Maximum and Minimum & Saddle Point

Lecture No.4

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LECTURE CONTENTS:

- Maximum Value.
- Minimum Value.
- Saddle Point.
- Examples.

- Let $f(x,y)$ be defined on a region (R) containing the point (a,b) then:

1- $f(a,b)$ is a local Maximum value of (f) if $f(a,b) \geq f(x,y)$ for all domain points (x,y) in an open disk centered at (a,b).

2- $f(a,b)$ is a local Minimum value of (f) if $f(a,b) \leq f(x,y)$ for all domain points in an open disk centered at (a,b).

- if $f(x,y)$ has a local maximum or minimum value at interior point (a,b) of its domain and if the first partial derivatives exist there, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

- An interior point of the domain of a function f(x,y) where both (fx) and (fy) are zero <u>or</u> where one or both of (fx) and (fy) do not exist is <u>a critical point</u> of (f).

- A differentiable function f(x,y) has <u>a saddle point</u> at a critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where f(x,y) > f(a,b) and domain points (x,y) where f(x,y) < f(a,b), the corresponding point (a,b,f(a,b)) on the surface z = f(x,y) is called saddle point of the surface.

- in mathmatics , a saddle point is a point on the surface of the graph of a function where the slopes (derivatives) in orthogonal direction are all zero (a critical point), but which is not a local extremum of the function , An example of a saddle point shown on the right is when there is a critical point with a relative minimum along one axial direction (between peacks) and at a relative max. alonge the crossing axis.

- suppose that $f(x,y)$ and its first and second partial derivatives are continuous through out at disk centered at (a,b) and that $f_x(a,b) = f_y(a,b) = 0$, then

1- (f) has <u>a local Maximum</u> at (a,b) if $\boxed{f_{xx} < 0 \text{ and } f_{xx} \cdot f_{yy} - f_{xy}^2 > 0}$ at (a,b).

2- (f) has <u>a local Minimum</u> at (a,b) if $\boxed{f_{xx} > 0 \text{ and } f_{xx} \cdot f_{yy} - f_{xy}^2 > 0}$ at (a,b).

3- (f) has <u>a saddle point</u> at (a,b) if $\boxed{f_{xx} \cdot f_{yy} - f_{xy}^2 < 0}$ at (a,b).

4- the test is <u>inconclusive</u> at (a,b) if $\boxed{f_{xx} \cdot f_{yy} - f_{xy}^2 = 0}$ at (a,b), in this case we must find some other way to determine the behavior of (f) at (a,b).

<u>خطوات الحل:</u>

١- نقوم بإيجاد الـ (critical points) او (stationary points) عن طريق المشتقة الجزئية الاولى لـ (X) و (Y) ومساواتها بـ (صفر) لايجاد قيم (X) و (Y) ، ويمكن ان يكون الناتج نقطة واحدة او أكثر.

٢- نجد قيمة (D) والتي تساوي :

$$D = f_{xx} \cdot f_{yy} - f_{xy}^2 \quad \text{at } (x_0, y_0)$$

يتم إيجاد قيمة (D) لجميع النقاط في حالة وجود أكثر من نقطة.

٣- نقارن قيمة (D) :

$D > 0$:
- $f_{xx} > 0 \longrightarrow$ local minimum $\cup$
- $f_{xx} < 0 \longrightarrow$ local maximum $\cap$

$D < 0 \longrightarrow$ saddle point

م: و (saddle point) لاهي (max) ولا (min).

Ex: Find all the local maximum and minimum and saddle points of the functions below:

1- $f(x,y) = \dfrac{1}{x^2+y^2-1}$

solution:

$f_x = \dfrac{-2x}{(x^2+y^2-1)^2} \Rightarrow \dfrac{-2x}{(x^2+y^2-1)^2} = 0 \rightarrow \boxed{x = 0}$

$f_y = \dfrac{-2y}{(x^2+y^2-1)^2} \Rightarrow \dfrac{-2y}{(x^2+y^2-1)^2} = 0 \rightarrow \boxed{y = 0}$

the critical point is $(0,0)$

نعوض قيم (x,y) التي تم إيجادها في المعادلة الاصلية $(0,0)$

$f(0,0) = \dfrac{1}{(0)^2+(0)^2-1} \Rightarrow f(0,0) = -1 \ @ (0,0)$

(23)

$$f_{xx} = \frac{((x^2+y^2-1)^2)*(-2)-(-2x)*(2(x^2+y^2-1)*(2x)}{(x^2+y^2-1)^4}$$

نعوض قيمة (x) و (y) ← (0,0):

$$f_{xx} = \frac{(-1)^2*(-2)+0}{(-1)^4} = \frac{-2}{1} = -2$$

$\Rightarrow f_{xx} = -2$ at $x=0, y=0$.

$$f_{yy} = \frac{(x^2+y^2-1)^2*(-2)-(-2y)*(2(x^2+y^2-1)*(2y)}{(x^2+y^2-1)^4}$$

$$f_{yy} = \frac{(-1)^2*(-2)+0}{(-1)^4} = -2$$

$\Rightarrow f_{yy} = -2$ at $x=0, y=0$.

$$f_{xy} = \frac{(x^2+y^2-1)^2 * 0 - (-2x) * (2(x^2+y^2-1) * (2y))}{(x^2+y^2-1)^4}$$

$$f_{xy} = \frac{0 + 0}{(-1)^4}$$

$\Rightarrow f_y = 0$ at $x = 0$ or $y = 0$

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (-2) * (-2) - (0)^2$$

$$D = 4 > 0$$

$$f_{xx} < 0$$

$D > 0$: $f_{xx} > 0$ min

$D > 0$: $f_{xx} < 0$ max

$\Rightarrow$ The point is local maximum.