7.3 The Exponential Function

Having developed the theory of the function $\ln x$, we introduce the exponential function $\exp x = e^x$ as the inverse of $\ln x$. We study its properties and compute its derivative and integral. Knowing its derivative, we prove the power rule to differentiate x^n when n is *any* real number, rational or irrational.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. As you can see in Figure 7.11,

$$\lim_{x\to\infty} \ln^{-1} x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} \ln^{-1} x = 0.$$

The function $\ln^{-1} x$ is also denoted by $\exp x$.

In Section 7.2 we defined the number e by the equation $\ln(e) = 1$, so $e = \ln^{-1}(1) = \exp(1)$. Although e is not a rational number, later in this section we see one way to express it as a limit. In Chapter 11, we will calculate its value with a computer to as many places of accuracy as we want with a different formula (Section 11.9, Example 6). To 15 places,

$$e = 2.718281828459045.$$

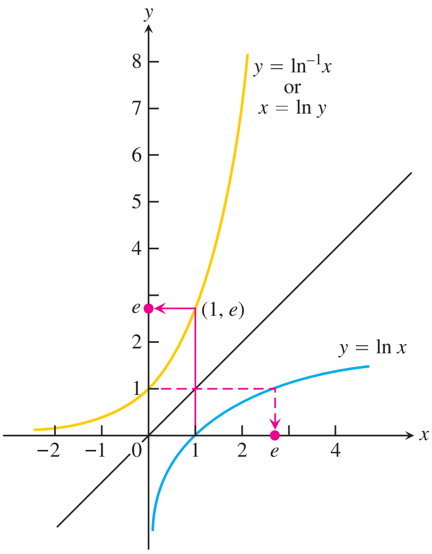

FIGURE 7.11 The graphs of $y = \ln x$ and $y = \ln^{-1} x = \exp x$. The number e is $\ln^{-1} 1 = \exp(1)$.

The Function $y = e^x$

We can raise the number e to a rational power r in the usual way:

$$e^2 = e \cdot e, \qquad e^{-2} = \frac{1}{e^2}, \qquad e^{1/2} = \sqrt{e},$$

and so on. Since e is positive, e^r is positive too. Thus, e^r has a logarithm. When we take the logarithm, we find that

$$\ln e^r = r \ln e = r \cdot 1 = r.$$

Since $\ln x$ is one-to-one and $\ln(\ln^{-1} r) = r$, this equation tells us that

$$e^r = \ln^{-1} r = \exp r \qquad \text{for } r \text{ rational.} \tag{1}$$

We have not yet found a way to give an obvious meaning to e^x for x irrational. But $\ln^{-1} x$ has meaning for any x, rational or irrational. So Equation (1) provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all x, so we use it to assign a value to e^x at every point where e^x had no previous definition.

Typical values of e^x

x	e^x (rounded)
−1	0.37
0	1
1	2.72
2	7.39
10	22026
100	2.6881×10^{43}

DEFINITION The Natural Exponential Function
For every real number x, $e^x = \ln^{-1} x = \exp x$.

For the first time we have a precise meaning for an irrational exponent. Usually the exponential function is denoted by e^x rather than $\exp x$. Since $\ln x$ and e^x are inverses of one another, we have

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \qquad (2)$$

$$\ln(e^x) = x \qquad (\text{all } x) \qquad (3)$$

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

EXAMPLE 1 Using the Inverse Equations

(a) $\ln e^2 = 2$

(b) $\ln e^{-1} = -1$

(c) $\ln \sqrt{e} = \frac{1}{2}$

(d) $\ln e^{\sin x} = \sin x$

(e) $e^{\ln 2} = 2$

(f) $e^{\ln(x^2+1)} = x^2 + 1$

(g) $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$ One way

(h) $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = 8$ Another way ■

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$e^{2k} = 10$$
$$\ln e^{2k} = \ln 10$$
$$2k = \ln 10 \qquad \text{Eq. (3)}$$
$$k = \frac{1}{2}\ln 10.$$ ■

Transcendental Numbers and Transcendental Functions

Numbers that are solutions of polynomial equations with rational coefficients are called **algebraic**: -2 is algebraic because it satisfies the equation $x + 2 = 0$, and $\sqrt{3}$ is algebraic because it satisfies the equation $x^2 - 3 = 0$. Numbers that are not algebraic are called **transcendental**, like e and π. In 1873, Charles Hermite proved the transcendence of e in the sense that we describe. In 1882, C.L.F. Lindemann proved the transcendence of π.

Today, we call a function $y = f(x)$ algebraic if it satisfies an equation of the form

$$P_n y^n + \cdots + P_1 y + P_0 = 0$$

in which the P's are polynomials in x with rational coefficients. The function $y = 1/\sqrt{x+1}$ is algebraic because it satisfies the equation $(x+1)y^2 - 1 = 0$. Here the polynomials are $P_2 = x + 1$, $P_1 = 0$, and $P_0 = -1$. Functions that are not algebraic are called transcendental.

The General Exponential Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can think of a^x as $(e^{\ln a})^x = e^{x \ln a}$. We therefore make the following definition.

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

HISTORICAL BIOGRAPHY

Siméon Denis Poisson
(1781–1840)

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$ ■

We study the calculus of general exponential functions and their inverses in the next section. Here we need the definition in order to discuss the laws of exponents for e^x.

Laws of Exponents

Even though e^x is defined in a seemingly roundabout way as $\ln^{-1} x$, it obeys the familiar laws of exponents from algebra. Theorem 3 shows us that these laws are consequences of the definitions of $\ln x$ and e^x.

THEOREM 3 **Laws of Exponents for e^x**

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

Proof of Law 1 Let

$$y_1 = e^{x_1} \quad \text{and} \quad y_2 = e^{x_2}. \tag{4}$$

Then

$$
\begin{aligned}
x_1 = \ln y_1 \quad &\text{and} \quad x_2 = \ln y_2 && \text{Take logs of both sides of Eqs. (4).} \\
x_1 + x_2 &= \ln y_1 + \ln y_2 \\
&= \ln y_1 y_2 && \text{Product Rule for logarithms} \\
e^{x_1 + x_2} &= e^{\ln y_1 y_2} && \text{Exponentiate.} \\
&= y_1 y_2 && e^{\ln u} = u \\
&= e^{x_1} e^{x_2}.
\end{aligned}
$$
■

The proof of Law 4 is similar. Laws 2 and 3 follow from Law 1 (Exercise 78).

EXAMPLE 4 Applying the Exponent Laws

(a) $e^{x + \ln 2} = e^x \cdot e^{\ln 2} = 2e^x$ Law 1

(b) $e^{-\ln x} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$ Law 2

(c) $\dfrac{e^{2x}}{e} = e^{2x-1}$ Law 3

(d) $(e^3)^x = e^{3x} = (e^x)^3$ Law 4 ■