— University of Mosul —

College of Petroleum & Mining Engineering

Second Order Derivative

Lecture No.2

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LECTURE CONTENTS:

- Second Order Partial Derivatives.
- Partial Derivatives of Still Higher Order.
- Examples.

- when we differentiate a function $f(x,y)$ twice, we produce its second derivatives, these derivatives are usually denoted by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x . \partial x} = f_{xx} \quad , \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y . \partial y} = f_{yy}$$

$$\frac{\partial^2 f}{\partial x . \partial y} = f_{yx} \quad , \quad \frac{\partial^2 f}{\partial y . \partial x} = f_{xy}$$

$$f_{yyx} = \frac{\partial^3 f}{\partial x \cdot \partial y \cdot \partial y} = \frac{\partial^3 f}{\partial x . \partial y^2}$$

ملاحظة: يفضل التعبير عن المشتقة الثانية بهذه الصيغة (f) بدلا
عن (∂).

Ex: find $\frac{\partial^2 z}{\partial x . \partial y}$ if the function $f(x,y) = x^2y^3 + x^4 . y$.

solution:

$$\frac{\partial^2 f}{\partial x . \partial y} = f_{yx}$$

$$f_y = 3x^2y^2 + x^4$$

$$f_{yx} = 6y^2x + 4x^3$$

Ex: if the function $f(x,y) = x.\cos y + y.e^x$, find the second order derivatives :

$$\frac{\partial^2 f}{\partial x^2} , \frac{\partial^2 f}{\partial y.\partial x} , \frac{\partial^2 f}{\partial y^2} , \frac{\partial^2 f}{\partial x.\partial y} .$$

solution:

$$\frac{\partial f}{\partial x} = \cos y + y.e^x.\underbrace{\ln e}_{1}.(1) = \cos y + y.e^x .$$

$$\frac{\partial f}{\partial y} = x.(-\sin y)*1 + e^x = -x.\sin y + e^x .$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\sin y *1 + e^x = -\sin y + e^x .$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^x \cdot \underbrace{\ln e}_{1} \cdot 1 = -\sin y + e^x .$$

$$\frac{\partial^2 f}{\partial x^2} = 0 + y \cdot e^x \cdot \underbrace{\ln e}_{1} * 1 = y \cdot e^x .$$

$$\frac{\partial^2 f}{\partial y^2} = -x \cdot \cos y * 1 + 0 = -x \cdot \cos y .$$

note: if $f(x,y)$ and its' partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throught on open region containing a point (a,b) and are all continuous at (a,b) then

$$\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$$

Ex: find all the second order partial derivatives of the function for the following below:

1- $S(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.

solution:

$$\frac{y}{x} = y \cdot x^{-1}$$

$$\frac{\partial S}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \left(-1 * y * x^{-2}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \frac{-y}{x^2}.$$

$$= \frac{-y}{x^2 + \left(\frac{y^2}{\cancel{x^2}}\right) * \cancel{x^2}} = \frac{-y}{x^2+y^2}.$$

$$\frac{\partial^2 s}{\partial x^2} = \frac{(x^2+y^2)*0-(-y)*2x}{(x^2+y^2)^2} = \boxed{\frac{2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial s}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} * \frac{1}{x} = \frac{1}{x + \frac{y^2}{x^{\cancel{2}}} \cdot \cancel{x}} = \frac{1}{x + \frac{y^2}{x}}$$

$$= \frac{1}{x + \frac{y^2}{x}} * \frac{x}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 S}{\partial y^2} = \frac{(x^2+y^2)*0 - x*2y}{(x^2+y^2)^2} = \boxed{\frac{-2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial^2 S}{\partial x \partial y} = \frac{(x^2+y^2)*1 - x*2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial^2 S}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

$$\frac{\partial^2 S}{\partial y \partial x} = \frac{(x^2+y^2)*-1-(-y)*2y}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial^2 S}{\partial y \partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}}$$

(Partial Derivatives of still higher order)

* Third and fourth-order derivatives denoted by symbols like:

$$\frac{\partial^3 f}{\partial x . \partial y^2} = f_{yyx} , \quad \frac{\partial^4 f}{\partial x^2 . \partial y^2} = f_{yyxx}$$

Ex: find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$.

Solution:

$f_y = -2x(2y) . z + x^2 \longrightarrow f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$