Kinematics of a Particle

12

CHAPTER OBJECTIVES

- To introduce the concepts of position, displacement, velocity, and acceleration.
- To study particle motion along a straight line and represent this motion graphically.
- To investigate particle motion along a curved path using different coordinate systems.
- To present an analysis of dependent motion of two particles.
- To examine the principles of relative motion of two particles using translating axes.

12.1 Introduction

Mechanics is a branch of the physical sciences that is concerned with the state of rest or motion of bodies subjected to the action of forces. Engineering mechanics is divided into two areas of study, namely, statics and dynamics. *Statics* is concerned with the equilibrium of a body that is either at rest or moves with constant velocity. Here we will consider *dynamics*, which deals with the accelerated motion of a body. The subject of dynamics will be presented in two parts: *kinematics*, which treats only the geometric aspects of the motion, and *kinetics*, which is the analysis of the forces causing the motion. To develop these principles, the dynamics of a particle will be discussed first, followed by topics in rigid-body dynamics in two and then three dimensions.

Historically, the principles of dynamics developed when it was possible to make an accurate measurement of time. Galileo Galilei (1564–1642) was one of the first major contributors to this field. His work consisted of experiments using pendulums and falling bodies. The most significant contributions in dynamics, however, were made by Isaac Newton (1642–1727), who is noted for his formulation of the three fundamental laws of motion and the law of universal gravitational attraction. Shortly after these laws were postulated, important techniques for their application were developed by Euler, D'Alembert, Lagrange, and others.

There are many problems in engineering whose solutions require application of the principles of dynamics. Typically the structural design of any vehicle, such as an automobile or airplane, requires consideration of the motion to which it is subjected. This is also true for many mechanical devices, such as motors, pumps, movable tools, industrial manipulators, and machinery. Furthermore, predictions of the motions of artificial satellites, projectiles, and spacecraft are based on the theory of dynamics. With further advances in technology, there will be an even greater need for knowing how to apply the principles of this subject.

Problem Solving. Dynamics is considered to be more involved than statics since both the forces applied to a body and its motion must be taken into account. Also, many applications require using calculus, rather than just algebra and trigonometry. In any case, the most effective way of learning the principles of dynamics is *to solve problems*. To be successful at this, it is necessary to present the work in a logical and orderly manner as suggested by the following sequence of steps:

1. Read the problem carefully and try to correlate the actual physical situation with the theory you have studied.
2. Draw any necessary diagrams and tabulate the problem data.
3. Establish a coordinate system and apply the relevant principles, generally in mathematical form.
4. Solve the necessary equations algebraically as far as practical; then, use a consistent set of units and complete the solution numerically. Report the answer with no more significant figures than the accuracy of the given data.
5. Study the answer using technical judgment and common sense to determine whether or not it seems reasonable.
6. Once the solution has been completed, review the problem. Try to think of other ways of obtaining the same solution.

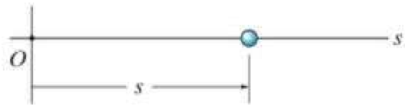
In applying this general procedure, do the work as neatly as possible. Being neat generally stimulates clear and orderly thinking, and vice versa.

12.2 Rectilinear Kinematics: Continuous Motion

We will begin our study of dynamics by discussing the kinematics of a particle that moves along a rectilinear or straight line path. Recall that a *particle* has a mass but negligible size and shape. Therefore we must limit application to those objects that have dimensions that are of no consequence in the analysis of the motion. In most problems, we will be interested in bodies of finite size, such as rockets, projectiles, or vehicles. Each of these objects can be considered as a particle, as long as the motion is characterized by the motion of its mass center and any rotation of the body is neglected.

Rectilinear Kinematics. The kinematics of a particle is characterized by specifying, at any given instant, the particle's position, velocity, and acceleration.

Position. The straight-line path of a particle will be defined using a single coordinate axis s, Fig. 12–1a. The origin O on the path is a fixed point, and from this point the *position coordinate* s is used to specify the location of the particle at any given instant. The magnitude of s is the distance from O to the particle, usually measured in meters (m) or feet (ft), and the sense of direction is defined by the algebraic sign on s. Although the choice is arbitrary, in this case s is positive since the coordinate axis is positive to the right of the origin. Likewise, it is negative if the particle is located to the left of O. Realize that position is a vector quantity since it has both magnitude and direction. Here, however, it is being represented by the algebraic scalar s since the direction always remains along the coordinate axis.

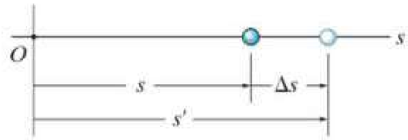


Position

(a)

Displacement. The *displacement* of the particle is defined as the *change* in its *position*. For example, if the particle moves from one point to another, Fig. 12–1b, the displacement is

$$\Delta s = s' - s$$



Displacement

(b)

Fig. 12–1

In this case Δs is *positive* since the particle's final position is to the *right* of its initial position, i.e., $s' > s$. Likewise, if the final position were to the *left* of its initial position, Δs would be *negative*.

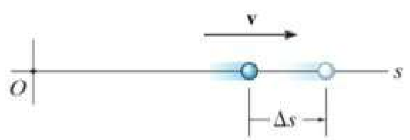
The displacement of a particle is also a *vector quantity*, and it should be distinguished from the distance the particle travels. Specifically, the *distance traveled* is a *positive scalar* that represents the total length of path over which the particle travels.

Velocity. If the particle moves through a displacement Δs during the time interval Δt, the *average velocity* of the particle during this time interval is

$$v_{avg} = \frac{\Delta s}{\Delta t}$$

If we take smaller and smaller values of Δt, the magnitude of Δs becomes smaller and smaller. Consequently, the *instantaneous velocity* is a vector defined as $v = \lim_{\Delta t \to 0} (\Delta s/\Delta t)$, or

$$(\overset{+}{\rightarrow}) \qquad v = \frac{ds}{dt} \qquad (12\text{–}1)$$



Velocity

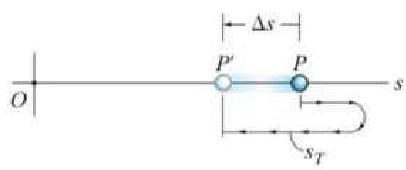
(c)

Since Δt or dt is always positive, the sign used to define the *sense* of the velocity is the same as that of Δs or ds. For example, if the particle is moving to the *right*, Fig. 12–1*c*, the velocity is *positive;* whereas if it is moving to the *left*, the velocity is *negative*. (This is emphasized here by the arrow written at the left of Eq. 12–1.) The *magnitude* of the velocity is known as the *speed*, and it is generally expressed in units of m/s or ft/s.

Occasionally, the term "average speed" is used. The *average speed* is always a positive scalar and is defined as the total distance traveled by a particle, s_T, divided by the elapsed time Δt; i.e.,

$$(v_{sp})_{avg} = \frac{s_T}{\Delta t}$$

For example, the particle in Fig. 12–1*d* travels along the path of length s_T in time Δt, so its average speed is $(v_{sp})_{avg} = s_T/\Delta t$, but its average velocity is $v_{avg} = -\Delta s/\Delta t$.



Average velocity and
Average speed

(d)

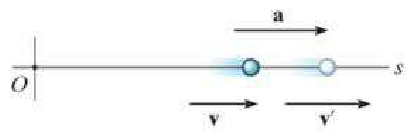
Fig. 12–1 (cont.)

Acceleration. Provided the velocity of the particle is known at two points, the *average acceleration* of the particle during the time interval Δt is defined as

$$a_{\text{avg}} = \frac{\Delta v}{\Delta t}$$

Here Δv represents the difference in the velocity during the time interval Δt, i.e., $\Delta v = v' - v$, Fig. 12–1*e*.

The *instantaneous acceleration* at time t is a vector that is found by taking smaller and smaller values of Δt and corresponding smaller and smaller values of Δv, so that $a = \lim_{\Delta t \to 0} (\Delta v/\Delta t)$, or



Acceleration

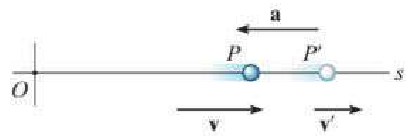
(e)

$$(\overset{+}{\rightarrow}) \qquad a = \frac{dv}{dt} \qquad (12\text{–}2)$$

Substituting Eq. 12–1 into this result, we can also write

$$(\overset{+}{\rightarrow}) \qquad a = \frac{d^2 s}{dt^2}$$

Both the average and instantaneous acceleration can be either positive or negative. In particular, when the particle is *slowing down*, or its speed is decreasing, the particle is said to be *decelerating*. In this case, v' in Fig. 12–1*f* is *less* than v, and so $\Delta v = v' - v$ will be negative. Consequently, a will also be negative, and therefore it will act to the *left*, in the *opposite sense* to v. Also, note that when the *velocity* is *constant*, the *acceleration is zero* since $\Delta v = v - v = 0$. Units commonly used to express the magnitude of acceleration are m/s^2 or ft/s^2.



Deceleration

(f)

Finally, an important differential relation involving the displacement, velocity, and acceleration along the path may be obtained by eliminating the time differential dt between Eqs. 12–1 and 12–2, which gives

$$(\overset{+}{\rightarrow}) \qquad a\,ds = v\,dv \qquad (12\text{–}3)$$

Although we have now produced three important kinematic equations, realize that the above equation is not independent of Eqs. 12–1 and 12–2.

Constant Acceleration, $a = a_c$. When the acceleration is constant, each of the three kinematic equations $a_c = dv/dt$, $v = ds/dt$, and $a_c\,ds = v\,dv$ can be integrated to obtain formulas that relate a_c, v, s, and t.

Velocity as a Function of Time. Integrate $a_c = dv/dt$, assuming that initially $v = v_0$ when $t = 0$.

$$\int_{v_0}^{v} dv = \int_0^t a_c\,dt$$

$(\overset{+}{\rightarrow})$

$$v = v_0 + a_c t \quad \text{Constant Acceleration} \qquad (12\text{–}4)$$

Position as a Function of Time. Integrate $v = ds/dt = v_0 + a_c t$, assuming that initially $s = s_0$ when $t = 0$.

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + a_c t)\,dt$$

$(\overset{+}{\rightarrow})$

$$s = s_0 + v_0 t + \tfrac{1}{2}a_c t^2 \quad \text{Constant Acceleration} \qquad (12\text{–}5)$$

Velocity as a Function of Position. Either solve for t in Eq. 12–4 and substitute into Eq. 12–5, or integrate $v\,dv = a_c\,ds$, assuming that initially $v = v_0$ at $s = s_0$.

$$\int_{v_0}^{v} v\,dv = \int_{s_0}^{s} a_c\,ds$$

$(\overset{+}{\rightarrow})$

$$v^2 = v_0^2 + 2a_c(s - s_0) \quad \text{Constant Acceleration} \qquad (12\text{–}6)$$

The algebraic signs of s_0, v_0, and a_c, used in the above three equations, are determined from the positive direction of the s axis as indicated by the arrow written at the left of each equation. Remember that these equations are useful *only when the acceleration is constant and when* $t = 0$, $s = s_0$, $v = v_0$. A typical example of constant accelerated motion occurs when a body falls freely toward the earth. If air resistance is neglected and the distance of fall is short, then the *downward* acceleration of the body when it is close to the earth is constant and approximately $9.81\ \text{m/s}^2$ or $32.2\ \text{ft/s}^2$. The proof of this is given in Example 13.2.

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EXAMPLE 12.1

The car in Fig. 12–2 moves in a straight line such that for a short time its velocity is defined by $v = (3t^2 + 2t)$ ft/s, where t is in seconds. Determine its position and acceleration when $t = 3$ s. When $t = 0$, $s = 0$.

Fig. 12–2

SOLUTION

Coordinate System. The position coordinate extends from the fixed origin O to the car, positive to the right.

Position. Since $v = f(t)$, the car's position can be determined from $v = ds/dt$, since this equation relates v, s, and t. Noting that $s = 0$ when $t = 0$, we have*

$$(\overset{+}{\rightarrow}) \qquad v = \frac{ds}{dt} = (3t^2 + 2t)$$

$$\int_0^s ds = \int_0^t (3t^2 + 2t)dt$$

$$s\Big|_0^s = t^3 + t^2\Big|_0^t$$

$$s = t^3 + t^2$$

When $t = 3$ s,

$$s = (3)^3 + (3)^2 = 36 \text{ ft} \qquad \textit{Ans.}$$

Acceleration. Since $v = f(t)$, the acceleration is determined from $a = dv/dt$, since this equation relates a, v, and t.

$$(\overset{+}{\rightarrow}) \qquad a = \frac{dv}{dt} = \frac{d}{dt}(3t^2 + 2t)$$

$$= 6t + 2$$

When $t = 3$ s,

$$a = 6(3) + 2 = 20 \text{ ft/s}^2 \rightarrow \qquad \textit{Ans.}$$

NOTE: The formulas for constant acceleration *cannot* be used to solve this problem, because the acceleration is a function of time.

*The *same result* can be obtained by evaluating a constant of integration C rather than using definite limits on the integral. For example, integrating $ds = (3t^2 + 2t)dt$ yields $s = t^3 + t^2 + C$. Using the condition that at $t = 0$, $s = 0$, then $C = 0$.

12.3 Rectilinear Kinematics: Erratic Motion

When a particle has erratic or changing motion then its position, velocity, and acceleration *cannot* be described by a single continuous mathematical function along the entire path. Instead, a series of functions will be required to specify the motion at different intervals. For this reason, it is convenient to represent the motion as a graph. If a graph of the motion that relates any two of the variables s, v, a, t can be drawn, then this graph can be used to construct subsequent graphs relating two other variables since the variables are related by the differential relationships $v = ds/dt$, $a = dv/dt$, or $a\,ds = v\,dv$. Several situations occur frequently.

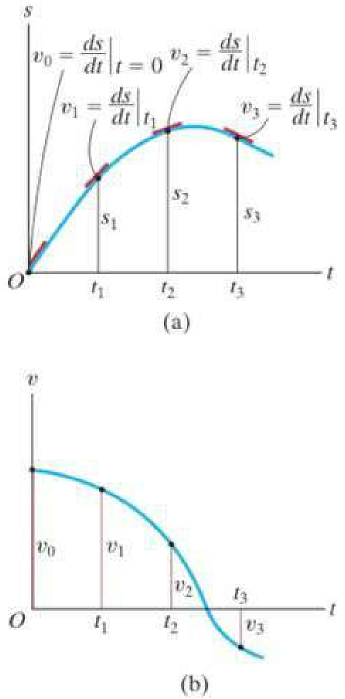


Fig. 12–7

The *s–t*, *v–t*, and *a–t* Graphs. To construct the v–t graph given the s–t graph, Fig. 12–7*a*, the equation $v = ds/dt$ should be used, since it relates the variables s and t to v. This equation states that

$$\frac{ds}{dt} = v$$

slope of s–t graph = velocity

For example, by measuring the slope on the s–t graph when $t = t_1$, the velocity is v_1, which is plotted in Fig. 12–7*b*. The v–t graph can be constructed by plotting this and other values at each instant.

The a–t graph can be constructed from the v–t graph in a similar manner, Fig. 12–8, since

$$\frac{dv}{dt} = a$$

slope of v–t graph = acceleration

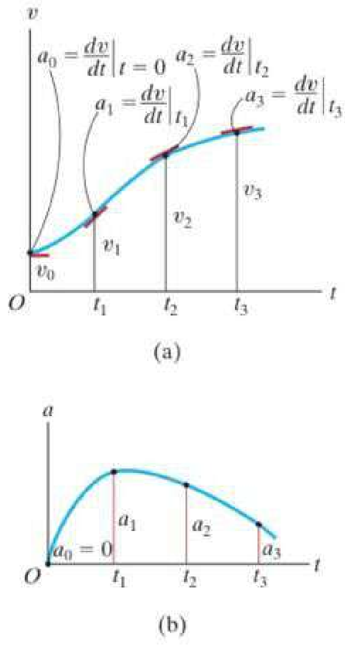


Fig. 12–8

Examples of various measurements are shown in Fig. 12–8*a* and plotted in Fig. 12–8*b*.

If the s–t curve for each interval of motion can be expressed by a mathematical function $s = s(t)$, then the equation of the v–t graph for the same interval can be obtained by differentiating this function with respect to time since $v = ds/dt$. Likewise, the equation of the a–t graph for the same interval can be determined by differentiating $v = v(t)$ since $a = dv/dt$. Since differentiation reduces a polynomial of degree n to that of degree $n - 1$, then if the s–t graph is parabolic (a second-degree curve), the v–t graph will be a sloping line (a first-degree curve), and the a–t graph will be a constant or a horizontal line (a zero-degree curve).

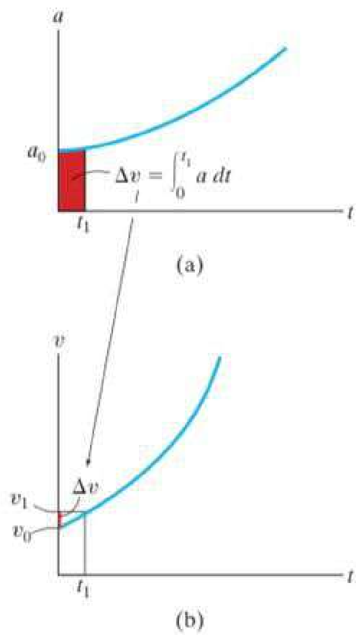


Fig. 12–9

If the a–t graph is given, Fig. 12–9a, the v–t graph may be constructed using $a = dv/dt$, written as

$$\Delta v = \int a\,dt$$

change in velocity = area under a–t graph

Hence, to construct the v–t graph, we begin with the particle's initial velocity v_0 and then add to this small increments of area (Δv) determined from the a–t graph. In this manner successive points, $v_1 = v_0 + \Delta v$, etc., for the v–t graph are determined, Fig. 12–9b. Notice that an algebraic addition of the area increments of the a–t graph is necessary, since areas lying above the t axis correspond to an increase in v ("positive" area), whereas those lying below the axis indicate a decrease in v ("negative" area).

Similarly, if the v–t graph is given, Fig. 12–10a, it is possible to determine the s–t graph using $v = ds/dt$, written as

$$\Delta s = \int v\,dt$$

displacement = area under v–t graph

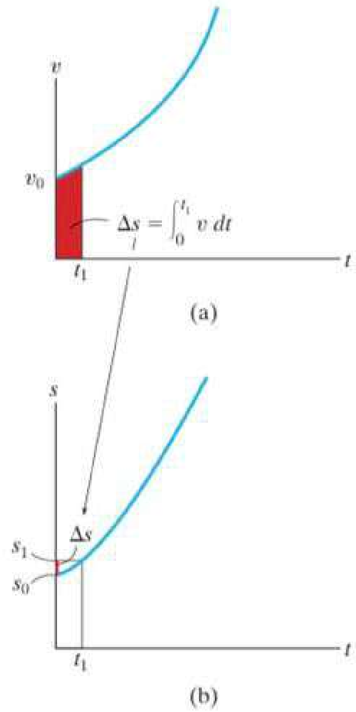


Fig. 12–10

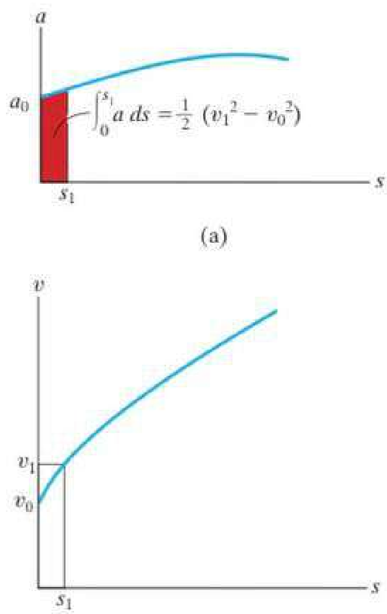
In the same manner as stated above, we begin with the particle's initial position s_0 and add (algebraically) to this small area increments Δs determined from the v–t graph, Fig. 12–10b.

If segments of the a–t graph can be described by a series of equations, then each of these equations can be *integrated* to yield equations describing the corresponding segments of the v–t graph. In a similar manner, the s–t graph can be obtained by integrating the equations which describe the segments of the v–t graph. As a result, if the a–t graph is linear (a first-degree curve), integration will yield a v–t graph that is parabolic (a second-degree curve) and an s–t graph that is cubic (third-degree curve).

The *v–s* and *a–s* Graphs. If the *a–s* graph can be constructed, then points on the *v–s* graph can be determined by using $v\,dv = a\,ds$. Integrating this equation between the limits $v = v_0$ at $s = s_0$ and $v = v_1$ at $s = s_1$, we have,

$$\tfrac{1}{2}(v_1^2 - v_0^2) = \int_{s_0}^{s_1} a\,ds$$

area under
a–s graph



(a)

(b)

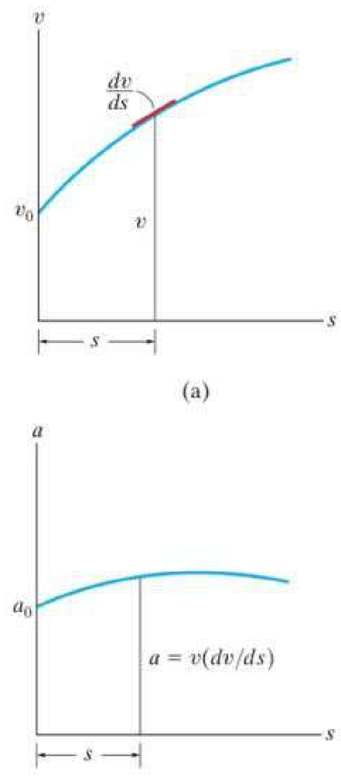
Fig. 12–11

Therefore, if the red area in Fig. 12–11*a* is determined, and the initial velocity v_0 at $s_0 = 0$ is known, then $v_1 = \left(2\int_{s_0}^{s_1} a\,ds + v_0^2\right)^{1/2}$, Fig. 12–11*b*. Successive points on the *v–s* graph can be constructed in this manner.

If the *v–s* graph is known, the acceleration *a* at any position *s* can be determined using $a\,ds = v\,dv$, written as

$$a = v\left(\frac{dv}{ds}\right)$$

acceleration = velocity times slope of *v–s* graph



(a)

(b)

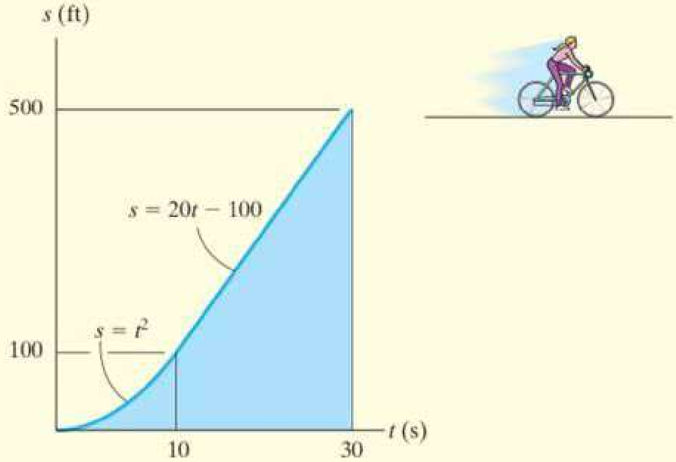
Fig. 12–12

Thus, at any point (s, v) in Fig. 12–12*a*, the slope dv/ds of the *v–s* graph is measured. Then with v and dv/ds known, the value of a can be calculated, Fig. 12–12*b*.

The *v–s* graph can also be constructed from the *a–s* graph, or vice versa, by approximating the known graph in various intervals with mathematical functions, $v = f(s)$ or $a = g(s)$, and then using $a\,ds = v\,dv$ to obtain the other graph.

EXAMPLE 12.6

A bicycle moves along a straight road such that its position is described by the graph shown in Fig. 12–13*a*. Construct the v–t and a–t graphs for $0 \le t \le 30$ s.



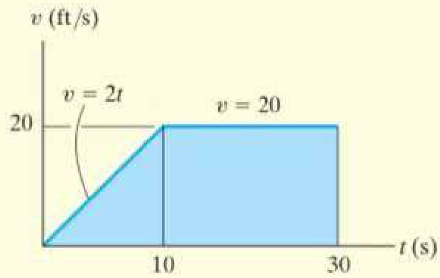
(a)

SOLUTION

v–t Graph. Since $v = ds/dt$, the v–t graph can be determined by differentiating the equations defining the s–t graph, Fig. 12–13*a*. We have

$$0 \le t < 10\text{ s}; \qquad s = (t^2)\text{ ft} \qquad v = \frac{ds}{dt} = (2t)\text{ ft/s}$$

$$10\text{ s} < t \le 30\text{ s}; \qquad s = (20t - 100)\text{ ft} \qquad v = \frac{ds}{dt} = 20\text{ ft/s}$$



(b)

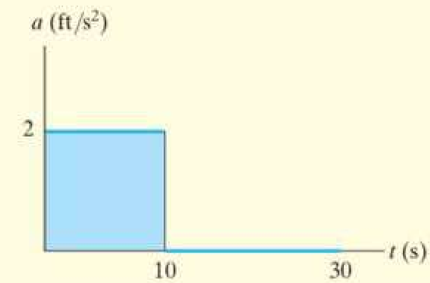
The results are plotted in Fig. 12–13*b*. We can also obtain specific values of v by measuring the *slope* of the s–t graph at a given instant. For example, at $t = 20$ s, the slope of the s–t graph is determined from the straight line from 10 s to 30 s, i.e.,

$$t = 20\text{ s}; \qquad v = \frac{\Delta s}{\Delta t} = \frac{500\text{ ft} - 100\text{ ft}}{30\text{ s} - 10\text{ s}} = 20\text{ ft/s}$$

a–t Graph. Since $a = dv/dt$, the a–t graph can be determined by differentiating the equations defining the lines of the v–t graph. This yields

$$0 \le t < 10\text{ s}; \qquad v = (2t)\text{ ft/s} \qquad a = \frac{dv}{dt} = 2\text{ ft/s}^2$$

$$10 < t \le 30\text{ s}; \qquad v = 20\text{ ft/s} \qquad a = \frac{dv}{dt} = 0$$



(c)

Fig. 12–13

The results are plotted in Fig. 12–13*c*.

NOTE: Show that $a = 2\text{ ft/s}^2$ when $t = 5$ s by measuring the slope of the v–t graph.

12.4 General Curvilinear Motion

Curvilinear motion occurs when a particle moves along a curved path. Since this path is often described in three dimensions, vector analysis will be used to formulate the particle's position, velocity, and acceleration.* In this section the general aspects of curvilinear motion are discussed, and in subsequent sections we will consider three types of coordinate systems often used to analyze this motion.

Position. Consider a particle located at a point on a space curve defined by the path function $s(t)$, Fig. 12–16*a*. The position of the particle, measured from a fixed point O, will be designated by the *position vector* $\mathbf{r} = \mathbf{r}(t)$. Notice that both the magnitude and direction of this vector will change as the particle moves along the curve.

Displacement. Suppose that during a small time interval Δt the particle moves a distance Δs along the curve to a new position, defined by $\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r}$, Fig. 12–16*b*. The *displacement* $\Delta\mathbf{r}$ represents the change in the particle's position and is determined by vector subtraction; i.e., $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$.

Velocity. During the time Δt, the *average velocity* of the particle is

$$\mathbf{v}_{\text{avg}} = \frac{\Delta\mathbf{r}}{\Delta t}$$

The *instantaneous velocity* is determined from this equation by letting $\Delta t \to 0$, and consequently the direction of $\Delta\mathbf{r}$ *approaches* the *tangent* to the curve. Hence, $\mathbf{v} = \lim_{\Delta t \to 0} (\Delta\mathbf{r}/\Delta t)$ or

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \qquad (12\text{–}7)$$

Since $d\mathbf{r}$ will be tangent to the curve, the *direction* of $\mathbf{v}$ is also *tangent to the curve*, Fig. 12–16*c*. The *magnitude* of $\mathbf{v}$, which is called the *speed*, is obtained by realizing that the length of the straight line segment $\Delta\mathbf{r}$ in Fig. 12–16*b* approaches the arc length Δs as $\Delta t \to 0$, we have $v = \lim_{\Delta t \to 0} (\Delta r/\Delta t) = \lim_{\Delta t \to 0} (\Delta s/\Delta t)$, or

$$v = \frac{ds}{dt} \qquad (12\text{–}8)$$

Thus, the *speed* can be obtained by differentiating the path function s with respect to time.

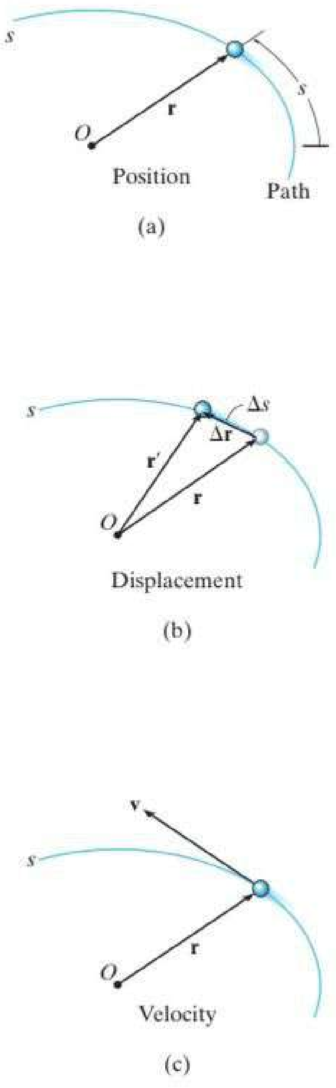
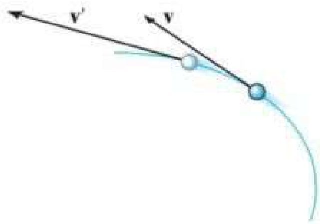


Fig. 12–16

*A summary of some of the important concepts of vector analysis is given in Appendix B.

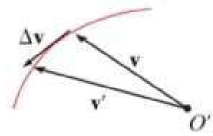
Acceleration. If the particle has a velocity $\mathbf{v}$ at time t and a velocity $\mathbf{v}' = \mathbf{v} + \Delta\mathbf{v}$ at $t + \Delta t$, Fig. 12–16*d*, then the *average acceleration* of the particle during the time interval Δt is



(d)

$$\mathbf{a}_{avg} = \frac{\Delta\mathbf{v}}{\Delta t}$$

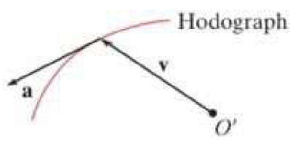
where $\Delta\mathbf{v} = \mathbf{v}' - \mathbf{v}$. To study this time rate of change, the two velocity vectors in Fig. 12–16*d* are plotted in Fig. 12–16*e* such that their tails are located at the fixed point O' and their arrowheads touch points on a curve. This curve is called a *hodograph*, and when constructed, it describes the locus of points for the arrowhead of the velocity vector in the same manner as the *path s* describes the locus of points for the arrowhead of the position vector, Fig. 12–16*a*.



(e)

To obtain the *instantaneous acceleration*, let $\Delta t \to 0$ in the above equation. In the limit $\Delta\mathbf{v}$ will approach the *tangent to the hodograph*, and so $\mathbf{a} = \lim_{\Delta t \to 0} (\Delta\mathbf{v}/\Delta t)$, or

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \qquad (12\text{–}9)$$

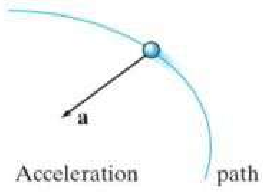


(f)

Substituting Eq. 12–7 into this result, we can also write

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}$$

By definition of the derivative, $\mathbf{a}$ acts *tangent to the hodograph*, Fig. 12–16*f*, and, *in general it is not tangent to the path of motion*, Fig. 12–16*g*. To clarify this point, realize that $\Delta\mathbf{v}$ and consequently $\mathbf{a}$ must account for the change made in *both* the magnitude *and* direction of the velocity $\mathbf{v}$ as the particle moves from one point to the next along the path, Fig. 12–16*d*. However, in order for the particle to follow any curved path, the directional change always "swings" the velocity vector toward the "inside" or "concave side" of the path, and therefore $\mathbf{a}$ *cannot* remain tangent to the path. In summary, $\mathbf{v}$ is always tangent to the *path* and $\mathbf{a}$ is always tangent to the *hodograph*.



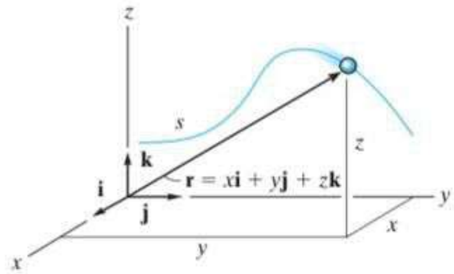
(g)

Fig. 12–16 (cont.)

12

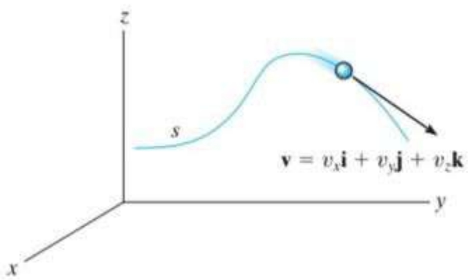
12.5 Curvilinear Motion: Rectangular Components

Occasionally the motion of a particle can best be described along a path that can be expressed in terms of its x, y, z coordinates.



Position

(a)



Velocity

(b)

Fig. 12–17

Position. If the particle is at point (x, y, z) on the curved path s shown in Fig. 12–17a, then its location is defined by the *position vector*

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \qquad (12\text{–}10)$$

When the particle moves, the x, y, z components of $\mathbf{r}$ will be functions of time; i.e., $x = x(t)$, $y = y(t)$, $z = z(t)$, so that $\mathbf{r} = \mathbf{r}(t)$.

At any instant the *magnitude* of $\mathbf{r}$ is defined from Eq. C–3 in Appendix C as

$$r = \sqrt{x^2 + y^2 + z^2}$$

And the *direction* of $\mathbf{r}$ is specified by the unit vector $\mathbf{u}_r = \mathbf{r}/r$.

Velocity. The first time derivative of $\mathbf{r}$ yields the velocity of the particle. Hence,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$

When taking this derivative, it is necessary to account for changes in *both* the magnitude and direction of each of the vector's components. For example, the derivative of the $\mathbf{i}$ component of $\mathbf{r}$ is

$$\frac{d}{dt}(x\mathbf{i}) = \frac{dx}{dt}\mathbf{i} + x\frac{d\mathbf{i}}{dt}$$

The second term on the right side is zero, provided the x, y, z reference frame is *fixed*, and therefore the *direction* (and the *magnitude*) of $\mathbf{i}$ does not change with time. Differentiation of the $\mathbf{j}$ and $\mathbf{k}$ components may be carried out in a similar manner, which yields the final result,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \qquad (12\text{–}11)$$

where

$$v_x = \dot{x} \quad v_y = \dot{y} \quad v_z = \dot{z} \qquad (12\text{–}12)$$

The "dot" notation $\dot{x}$, $\dot{y}$, $\dot{z}$ represents the first time derivatives of $x = x(t)$, $y = y(t)$, $z = z(t)$, respectively.

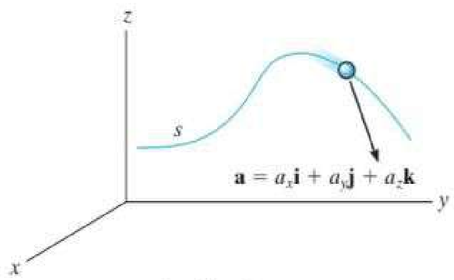
The velocity has a *magnitude* that is found from

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

and a *direction* that is specified by the unit vector $\mathbf{u}_v = \mathbf{v}/v$. As discussed in Sec. 12–4, this direction is *always tangent to the path*, as shown in Fig. 12–17*b*.

Acceleration. The acceleration of the particle is obtained by taking the first time derivative of Eq. 12–11 (or the second time derivative of Eq. 12–10). We have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{12–13}$$



Acceleration

(c)

where

$$\begin{aligned} a_x &= \dot{v}_x = \ddot{x} \\ a_y &= \dot{v}_y = \ddot{y} \\ a_z &= \dot{v}_z = \ddot{z} \end{aligned} \tag{12–14}$$

Here a_x, a_y, a_z represent, respectively, the first time derivatives of $v_x = v_x(t)$, $v_y = v_y(t)$, $v_z = v_z(t)$, or the second time derivatives of the functions $x = x(t)$, $y = y(t)$, $z = z(t)$.

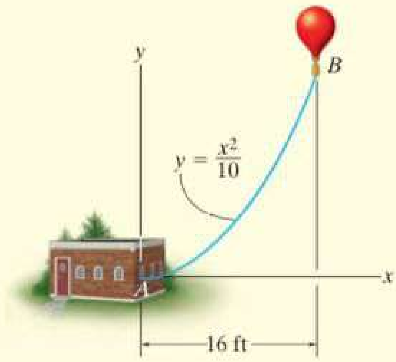
The acceleration has a *magnitude*

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

and a *direction* specified by the unit vector $\mathbf{u}_a = \mathbf{a}/a$. Since $\mathbf{a}$ represents the time rate of *change* in both the magnitude and direction of the velocity, in general $\mathbf{a}$ will *not* be tangent to the path, Fig. 12–17*c*.

EXAMPLE 12.9

At any instant the horizontal position of the weather balloon in Fig. 12–18*a* is defined by $x = (8t)$ ft, where t is in seconds. If the equation of the path is $y = x^2/10$, determine the magnitude and direction of the velocity and the acceleration when $t = 2$ s.



(a)

SOLUTION

Velocity. The velocity component in the x direction is

$$v_x = \dot{x} = \frac{d}{dt}(8t) = 8 \text{ ft/s} \rightarrow$$

To find the relationship between the velocity components we will use the chain rule of calculus. (See Appendix A for a full explanation.)

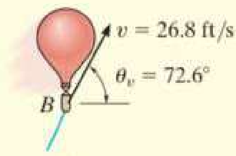
$$v_y = \dot{y} = \frac{d}{dt}(x^2/10) = 2x\dot{x}/10 = 2(16)(8)/10 = 25.6 \text{ ft/s} \uparrow$$

When $t = 2$ s, the magnitude of velocity is therefore

$$v = \sqrt{(8 \text{ ft/s})^2 + (25.6 \text{ ft/s})^2} = 26.8 \text{ ft/s} \qquad \textit{Ans.}$$

The direction is tangent to the path, Fig. 12–18*b*, where

$$\theta_v = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{25.6}{8} = 72.6° \qquad \textit{Ans.}$$



(b)

Acceleration. The relationship between the acceleration components is determined using the chain rule. (See Appendix C.) We have

$$a_x = \dot{v}_x = \frac{d}{dt}(8) = 0$$

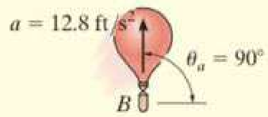
$$a_y = \dot{v}_y = \frac{d}{dt}(2x\dot{x}/10) = 2(\dot{x})\dot{x}/10 + 2x(\ddot{x})/10$$
$$= 2(8)^2/10 + 2(16)(0)/10 = 12.8 \text{ ft/s}^2 \uparrow$$

Thus,

$$a = \sqrt{(0)^2 + (12.8)^2} = 12.8 \text{ ft/s}^2 \qquad \textit{Ans.}$$

The direction of **a**, as shown in Fig. 12–18*c*, is

$$\theta_a = \tan^{-1}\frac{12.8}{0} = 90° \qquad \textit{Ans.}$$



(c)

Fig. 12–18

NOTE: It is also possible to obtain v_y and a_y by first expressing $y = f(t) = (8t)^2/10 = 6.4t^2$ and then taking successive time derivatives.

12.6 Motion of a Projectile

The free-flight motion of a projectile is often studied in terms of its rectangular components. To illustrate the kinematic analysis, consider a projectile launched at point (x_0, y_0), with an initial velocity of $\mathbf{v}_0$, having components $(\mathbf{v}_0)_x$ and $(\mathbf{v}_0)_y$, Fig. 12–20. When air resistance is neglected, the only force acting on the projectile is its weight, which causes the projectile to have a *constant downward acceleration* of approximately $a_c = g = 9.81\ \text{m/s}^2$ or $g = 32.2\ \text{ft/s}^2$.*

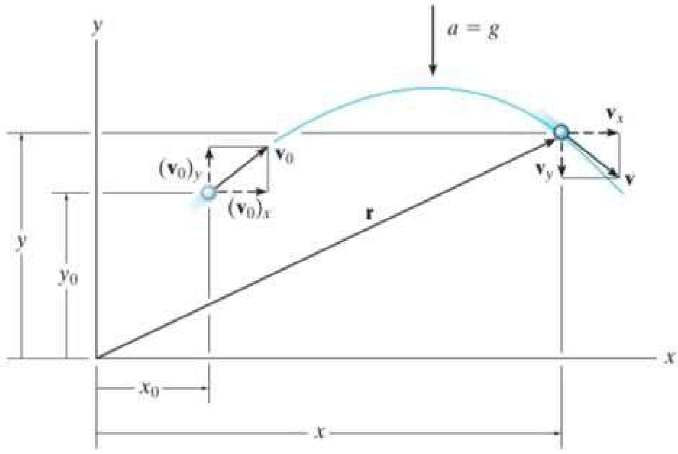
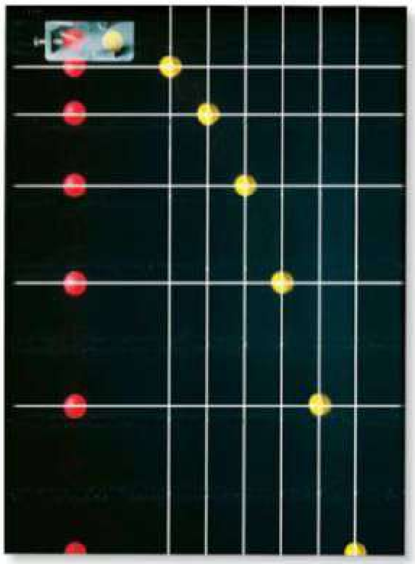


Fig. 12–20



Each picture in this sequence is taken after the same time interval. The red ball falls from rest, whereas the yellow ball is given a horizontal velocity when released. Both balls accelerate downward at the same rate, and so they remain at the same elevation at any instant. This acceleration causes the difference in elevation between the balls to increase between successive photos. Also, note the horizontal distance between successive photos of the yellow ball is constant since the velocity in the horizontal direction remains constant.

Horizontal Motion. Since $a_x = 0$, application of the constant acceleration equations, 12–4 to 12–6, yields

$$(\overset{+}{\rightarrow}) \qquad v = v_0 + a_c t; \qquad v_x = (v_0)_x$$

$$(\overset{+}{\rightarrow}) \qquad x = x_0 + v_0 t + \tfrac{1}{2} a_c t^2; \qquad x = x_0 + (v_0)_x t$$

$$(\overset{+}{\rightarrow}) \qquad v^2 = v_0^2 + 2a_c(x - x_0); \qquad v_x = (v_0)_x$$

The first and last equations indicate that *the horizontal component of velocity always remains constant during the motion.*

Vertical Motion. Since the positive y axis is directed upward, then $a_y = -g$. Applying Eqs. 12–4 to 12–6, we get

$$(+\uparrow) \qquad v = v_0 + a_c t; \qquad v_y = (v_0)_y - gt$$

$$(+\uparrow) \qquad y = y_0 + v_0 t + \tfrac{1}{2} a_c t^2; \qquad y = y_0 + (v_0)_y t - \tfrac{1}{2} g t^2$$

$$(+\uparrow) \qquad v^2 = v_0^2 + 2a_c(y - y_0); \qquad v_y^2 = (v_0)_y^2 - 2g(y - y_0)$$

Recall that the last equation can be formulated on the basis of eliminating the time t from the first two equations, and therefore *only two of the above three equations are independent of one another.*

* This assumes that the earth's gravitational field does not vary with altitude.

To summarize, problems involving the motion of a projectile can have at most three unknowns since only three independent equations can be written; that is, *one* equation in the *horizontal direction* and *two* in the *vertical direction*. Once $\mathbf{v}_x$ and $\mathbf{v}_y$ are obtained, the resultant velocity $\mathbf{v}$, which is *always tangent* to the path, can be determined by the *vector sum* as shown in Fig. 12–20.

Procedure for Analysis

Coordinate System.

- Establish the fixed x, y coordinate axes and sketch the trajectory of the particle. Between any *two points* on the path specify the given problem data and identify the *three unknowns*. In all cases the acceleration of gravity acts downward and equals 9.81 m/s^2 or 32.2 ft/s^2. The particle's initial and final velocities should be represented in terms of their x and y components.
- Remember that positive and negative position, velocity, and acceleration components always act in accordance with their associated coordinate directions.

Kinematic Equations.

- Depending upon the known data and what is to be determined, a choice should be made as to which three of the following four equations should be applied between the two points on the path to obtain the most direct solution to the problem.

Horizontal Motion.

- The *velocity* in the horizontal or x direction is *constant*, i.e., $v_x = (v_0)_x$, and

$$x = x_0 + (v_0)_x t$$

Vertical Motion.

- In the vertical or y direction *only two* of the following three equations can be used for solution.

$$v_y = (v_0)_y + a_c t$$

$$y = y_0 + (v_0)_y t + \tfrac{1}{2} a_c t^2$$

$$v_y^2 = (v_0)_y^2 + 2a_c(y - y_0)$$

For example, if the particle's final velocity v_y is not needed, then the first and third of these equations will not be useful.

Gravel falling off the end of this conveyor belt follows a path that can be predicted using the equations of constant acceleration. In this way the location of the accumulated pile can be determined. Rectangular coordinates are used for the analysis since the acceleration is only in the vertical direction.

EXAMPLE 12.11

A sack slides off the ramp, shown in Fig. 12–21, with a horizontal velocity of 12 m/s. If the height of the ramp is 6 m from the floor, determine the time needed for the sack to strike the floor and the range R where sacks begin to pile up.

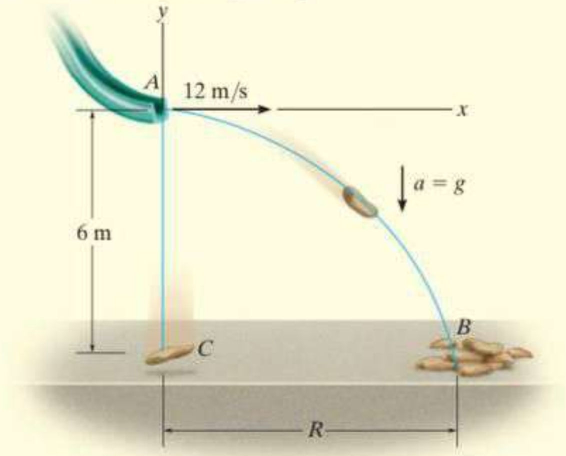


Fig. 12–21

SOLUTION

Coordinate System. The origin of coordinates is established at the beginning of the path, point A, Fig. 12–21. The initial velocity of a sack has components $(v_A)_x = 12\text{ m/s}$ and $(v_A)_y = 0$. Also, between points A and B the acceleration is $a_y = -9.81\text{ m/s}^2$. Since $(v_B)_x = (v_A)_x = 12\text{ m/s}$, the three unknowns are $(v_B)_y$, R, and the time of flight t_{AB}. Here we do not need to determine $(v_B)_y$.

Vertical Motion. The vertical distance from A to B is known, and therefore we can obtain a direct solution for t_{AB} by using the equation

$(+\uparrow)$

$$y_B = y_A + (v_A)_y t_{AB} + \tfrac{1}{2} a_c t_{AB}^2$$

$$-6\text{ m} = 0 + 0 + \tfrac{1}{2}(-9.81\text{ m/s}^2)t_{AB}^2$$

$$t_{AB} = 1.11\text{ s} \qquad \textit{Ans.}$$

Horizontal Motion. Since t_{AB} has been calculated, R is determined as follows:

$(\overset{+}{\rightarrow})$

$$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 12\text{ m/s}\,(1.11\text{ s})$$

$$R = 13.3\text{ m} \qquad \textit{Ans.}$$

NOTE: The calculation for t_{AB} also indicates that if a sack were released *from rest* at A, it would take the same amount of time to strike the floor at C, Fig. 12–21.