University of Mosul
College of Petroleum & Mining Engineering

Mathematics I

Lecture (3)

Limits

Petroleum and Refining Engineering Department
Ass.L. Ghufran Faris Alrahhawi
Email: Ghufranalrahhawi@uomosul.edu.iq

LECTURE CONTENTS

Limits

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The limit Laws

To calculate limits of the functions that are arithmetic combinations of functions having known limits, we can use several fundamental rules.

If L , M, C and K are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ then

1. Sum Rule $\lim_{x\to c}\big(f(x) + g(x)\big) = L + M$
2. Difference Rule $\lim_{x\to c}\big(f(x) - g(x)\big) = L - M$
3. Constant multiple Rule $\lim_{x\to c} 9k.f(x)) = K.L$
4. Product Rule $\lim_{x\to c}\big(f(x).g(x)\big) = L.M$
5. Quotient Rule $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$
6. Power Rule $\lim_{x\to c}[f(x)]^n = L^n, n$ a positive integer
7. Root Rule $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, n$ a positive integer

If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$

Example (1)

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ and fundamental rules of limits to find the following limits

a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$

b) $\lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}$

c) $\lim_{x\to -2}\sqrt{4x^2 - 3}$

Solution

a) $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$

$= c^3 + 4c^2 - 3$

Ass.L. Ghufran Faris Alrahhawi

Sum and Difference Rule & Power and Multiple Rules

b) $\lim_{x \to c} \frac{x^4+x^2-1}{x^2+5} = \frac{\lim_{x \to c}(x^4+x^2-1)}{\lim_{x \to c}(x^2+5)}$ Quotient Rule

$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ Sum and difference Rule

$= \frac{c^4+c^2-1}{c^2+5}$ Power or Product Rule

c) $\lim_{x \to -2} \sqrt{4x^2-3} = \sqrt{\lim_{x \to -2}(4x^2-3)}$ Root Rule with n=2

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ Difference Rule

Product and Multiple $= \sqrt{4(-2)^2-3} = \sqrt{13}$

Limits at Infinity

General Rules

1. $\lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to -\infty} \frac{1}{x} = 0$
2. $\lim_{x \to \infty} k = k$, $\lim_{x \to -\infty} k = k$
3. $\lim_{x \to \infty} \frac{sin\theta}{\theta} = 0$ to prove it

$-1 \leq sin\theta \leq 1$ $[\div \theta]$ $\frac{-1}{\theta} \leq sin\theta \leq \frac{1}{\theta}$

$\lim_{\theta \to \infty} \frac{-1}{\theta} = 0$ and $\lim_{\theta \to \infty} \frac{1}{\theta} = 0$

Limits at Infinity of Rational Functions

There are two methods: -

1. Divide both the numerator and the denominator by the highest power of x in denominator.
2. Suppose that x=1/h and find limit as h approaches zero.

Note: - For rational function $\frac{f(x)}{g(x)}$

1. If degree of $f(x)$ less than degree of $g(x)$, then $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$

2. If degree of $f(x)$ equal degree of $g(x)$ then $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite.

3. If degree of $f(x)$ greater than degree of $g(x)$ then $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite.

Example (2)

Find $\lim_{x \to \infty} \frac{x^3 - 4x^2 + 7}{2x^2 - 3}$

Method 1:

$$\lim_{x \to \infty} \frac{\frac{x^3}{x^2} - \frac{4x^2}{x^2} + \frac{7}{x^2}}{\frac{2x^2}{x^2} - \frac{3}{x^2}} = \lim_{x \to \infty} \frac{x - 4 + \frac{7}{x^2}}{2 - \frac{3}{x^2}}$$

$$= \frac{\infty - 4 + 0}{2 - 0} = \frac{\infty}{2} = \infty$$

Method 2:

$$\lim_{h \to 0} \frac{(\frac{1}{h})^3 - 4\left(\frac{1}{h}\right)^2 + 7}{2(\frac{1}{h})^2 - 3} = \lim_{h \to 0} \frac{\frac{1}{h^3} - \frac{4}{h^2} + 7}{\frac{2}{h^2} - 3}$$

$$\lim_{h \to 0} \frac{\frac{1 - 4h + 7h^3}{h^3}}{\frac{2 - 3h^2}{h^2}}$$

$$\lim_{h \to 0} \frac{1 - 4h + 7h^3}{h(2 - 3h^2)} = \frac{1 - 4(0) + 7(0)}{0(2 - 3(0))} = \frac{1}{0} = \infty$$

Some Rules about Limits

$$\lim_{\theta \to 0} sin\theta = 0$$

$$\lim_{\theta \to 0} cos\theta = 1$$

$$\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{\theta}{sin\theta} = 1$$

$$\lim_{\theta \to 0} \frac{tan\theta}{\theta} = 1$$

Ĺ Hopitals Rule

Ĺ Hopitals rule is a general method of evaluating indeterminate form such as 0/0 or ∞/∞ . To evaluate the limits of indeterminate forms for derivatives in calculus, Ĺ Hopitals rule is used. Ĺ Hopital rule can be applied more than once. You can apply this rule still it holds any indefinite form every time after its applications. If the problem is out of the indeterminate forms, you cant be able to apply Ĺ Hopitals Rule.

Suppose that we have one of the following cases.

$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ OR $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$

Where a can be any real number, infinity or negative infinity. In these cases we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{\acute{f}(x)}{\acute{g}(x)}$$

Examples

Evaluate each of the following limits

a) $\lim_{x\to 0}\frac{\sin x}{x}=\left[\frac{0}{0}\right]$

$=\lim_{x\to 0}\cos x\,\frac{\cos x}{1}=1$

b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{1}{2}x}{x^2}=\left[\frac{0}{0}\right]$

$=\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}-\frac{1}{2}}{2x}=\left[still \quad \frac{0}{0}\right]$

$=\lim_{x\to 0}\frac{\frac{-1}{4}(1+x)^{-\frac{3}{2}}}{2}=-\frac{1}{8}$

c) $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\left[\frac{0}{0}\right]$

$\lim_{x\to 0}\frac{0-(-\sin x)}{1+2x}=\frac{0}{1}=0$

Or $\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1}{(1+x)}$

$=0*\frac{1}{(1+0)}=0$

Ass.L. Ghufran Faris Alrahhawi