1 Area

a) Area between f(x) and the axis

$$A=\int_a^b f(x)dx=\int_a^b y\,dx \quad , \quad area\ respected\ to\ x-axis$$

$$A=\int_c^d g(y)dy=\int_a^b x\,dy \quad , \quad area\ respected\ to\ y-axis$$

b) Area between two curves

$y_1=f(x_1)$, $y_2=f(x_2)$, $x_1=g(y_1)$, $x_2=g(y_2)$

$$A=\int_a^b (y_1-y_2)dx \quad ,x-axis \quad ,y_1\geq y_2$$

$$A=\int_c^d (x_1-x_2)dy \quad ,y-axis \quad ,x_1\geq x_2$$

ملاحظات هامة :

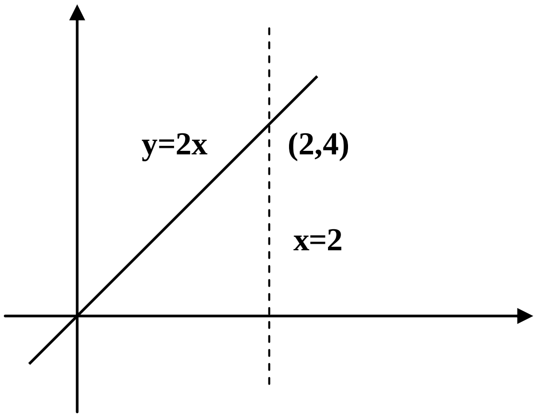
1) اذا اعطانا دالة وحدود تكامل فالحل يكون مباشر .
2) اذا اعطانا دالتين بدون حدود تكامل فنقاطع الدالتين .
3) اذا اعطانا دالة فقط بدون حدود تكامل ، فالحل يكون بالاعتماد على المحور ، فاذا اعطى المساحة بالنسبة للمحور x نضيف معادلة y=0 واذا اعطى المساحة بالنسبة للمحور y نضيف معادلة x=0 ، ثم نقاطع الدالتين .
4) اذا طلب المساحة بالنسبة للمحور x وحدود التكامل معطاة بالنسبة للمحور y فيجب تحويل حدود التكامل بدلالة x والعكس صحيح .

EXAM :- Find the area bounded by the line y=2x and the x-axis from x=0 , x=2 and check the result by geometrically.

$$A=\int_a^b y\,dx=\int_0^2 2x\,dx=x^2\Big|_0^2=4\,unit^2$$

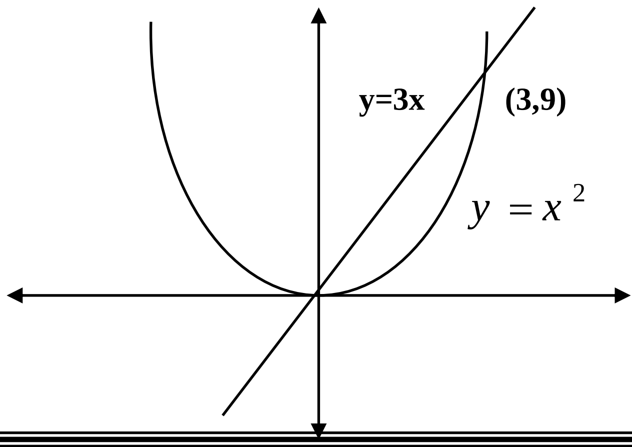
$In\,Geometry$

$$\triangle A=\frac{1}{2}b.h=\frac{1}{2}(2)(4)=4$$



EXAM :- Find the area between the curves :

$y=x^2$ and the line $y=3x$



Solution :

$y=x^2 \quad \ldots\ldots\ldots(1)$

$y=3x \quad \ldots\ldots\ldots(2)$

$x^2=3x \Rightarrow x^2-3x=0$

$$x(x-3)=0 \Rightarrow x=0\,, x=3$$

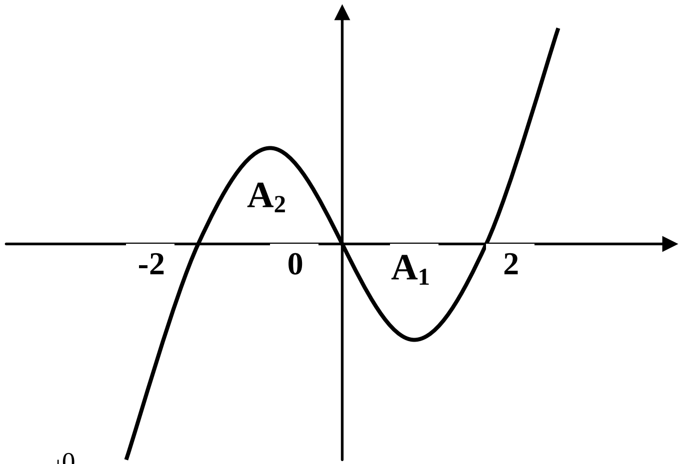
$$A=\int_a^b (y_1-y_2)dx \qquad , y_1 \ge y_2$$

$$=\int_0^3 (3x-x^2)dx = \frac{3}{2}x^2-\frac{1}{3}x^3\Big|_0^3 = 4.5\,unit^2$$

EXAM :- Find the area between the curves :

$y=x(x^2-4)$ **and the x-axis**

Solution :



$$y=x(x^2-4)$$

$$y=0$$

$$\therefore x(x^2-4)=0 \Rightarrow x=0, x=-2, x=2$$

$$A=|A_1|+|A_2|$$

$$A_1=\int_{-2}^{0} x(x^2-4)dx = \int_{-2}^{0}(x^3-4x)dx = \frac{1}{4}x^4-2x^2\Big|_{-2}^{0} = 4$$

$$A_2=\int_0^2 x(x^2-4)dx = \int_0^2 (x^3-4x)dx = \frac{1}{4}x^4-2x^2\Big|_0^2 = -4$$

$$A=|4|+|-4|=8$$

EXAM :- Find the area between the line x-y=10 and the

a) x-axis b)y-axis

from x=0 , x=5

$$a) \quad A=\int_a^b y\,dx \qquad , \quad x-y=10 \Rightarrow y=x-10$$

$$=\int_0^5 (x-10)dx = \frac{x^2}{2}-10x\Big|_0^5 = |-37.5| = 37.5$$

$$b) \quad A=\int_c^d x\,dy \qquad , \quad x-y=10 \Rightarrow x=y+10$$

$$A=\int_{-10}^{-5}(y+10)dy \qquad x=0 \rightarrow y=-10 \quad \& \, x=5 \rightarrow y=-5$$

$$=\frac{1}{2}y^2+10y\Big|_{-10}^{-5} = ?$$

2 Volumes

DEF: if f(x) ≥ 0 cont. on [a,b] if the area bounded by f(x) and the x-axis from x=a to x=b rotated about the x-axis then :

$$V_x = \int_a^b \pi[f(x)]^2 dx = \int_a^b \pi y^2 dx$$

DEF: if g(y) ≥ 0 cont. on [c,d] if the area bounded by g(y) and the y-axis fromy=c to y=d rotated about the y-axis then :

$$V_y = \int_c^d \pi[g(y)]^2 dy = \int_c^d \pi x^2 dy$$

DEF : if the area bounded between two curves is rotated about x-axis

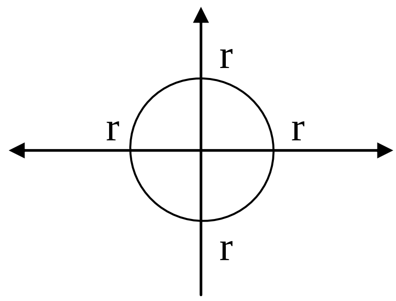
$$V_x = \int_a^b \pi(y_1^2 - y_2^2)dx \quad , y_1 \ge y_2$$

DEF : if the area bounded between two curves is rotated about y-axis

$$V_y = \int_c^d \pi(x_1^2 - x_2^2)dy \quad , x_1 \ge x_2$$

<u>EXAM</u> :- The area bounded by the circle with radius r and center is the origin is rotated about the x-axis , find the volume generated and check the result by geom.

$$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$$



$$V_x = \int_a^b \pi y^2 dx = \int_{-r}^{r} \pi(r^2 - x^2)dx$$

$$= 2\pi\int_0^r (r^2 - x^2)dx = 2\pi(r^2 x - \frac{1}{3}x^3)\Big|_0^r$$

$$= \frac{4}{3}\pi r^3 \; unit^3$$

In Geometry

the volume of sphare is $\frac{4}{3}\pi r^3 \; unit^3$

<u>EXAM</u> :- The area bounded by the function $x^2 y^2 = 1$ is rotated about the y-axis, find the volume generated from y=1 , y=2 .

$$V_y = \pi\int_c^d x^2 dy \qquad , \quad x^2 y^2 = 1 \rightarrow x^2 = y^{-2}$$

$$= \pi\int_c^d y^{-2} dy = \frac{-1}{y}\Big|_1^2 = \frac{1}{2}\pi \; unit^3$$

EXAM **:- The area bounded by the line y=x-1**
a) is rotated about the x-axis b) is rotated about the y-axis
find the volume generated from those rotation from x=0,1

$$a)V_x=\pi\int_a^b y^2dx \quad , \quad y=x-1\Rightarrow y^2=(x-1)^2$$

$$=\pi\int_0^1 (x-1)^2dx=\frac{\pi}{3}(x-1)^3\Big|_0^1=\frac{\pi}{3}unit^3$$

$$b)V_y=\pi\int_c^d x^2dy \quad , \quad y=x-1\Rightarrow x=y+1\Rightarrow x^2=(y+1)^2$$

$$x=0\rightarrow y=-1 \quad \& x=1\rightarrow y=0$$

$$=\pi\int_{-1}^0 (y+1)^2dx=\frac{\pi}{3}(y+1)^3\Big|_{-1}^0=\frac{\pi}{3}unit^3$$

3 Arc Length

If y=f(x) is continous with continous derivative at each point of the curve from (a,f(a)) to (b,f(b)) then :

$$S=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}dx \quad ,if \quad \frac{dy}{dx}\ is\ cont.$$

$$S=\int_c^d\sqrt{1+(\frac{dx}{dy})^2}dy \quad ,if \quad \frac{dy}{dx}\ is\ not\ cont.$$

EXAM **:- Find the length of the segment of curve** $y=\frac{1}{3}(x^2+2)^{3/2} \quad ,from\ x=0,3$

$$\frac{dy}{dx}=\frac{1}{3}\frac{3}{2}(x^2+2)^{1/2}(2x)=x(x^2+2)^{1/2}\ cont.on[0,3]$$

$$S=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}dx=\int_0^3\sqrt{1+(x(x^2+2)^{1/2})^2}dx$$

$$=\int_0^3\sqrt{1+x^2(x^2+2)}dx=\int_0^3\sqrt{x^4+2x^2+1}dx$$

$$=\int_0^3\sqrt{(x^2+1)^2}dx=\int_0^3(x^2+1)dx$$

$$=\frac{x^3}{3}+x\Big|_0^3=?$$

<u>EXAM</u> :- Find the length of the segment of curve $y = x^{2/3} \quad , from\ x = -1,8$

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{-1/3}} \quad not\ cont.on\ [-1,8]$$

$$\frac{dx}{dy} = \frac{3}{2}x^{1/3} \Rightarrow\Rightarrow x = -1 \rightarrow y = 1 \,\&\, x = 8 \rightarrow y = 4$$

$$S = \int_1^4 \sqrt{1 + (\frac{3}{2}x^{1/3})^2}\,dy = \int_1^4 \sqrt{1 + \frac{9}{4}x^{2/3}}\,dy$$

$$S = \int_1^4 \sqrt{1 + \frac{9}{4}y}\,dy = \frac{4}{9}\frac{2}{3}(1 + \frac{9}{4}y)^{3/2}\Big|_1^4$$

$$= \frac{8}{27}[(10)^{3/2} - (\frac{13}{4})^{3/2}] = ?$$

HOME WORK

1- Find the area bounded by the line 2x-5y=10 from x=0 , 10 about the x&y – axis.

2- Find the area bounded by the curves :

$y = x^2 \quad , \quad y = \sqrt{x} \quad , x = 1,2 \quad , x - axis$

$y = \sec^2 x \quad , \quad y = x \quad , x = -45, 45 \quad , x - axis$

$y = x^2 = y \quad , \quad x = y - 2 \quad , x - axis$

$y = x^3 - 2x^2 \quad , \quad y = 2x^2 - 3x \quad , x - axis$

3- Find the volume generated from rotation between curves :

$y = \sec x \quad , \quad y = 0 \quad , x = 45, 60 \quad , x - axis$

$y = \sqrt{25 - x^2} \quad , \quad y = 3 \quad , x - axis$

$y = c\,\text{sc}\,x \quad , \quad y = 2 \quad , x = 45, 60 \quad , x - axis$

$x = 1 - y^2 \quad , \quad x = 2 + y^2 \quad , y = -1,1 \quad , y - axis$

$y = \sin x \quad , \quad y = \cos x \quad , x = 0, 45 \quad , x - axis$

$y = \tan x \quad , \quad y = -1 \quad , x = 45, 60 \quad , x - axis$

4- Find the horizontal line y=k that divides the area between $y = x^2$ & $y = 9$ **into two equal parts .**

5- Find the vertical line x=k that divides the area between $x = \sqrt{y}$ & $x = 2$ **into two equal parts .**

6- Find the volume of the solid that result when the reigon above the x-axis and below $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , a, b > 0$ **to the ellipse**

7- Let V be the volume of the solid that result when the reigon enclosed by

$y = \frac{1}{x} \quad , y = 0 \quad , x = 2 \quad , x = b \quad (0 < b < 2)$

is revolved about the x-axis , find the value of " b " for which V=3.

8- Find the exact arc length of the curve over the stated interval : -

$y = 3x^{3/2} - 1 \qquad x = 0,1$

$24xy = y^4 + 48 \qquad , y = 2, 4$

$x = \frac{1}{8} y^4 + \frac{1}{4} y^{-2} \qquad y = 1, 4$