Conservation of momentum

Conservation of momentum is governed by the Navier-Stokes equations, but is normally simplified for low velocity flow in porous materials to be described by the semi-empirical Darcy's equation, which for single phase, one dimensional, horizontal flow is:

$$u = -\frac{k}{\mu}\frac{\partial P}{\partial x}.$$

Alternative equations are the Forchheimer equation, for high velocity flow

$$-\frac{\partial P}{\partial x} = u\frac{\mu}{k} + \beta u^n,$$

where *n* was proposed by Muscat to be 2, and the Brinkman equation, which applies to both porous and non-porous flow:

$$-\frac{\partial P}{\partial x} = u\frac{\mu}{k} - \mu\frac{\partial^2 u}{\partial x^2}.$$

To understand the basic concept of the derivation of the Conversation of Momentum, we start by the Netwon's Second Law of Motion:

Where $\sum F_i = ma_i$

Where $\sum F_i$ equals to summation of forces in i-direction

i direction includes all directions i in x, y, z direction

ai = accelration in i direction

m = mass of fluid particle.

Lets understand the Netwon's Second Law:

For the left hand side F (Forces): There are two types of Forces:

1- Surface Forces
2- Body Forces

Reservoir Simulation 23/24
Derivation of Fluid Flow Equations

- Surface Force

Surface Force means that there are action on the surface of the element. There are two types of surface force: Normar and Tangential (e.g. Pressure Force and Viscous Force)

- Body Force

Body Force means that there are action caused from the material of the element (e.g gravity force, centrifugal force and electromagnetic force.

We will use some notation for this derivation:

σ_{ij} and T_{ij}

i represents the direction of the normal line to the surface on which the stress is acting.

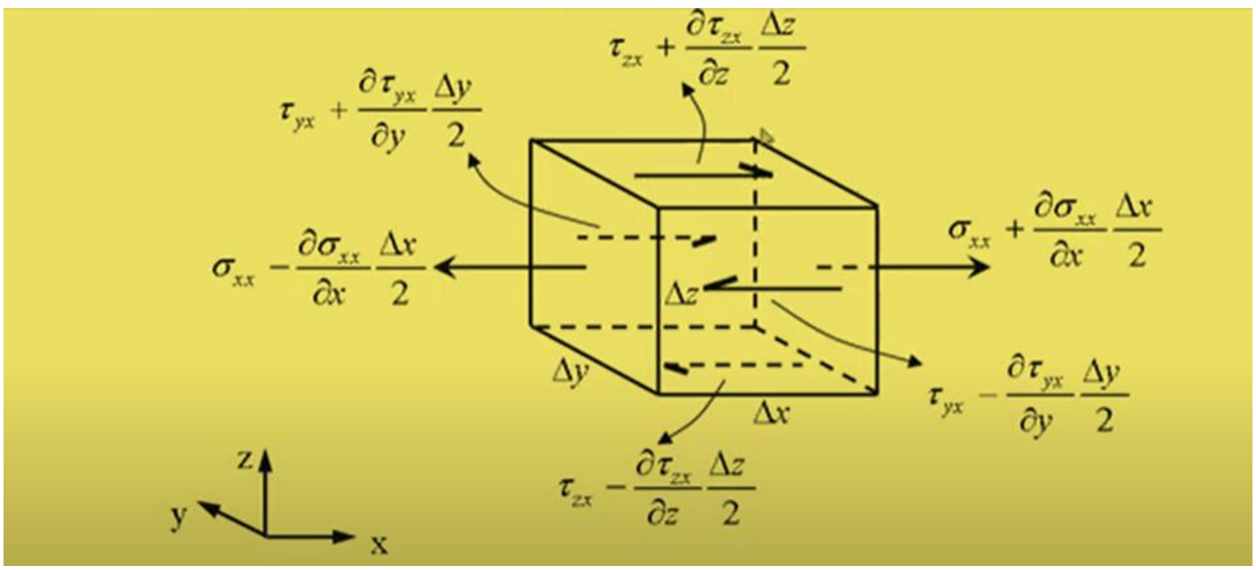
j represents the stress direction

T (*Tao*) represents tangential stress

σ (*Sigma*) represents normal stress

In order to apply Newton's Second Law, it is required to obtain the resultant of forces in x , y, and z direction:

- To find the summation of forces $\sum F_x$ at x direction we use the 3D block to derive the equation of force at x direction only then we do the same for y and z direction



$$
\begin{aligned}
\mathrm{F}_x = &\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\,\Delta z - \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y\,\Delta z \\
&+\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\,\Delta z - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\Delta x\,\Delta z \\
&+\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\,\Delta y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x\,\Delta y
\end{aligned}
$$

$$
\therefore \mathrm{F}_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\Delta x\,\Delta y\,\Delta z
$$

The resultant of surface forces in x-direction per unit volume;

$$
\mathrm{f}_x = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}
$$

Similarly;

$$
\mathrm{f}_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}
$$

$$
\mathrm{f}_z = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}
$$

Body forces

The mass of the fluid particle $= \rho\, \Delta x\, \Delta y\, \Delta z$

let a body force per unit volume in x - direction $= X$,

a body force per unit volume in y - direction $= Y$,

a body force per unit volume in z - direction $= Z$.

- If gravity is the only body force $X = \rho g_x$

$$\therefore \sum F_x \text{ (per unit volume)} = X + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\therefore \sum F_y \text{ (per unit volume)} = Y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\therefore \sum F_z \text{ (per unit volume)} = Z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

$a_x = \dfrac{Du}{Dt}$ since $u = f(x, y, z, t)$

$$\Delta u = \frac{\partial u}{\partial t}\Delta t + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$\frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial u}{\partial z}\frac{\Delta z}{\Delta t}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\Delta u}{\Delta t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\therefore a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Similarly;

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \underbrace{\frac{\partial w}{\partial t}}_{\boxed{\text{Local Acc.}}} + \underbrace{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}_{\boxed{\text{Convective term}}}$$

Where $\left(\frac{D}{Dt}\right)$ is called "**Substantive derivative or Material derivative.**

Apply Newton second law in x - direction;

$\sum F_x = m a_x$ where the mass is; $m = \rho\,\Delta x\,\Delta y\,\Delta z$

$$\sum F_{x(\text{per unit volume})} = \rho a_x$$

$$\rho\frac{Du}{Dt} = X + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \quad \text{in x-direction} \qquad \ldots(1a)$$

Similarly;

$$\rho\frac{Dv}{Dt} = Y + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \quad \text{in y-direction} \qquad \ldots(1b)$$

$$\rho\frac{Dw}{Dt} = Z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right) \quad \text{in z-direction} \qquad \ldots(1c)$$

End of lecture