The Derivative : is the slope and we can define as :

$$f(x)=\frac{dy}{dx}=m=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Where " m " is the slope , if this derivative exists then we say f is " differentiable " .

EXAM : Find to the function $\frac{dy}{dx}$ $f(x)=\sqrt{x}$

Solution :

$$\frac{dy}{dx}=f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\bullet\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$=\lim_{\Delta x\to 0}\frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

EXAM : Find $\frac{dy}{dx}$ to the function

1) $f(x)=x^2$

Solution :

$$f(x+\Delta x)=(x+\Delta x)^2=x^2+2x\Delta x+\Delta^2 x$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cancel{x^2}+2x\Delta x+\Delta^2 x-\cancel{x^2}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta^2 x}{\Delta x}=\lim_{\Delta x\to 0}\frac{\cancel{\Delta x}[2x+\Delta x]}{\cancel{\Delta x}}$$

$$=\lim_{\Delta x\to 0}2x+\Delta x=2x+0=2x$$

2) $f(x)=\frac{1}{x}$

Solution :

$$f(x+\Delta x)=\frac{1}{x+\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\frac{x-x-\Delta x}{x(x+\Delta x)}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{-1}{x(x+\Delta x)}=\frac{-1}{x^2}$$

RULES OF DERIVATIVE :

1) $y=f(x)=C\Rightarrow y'=0$

2) $y=f(x)=x\Rightarrow y'=1$

3) $y=f(x)=x^n\Rightarrow y'=nx^{n-1}$

4) $y=f(x)=Cg(x)\Rightarrow y'=Cg'(x)$

5) $y=f(x)\pm g(x)\Rightarrow y'=f'(x)\pm g'(x)$

6) $y=f(x)\bullet g(x)\Rightarrow y'=f(x)g'(x)+g(x)f'(x)$

7) $y=\dfrac{f(x)}{g(x)}\Rightarrow y'=\dfrac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}$

8) $y=f(x)=C(g(x))^n\Rightarrow y'=C(g(x))^{n-1}\bullet g'(x)$

Proof :

1) $f(x)=C\Rightarrow f(x+\Delta x)=C$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{C-C}{\Delta x}=0$$

2) $f(x)=x\Rightarrow f(x+\Delta x)=x+\Delta x$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\cancel{x}+\Delta x-\cancel{x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1$$

NOTE

$$(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{2!}x^{n-2}y^2+\frac{n(n-1)(n-2)}{3!}x^{n-3}y^3+\cdots+y^n$$

3) $f(x) = x^n$

$$f(x+\Delta x) = (x+\Delta x)^n = x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta^2 x + \ldots + \Delta^n x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{x^n} + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta^2 x + \ldots + \Delta^n x - \cancel{x^n}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}\,[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}\Delta^2 x + \ldots + \Delta^{n-1} x]}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}\Delta^2 x + \ldots + \Delta^{n-1} x$$

$$= nx^{n-1} + 0 + 0 + \ldots + 0 = nx^{n-1}$$

EXAM : Find $\frac{dy}{dx}$ to the function

1) $y = x^6 \Rightarrow y' = 6x^5$

2) $y = 3x^4 \Rightarrow y' = 3.4x^3 = 12x^3$

3) $y = 3x^4 + 2x^2 - 7x + 10 \Rightarrow y' = 12x^3 + 4x - 7$

4) $y = (x+3)(x^2+2) \Rightarrow y' = (x+3)(2x) + (x^2+2).1$

5) $y = \frac{(x+3)}{(x^2+2)} \Rightarrow y' = \frac{(x^2+2).1 - (x+3)(2x)}{(x^2+2)^2}$

6) $y = (3x^2+4x)^5 \Rightarrow y' = 5(3x^2+4x)^4(6x+4)$

7) $y = (3x^2+4x)^5(x^3+7)^3 \Rightarrow y' = (3x^2+4x)^5.3(x^3+7)^2.(3x^2) +$

$+ (x^3+7)^3 5(3x^2+4x)^4(6x+4)$

IMPLICIT DIFFERENTIATION الاشتقاق الضمنى

نستخدم الاشتقاق الضمني في حالة إعطاء الدالة بمتغيرين او أكثر ، حيث نشتق المتغير المستقل(Independent) اشتقاقاً صريحاً ، اما المتغير المعتمد(Dependent) نشتقه اشتقاقاً ضمنياً ، حيث أن المتغير المعتمد هو المتغير الذي يعتمد على متغيرات اخرى اما المتغير المستقل فهو المتغير الذي يُشتق بالنسبة له . مثل :

$\frac{dy}{dx}$ y (dependent) & x (independent)

$\frac{dz}{dy}$ z (dependent) & y (independent)

EXAM :

If $x^2 + y^2 = 9$ *, Find* $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

Solution :

$$2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow \boxed{y' = -\frac{x}{y}}$$

$$y'' = -\frac{(y.1 - xy')}{y^2} = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{\frac{x^2+y^2}{y}}{y^2}$$

$$= -\frac{x^2+y^2}{y^3} = -\frac{9}{y^3}$$

EXAM :

If $2y^2 = x^3$ *,then prove that* $\frac{d^2y}{dx^2} = \frac{3}{8}\frac{x}{y}$

solution :

$$4yy' = 3x^2 \Rightarrow \boxed{y' = \frac{3}{4}\frac{x^2}{y}}$$

$$y'' = \frac{3}{4}\frac{y.2x - x^2.y'}{y^2} = \frac{3}{4}\frac{2yx - \frac{3}{4}\frac{x^4}{y}}{y^2} = \frac{3}{4}\frac{\frac{8xy^2 - 3x^4}{4y}}{y^2}$$

$$= \frac{3}{4}\frac{x(8y^2 - 3x^3)}{4y^3} = \frac{3}{16}\frac{x(8y^2 - 6y^2)}{y^3} \qquad \boxed{\because x^3 = 2y^2}$$

$$= \frac{3}{16}\frac{x(2y^2)}{y^3} = \frac{3}{8}\frac{x}{y}$$

EXAM :

If $y^3 - x^3 = a^3$ *, prove that :*

$$\frac{d^2x}{dy^2} = -\frac{2ya^3}{x^5}$$

a is constant.

$$3y^2 - 3x^2 x' = 0 \Rightarrow 3y^2 = 3x^2 x' \Rightarrow \boxed{x' = \frac{y^2}{x^2}}$$

$$x'' = \frac{x^2.2y - y^2.2xx'}{x^4} = \frac{2yx^2 - .2xy^2 \frac{y^2}{x^2}}{x^4} = \frac{2yx^2 - \frac{2y^4}{x}}{x^4}$$

$$= \frac{2yx^3 - 2y^4}{x^5} = \frac{2y(x^3 - y^3)}{x^5} = \frac{2y(-a^3)}{x^5} = \frac{-2ya^3}{x^5}$$

CHAIN RULE قاعدة السلسلة

Let y=f(t) and x=g(t) , the chain rule may be written as :

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

EXAM :

$$If \quad y = t^3 + 6 \quad , \quad x = 2t + 4$$

$$Find \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}$$

Solution :

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dt} = 3t^2 \quad , \frac{dx}{dt} = 2 \Rightarrow \frac{dt}{dx} = \frac{1}{2}$$

$$\frac{dy}{dx} = 3t^2.\frac{1}{2} \quad = \boxed{\frac{3}{2}t^2} \Leftarrow A$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}A = \frac{dA}{dx} = \frac{dA}{dt}.\frac{dt}{dx}$$

$$= 3t.\frac{1}{2} = \frac{3t}{2}$$

EXAM :

$$If \ \ y = t^2 + 1 \quad , \quad x = t^2 - 1$$

$$Find \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}$$

Solution :

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dt} = 2t \quad , \frac{dx}{dt} = 2t \Rightarrow \frac{dt}{dx} = \frac{1}{2t}$$

$$\frac{dy}{dx} = 2t.\frac{1}{2t} \quad = 1 \Leftarrow A$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}A = \frac{dA}{dx} = \frac{dA}{dt}.\frac{dt}{dx}$$

$$= 0.\frac{1}{2t} = 0$$

EXAM :

$$If \ \ y = Cost \quad , \quad x = \sin t$$

$$Find \quad \frac{dy}{dx}, \frac{d^2y}{dx^2}$$

Solution :

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dt} = -\sin t \quad , \frac{dx}{dt} = Cost \Rightarrow \frac{dt}{dx} = \frac{1}{Cost}$$

$$\frac{dy}{dx} = -\sin t.\frac{1}{Cost} \quad = \boxed{-\tan t} \Leftarrow A$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}A = \frac{dA}{dx} = \frac{dA}{dt}.\frac{dt}{dx}$$

$$= -Sec^2 t.\frac{1}{Cost} = -Sec^3 t$$

Partial Derivative :

DEF:

Let f be a function with two variables x, y , then the partial derivative for f respect to x is the function f_x or and its value at any point (x,y) in domain f is : $\frac{\partial f}{\partial x}$

$$f_x(x,y)=\frac{\partial}{\partial x}f(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

DEF:

Let f be a function with two variables x, y , then the partial derivative for f respect to x is the function f_x or and its value at any point (x,y) in domain f is : $\frac{\partial f}{\partial y}$

$$f_y(x,y)=\frac{\partial}{\partial y}f(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

DEF:

Let f be a function with three variables x, y ,z then the partial derivative for f :

$$f_x(x,y,z)=\frac{\partial}{\partial x}f(x,y,z)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}$$

$$f_y(x,y,z)=\frac{\partial}{\partial y}f(x,y,z)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y}$$

$$f_z(x,y,z)=\frac{\partial}{\partial z}f(x,y,z)=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y,z)}{\Delta z}$$

Note :

$$\frac{\partial^2 f}{\partial x^2}=f_{xx} \qquad \frac{\partial^2 f}{\partial xy}=f_{xy}$$

$$\frac{\partial^2 f}{\partial yx}=f_{yx} \qquad \frac{\partial^2 f}{\partial y^2}=f_{yy}$$

$$f_{xy}=f_{yx}$$

and in three variables

$$\frac{\partial^3 f}{\partial x\,\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)=f_{xyy}\cdots\cdots\cdots etc.$$

EXAM :

$$f(x,y)=x^2-5xy+y^3$$

$$\frac{\partial f}{\partial x}=2x-5y$$

$$\frac{\partial f}{\partial y}=0-5x+3y^2=-5x+3y^2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = -5 = \frac{\partial^2 f}{\partial y\,\partial x}$$

EXAM :

$$f(x, y) = (3x + 5y)^5$$

$$\frac{\partial f}{\partial x} = 5(3x + 5y)^4.3 = 15(3x + 5y)^4$$

$$\frac{\partial f}{\partial y} = 5(3x + 5y)^4.5 = 25(3x + 5y)^4$$

$$\frac{\partial^2 f}{\partial x^2} = 60(3x + 5y)^3.3 = 180(3x + 5y)^3$$

$$\frac{\partial^2 f}{\partial y^2} = 100(3x + 5y)^3.5 = 500(3x + 5y)^3$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = 60(3x + 5y)^3.5 = 300(3x + 5y)^3 = \frac{\partial^2 f}{\partial y\,\partial x}$$

EXAM :

$$f(x, y) = xy^3 - 5x^2yz^4$$

$$\frac{\partial f}{\partial x} = y^3 - 10xyz^4$$

$$\frac{\partial f}{\partial y} = 3y^2x - 5x^2z^4$$

$$\frac{\partial f}{\partial z} = -20x^2yz^3$$

$$\frac{\partial^2 f}{\partial x^2} = -10yz^4$$

$$\frac{\partial^2 f}{\partial y^2} = 6xy$$

$$\frac{\partial^2 f}{\partial z^2} = -60x^2yz^2$$

$$\frac{\partial^3 f}{\partial x^3} = 0$$

$$\frac{\partial^3 f}{\partial y^3} = 6x$$

$$\frac{\partial^3 f}{\partial z^3} = -120x^2yz$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = 3y^2 - 10xz^4$$

$$\frac{\partial^2 f}{\partial x\,\partial z} = -40xyz^3$$

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial z}\right)\right)$$

$$\frac{\partial^3 f}{\partial z\,\partial y^2} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial y^2}\right) = 0$$

DEF :

Let u(x,y) , v(x,y) be a functions then we say that u & v satisfy the Cauchy Riemann Equation (C.R.E) iff

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$

EXAM :

$$u=x^2-y^2 \quad , \quad v=2xy$$

$$\frac{\partial u}{\partial x}=2x \quad , \quad \frac{\partial v}{\partial y}=2x \quad \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=-2y \quad , \quad \frac{\partial v}{\partial x}=2y \quad \Rightarrow \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\therefore \quad u \,\&v \quad satisfy \; C.R.E$$

DEF :

When the function f(x,y) satisfy Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

then f(x,y) is said to be harmonic .

EXAM :

$$f(x,y)=x^3-3xy^2$$

$$\frac{\partial f}{\partial x}=3x^2-3y^2 \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2}=6x$$

$$\frac{\partial f}{\partial y}=-6xy \quad \Rightarrow \quad \frac{\partial^2 f}{\partial y^2}=-6x$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=6x-6x=0$$

$$\therefore \quad f \quad harmonic$$

HOME WORK

1 By using the definition of the derivative find to $f'(x)$

$3x^3 \quad , \quad x^2+1 \quad , \quad x^5$

$\frac{1}{x+1} \quad , \quad \frac{1}{x^2} \quad , \quad \sqrt{x+1}$

2

$$Find\ \frac{d^2y}{dx^2} \,\&\, \frac{d^2x}{dy^2}\ to\ the:$$

$x^2y^2 - y = 7$

$2x^2y - 4y^3 + 1 = 0$

$x^{2/3} + y^{2/3} = a^{2/3} \qquad , a\ is\ cons.$

$x^{1/3} + y^{1/3} = a^{1/3} \qquad , a\ is\ cons.$

$x^3 - 4y^2 + 3 = 0$

3

$If\ \ y = t^3 + 3t^2 + 1$

$x = 4t^4 + 2t^2$

$then\ Find\ \ \frac{d^2y}{dx^2}$