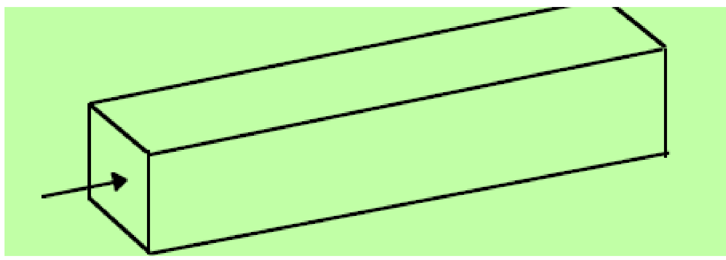
Derivation of Fluid Flow Equations

Review of basic steps

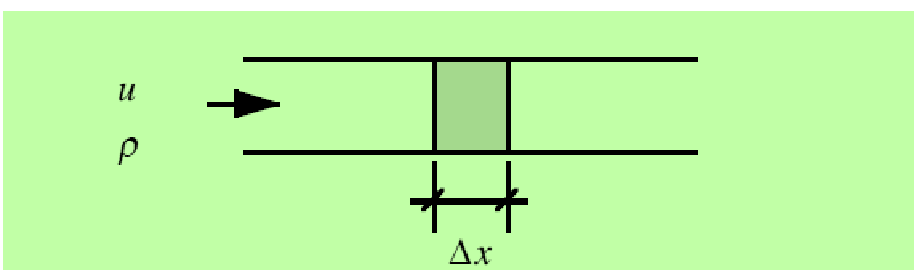
Generally speaking, flow equations for flow in porous materials are based on a set of mass, momentum, and constitutive equations for the fluids and the porous material involved. For simplicity, we will in the following assume isothermal conditions, so that we not have to involve an energy conservation equation. However, in cases of changing reservoir temperature, such as in the case of cold water injection into a warmer reservoir, this may be of importance.
Below, equations are initially described for single phase flow in linear, one-dimensional, horizontal systems, but are later on extended to multi-phase flow in two and three dimensions, and to other coordinate systems.

Conservation of mass

Consider the following one-dimensional rod of porous material:



Mass conservation may be formulated across a control element of the slab, with one fluid of density ρ is flowing through it at a velocity u:



The mass balance for the control element is then written as:

$$\begin{Bmatrix}\text{Mass into the}\\ \text{element at } x\end{Bmatrix} - \begin{Bmatrix}\text{Mass out of the}\\ \text{element at } x+\Delta x\end{Bmatrix} = \begin{Bmatrix}\text{Rate of change of mass}\\ \text{inside the element}\end{Bmatrix},$$

or

$$\{u\rho A\}_x - \{u\rho A\}_{x+\Delta x} = \frac{\partial}{\partial t}\{\phi A \Delta x \rho\}.$$

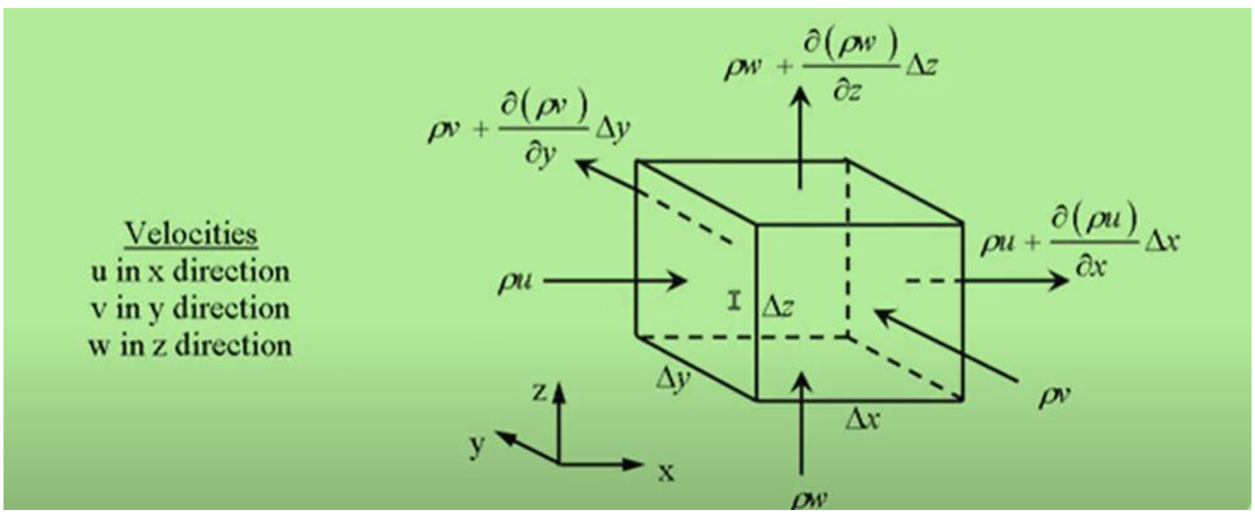
Dividing by Δx, and taking the limit as Δx approaches zero, we get the conservation of mass, or continuity equation:

$$-\frac{\partial}{\partial x}\left(A\rho u\right)=\frac{\partial}{\partial t}\left(A\phi\rho\right).$$

For constant cross-sectional area, the continuity equation simplifies to:

$$-\frac{\partial}{\partial x}\left(\rho u\right)=\frac{\partial}{\partial t}\left(\phi\rho\right).$$

In 3D Concept, the conversation of mass will become like below:



Mass in flow - Mass out flow = Rate of increase of mass

The mass balance for the fluid element is given by;

In x - dir. $$(\rho u)\Delta y\,\Delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y\,\Delta z$$

In y - dir. $$+(\rho v)\Delta x\,\Delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\Delta y\right)\Delta x\,\Delta z$$

In z - dir. $$+(\rho w)\Delta x\,\Delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\Delta z\right)\Delta x\,\Delta y = \frac{\partial \rho}{\partial t}(\Delta x\,\Delta y\,\Delta z)$$

Rearrangement the above equ. we have;

$$-\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\Delta x\,\Delta y\,\Delta z = \frac{\partial \rho}{\partial t}(\Delta x\,\Delta y\,\Delta z)$$

Finally, divided by the element volume, we get general form of continuity in Cartesian coordinate form as;

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\boxed{\text{the rate of change in time}}} + \underbrace{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}}_{\boxed{\text{Convective term}}} = 0 \qquad (1)$$