = y_2 - y_1$$

$$\Delta y_2 = y_3 - y_2$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

The symbol Δ doenotes an operation, called ***forward difference operator***.

Higher differences: The ***second*** and higher difference are defined as below:

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta\,(y_1 - y_0) = \Delta y_1 - \Delta\, y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta\,(y_2 - y_1) = \Delta y_2 - \Delta\, y_1$$

$$\ldots\ldots$$

$$\Delta^2 y_{n-1} = \Delta(\Delta y_{n-1}) = \Delta\,(y_n - y_{n-1}) = \Delta y_n - \Delta\, y_{n-1}$$

Here, Δ^2 is an operator called, second order forwarded difference operator.

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In the same way, the third order forward difference operator Δ^3 is as follows:

That is, $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$

…………………….. etc.

In general $\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i$

Though the ***arguments*** x_0, x_1, x_2 .. … need not, in general be equally spaced, for purposes of practical work, we take them equally spaced.

Usually, the ***arguments*** are taken as:

$$x_0 ,\ x_0+h ,\ x_0+2h,\ x_0+3h \ldots\ldots\ldots$$

So that $x_1-x_0 = x_2-x_1 = x_3-x_2 = \ldots\ldots\ldots = h$

Here, h is called the ***interval of differencing.***

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Operators, We have already defined the forward difference operator Δ. We will now see some more operators and the relations connecting them.

Backward difference operator (∇) :

Backward difference operator (∇) is defined as:

$$\nabla f(x) = f(x) - f\,(x - h)$$

By definition $\nabla\, y_1 = y_1 - y_0$

$$\nabla\, y_2 = y_2 - y_1 \quad \text{etc}$$

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Finite Differences

Consider the function $y = f(x)$

Suppose $y_0 = f(a)$, $y_1 = f(a+h)$

$y_1 - y_0$ **is called the first difference of** y_0

$y_1 - y_0 = \Delta y_0$

$\Delta \rightarrow$ **denote the difference operator** معامل اوعلامة تدل على الاختلاف

$\Delta^2 \rightarrow$ **the second difference operato**

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Values of x	Values of y	First difference	Second difference	Third difference	Fourth difference
a (x_0)	y_0				
		Δy_0			
a+h (x_1)	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
a+2h (x_2)	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
a+3h (x_3)	y_3		$\Delta^2 y_2$		
		Δy_3			
a+4h (x_4)	y_4				

Newtons Forward Interpolation Formula

$$f(a+h) = E\,.\,f(a) = (1+\Delta)\,.\,f(a)$$

$$f(a+2h) = E\,.\,f(a+h) = E.E\,.\,f(a) = E^2\,f(a) = (1+\Delta)^2\,f(a)$$

$$f(a+3h) = (1+\Delta)^3\,f(a)$$

$$f(a+nh) = (1+\Delta)^n\,f(a)$$

On expanding $(1+\Delta)^n$ by Binomial theorem, we get

$$f(a+nh) = f(a) + n\,\Delta f(a) + \frac{n(n-1)}{2!}\,\Delta^2\,f(a) + \ldots\ldots\ldots\ldots$$

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Example

Using Newtons forward interpolation formula, and the give table of values:

x	1.1	1.3	1.5	1.7	1.9
$f(x)$	0.21	0.69	1.25	1.89	2.61

Obtain the value of $f(x)\ when\ \ x = 1.4$

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x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
a 1.1	F(a) 0.21	ΔF(a)			
		0.48	Δ^2F(a)		
1.3	0.69		0.08	Δ^3F(a)	
		0.56		0	Δ^4F(a)
1.5	1.25		0.08		0
		0.64		0	
1.7	1.89		0.08		
		0.72			
1.9	2.61				

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$$a = 1.1 \quad , \qquad f(a) = 0.21$$

$$\Delta f(a) = 0.48 \quad , \qquad \Delta^2 f(a) = 0.08 \quad , \qquad \Delta^3 f(a) = 0$$

$$h = 0.2$$

$$a + nh = x \quad \rightarrow \quad 1.1 + n \times 0.2 = 1.4 \quad \rightarrow \quad n = 1.5$$

$$f(a+nh) = f(a) + n\,\Delta f(a) + \frac{n\,(n-1)}{2!}\,\Delta^2\, f(a) + \ldots\ldots\ldots.$$

$$f(1.4) = 0.21 + 1.5 \times 0.48 + \frac{1.5\,(1.5-1)}{2 \times 1} \times 0.08 = 0.96$$

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Example

The following are data from table:

$Temperature$ °C	140	150	160	170	180
$Pressure$ kg/cm^2	3.685	4.854	6.302	8.076	10.225

Using Newtons forward interpolation formula, find the pressure at the temperature of 142 °C

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Temp.	*Press.*	Δ	Δ^2	Δ^3	Δ^4
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.002
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

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$$a + nh = 142 \quad \rightarrow \quad 140 + n \times 10 = 142 \quad \rightarrow \quad n = 0.2$$

$$f(a + nh) = f(a) + n\,\Delta f(a) + \frac{n\,(n-1)}{2!}\,\Delta^2 f(a) + \frac{n\,(n-1)(n-2)}{3!}\,\Delta^3 f(a)$$

$$+ \frac{n\,(n-1)(n-2)(n-3)}{4!}\,\Delta^4 f(a)$$

$$f(142) = 3.685 + 0.2 \times 1.169 + \frac{0.2\,(0.2-1)}{2 \times 1} \times 0.279$$

$$+ \frac{0.2(0.2-1)(0.2-2)}{3 \times 2 \times 1} \times 0.047 + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4 \times 3 \times 2 \times 1} \times 0.002$$

$$= 3.899$$

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Example

From the following table, find the value of $tan\ 45.25$

x	45	46	47	48	49	50
$tan\ x$	1	1.03553	1.07237	1.11061	1.15037	1.19175

Using Newtons forward interpolation formula

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x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
45	1					
		0.03553				
46	1.03553		0.00131			
		0.03684		0.00009		
47	1.07237		0.0014		0.00003	
		0.03824		0.00012		-0.00005
48	1.11061		0.00152		-0.00002	
		0.03976		0.0001		
49	1.15037		0.00162			
		0.04138				
50	1.19175					

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$$h = 1$$

$$a + nh = x \qquad 45 + n \times 1 = 45.25 \quad n = 45.25 - 45 = 0.25$$

$$f(a + nh) = f(a) + n\,\Delta f(a) + \frac{n\,(n-1)}{2!}\,\Delta^2 f(a) + \ldots\ldots\ldots.$$

$$f(45.25) = 1 + 0.25 \times 0.03553 + \frac{0.25\,(0.25-1)}{2 \times 1} \times 0.00131$$

$$+ \frac{0.25(0.25-1)(0.25-2)}{3 \times 2 \times 1} \times 0.00009$$

$$+ \frac{0.25(0.25-1)(0.25-2)(0.25-3)}{4 \times 3 \times 2 \times 1} \times 0.00003$$

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$$+\frac{0.25(0.25-1)(0.25-2)(0.25-3)(0.25-4)}{5\times4\times3\times2\times1}\times(-0.00005)$$

$$=1.00876$$

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Newtons Formula for Backward differences

$$\nabla f(a) = f(a) - f(a-h)$$

$$f(a-h) = (1-\nabla)f(a)$$

$$f(a-2h) = (1-\nabla)^2 f(a)$$

$$f(a-nh) = (1-\nabla)^n f(a)$$

$$f(a-nh) = f(a) - n\nabla f(a) + \frac{n(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \dots\dots..$$

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Example

Use the Newtons formula for backward difference to estimate $f(7.5)$ from the following data

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

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x	$f(x)$	∇	∇^2	∇^3	∇^4	∇^5	∇^6	∇^7
1	1							
		7						
2	8		12					
		19		6				
3	27		18		0			
		37		6		0		
4	64		24		0		0	
		61		6		0		0
5	125		30		0		0	
		91		6		0		
6	216		36		0			
		127		6				
7	343		42					
		169						
8	512							

$$a = 8 \qquad , \qquad h = 1$$

$$a - nh = 7.5 \quad \rightarrow \quad 8 - n \times 1 = 7.5 \quad \rightarrow \quad n = 0.5$$

$$f(a - nh) = f(a) - n\nabla f(a) + \frac{n\,(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \dots\dots.$$

$$f(7.5) = 512 - 0.5 \times 169 + \frac{0.5 \times (0.5 - 1)}{2 \times 1} \times 42 - \frac{0.5 \times (0.5 - 1) \times (0.5 - 2)}{3 \times 2 \times 1} \times 6$$

$$= 421.875$$

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Example

From the following compute the value $sin\ 38$, Use the Newtons formula for backward difference

x	0	10	20	30	40
$\sin x$	0	0.17365	0.34202	0.5	0.64279

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x	$\sin x$	∇	∇^2	∇^3	∇^4
0	0				
		0.17365			
10	0.17365		-0.00528		
		0.16837		-0.00511	
20	0.34202		-0.01039		0.00031
		0.15798		-0.0048	
30	0.5		-0.01519		
		0.14279			
40	0.64279				

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$$a = 40 \qquad , \qquad h = 10$$

$$a - nh = 38 \quad \rightarrow \quad 40 - n \times 10 = 38 \quad \rightarrow \quad n = 0.2$$

$$f(a - nh) = f(a) - n\nabla f(a) + \frac{n\,(n-1)}{2!}\nabla^2 f(a) - \frac{n(n-1)(n-2)}{3!}\nabla^3 f(a) + \ldots\ldots..$$

$$\sin 38 = 0.64279 - 0.2 \times 0.14279 + \frac{0.2 \times (0.2 - 1)}{2 \times 1} \times (-0.01519)$$

$$- \frac{0.2 \times (0.2 - 1) \times (0.2 - 2)}{3 \times 2 \times 1} \times (-0.0048)$$

$$+ \frac{0.2 \times (0.2 - 1)(0.2 - 2)(0.2 - 3)}{4 \times 3 \times 2 \times 1} \times 0.00031 = 0.61566$$

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