University of Mosul

College of Petroleum & Mining Engineering

Mathematics I

Lecture (5)

Area And Volume Under The Curve

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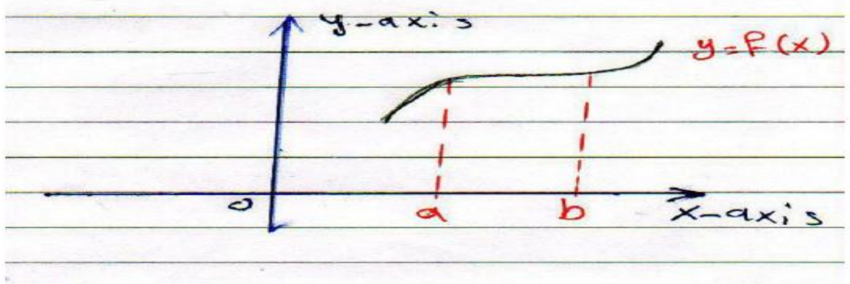
LECTURE CONTENTS

Area Under The Curve

Area Under the Curve

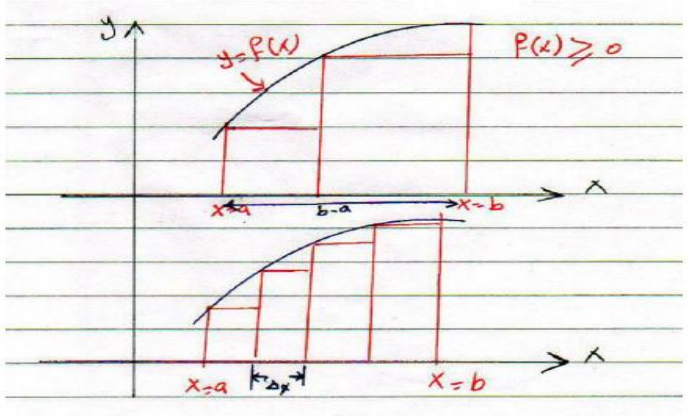
The area under curve between two points is found out by doing a definite integral between the two points. To find the area under the curve $y = f(x)$ between $x = a$ & $x = b$ integrate $y = f(x)$ between the limits a and b

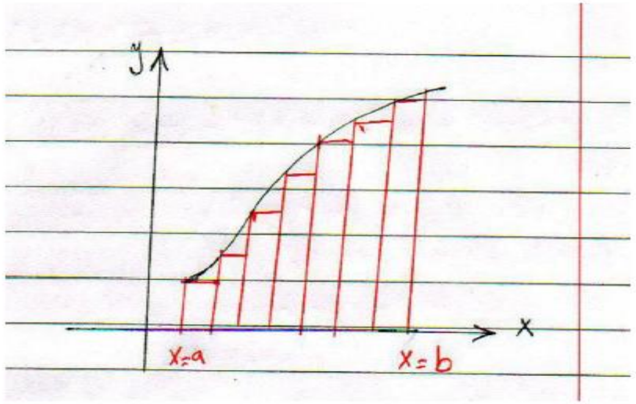


Methods to Find Area Under the Curve

Method 1 (Inscribed rectangle)

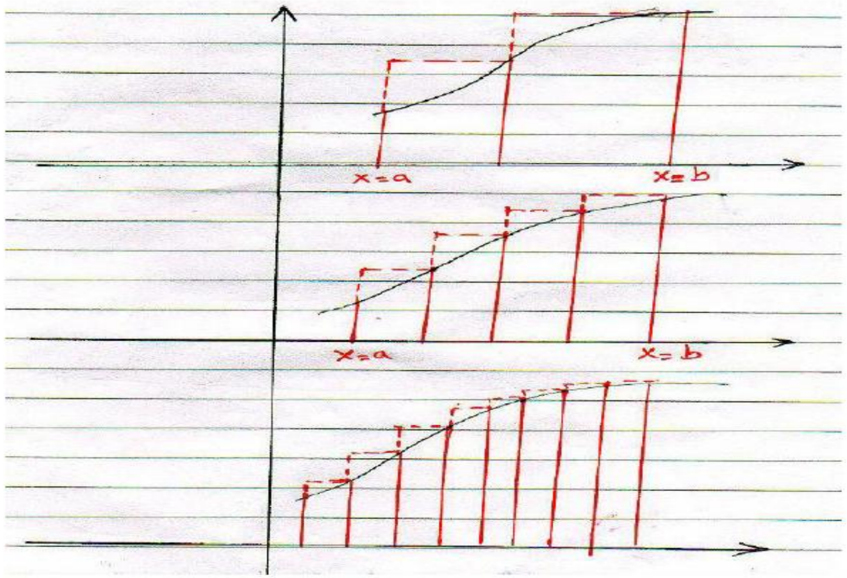
The summation of the area of these rectangles gives the area under the curve.





Method 2 (Circum scribed Rectangle)

Is the smallest rectangle that can be formed around a group of points with all of the points or on one of its sides.



$\Delta x = \frac{b-a}{n}$

$n =$ No. of rectangle

$\Delta x =$ Width of rectangle

$\Delta A_i = y_i . \Delta x = f(x_i) * \Delta x$

$A_T = \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} y_i . \Delta x = \sum_{i=1}^{n} f(x_i) \Delta x$

When $n \to \infty$ $\Delta x \to 0$

$A = \lim_{\Delta x \to 0} A_T = \sum_{\Delta x \to 0}^{n} \Delta A_i = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i) \, \Delta x$

When $n \to \infty$ $\Delta x \to 0$

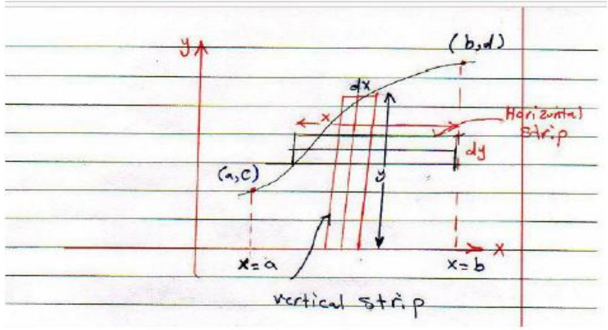
$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$A = \lim \int_a^b f(x) dx = \int_a^b y . dx$

Area Under the Curve

$A = \int_a^b f(x)dx = \int_a^b y.dx$ **Vertical strip**

$A = \int_c^d f(y)dy = \int_a^b x.dy$ **Horizontal strip**



Area Between Two Curves

The area between two curves can be conveniently calculated by taking the difference of the areas of one curve form the area of another curve. Here the boundary with respect to the axis for the curves is the same. The below figures show two curves $y_1 = f(x)$, $y_2 = f(x)$

And the objective is to find the area between these two curves. Here we take the integral of the difference of the two curves and apply the boundaries to find the resultant.

Example 1

Find the area of the region between the curve $y = \sqrt{x}$, x- axis , and the line x=4

a) Use vertical strip, b) Horizontal strip

Solution

$y = \sqrt{x}$, x=4

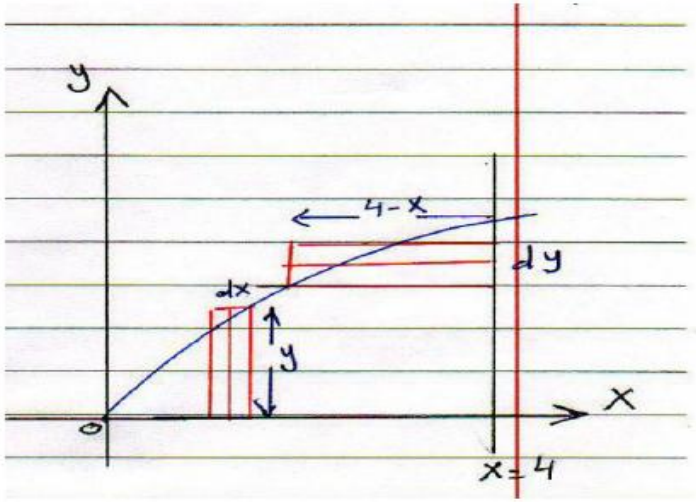
At x=4 $\Longrightarrow y = \sqrt{4} = 2$

a) Vertical strip

$A = \int_0^4 y\,dx = \int_0^4 \sqrt{x}dx$

$= \frac{2}{3}\ \ x^{3/2}\Big|_0^4$

$= \frac{2}{3}\ \ (4)^{3/2} = \frac{2}{3} * 8 = \frac{16}{3}$ sq.unit



b) Horizontal strip

$A = \int_0^2 f(y)dy = \int_0^2 4 - x\,dy = \int_0^2 4 - y^2dy$

$= 4y - \frac{1}{3}y^3|_0^2$

$A = 8 - \frac{8}{3} = 16/3$ sq. unit

Volumes by integration

If a region in the plane is revolved a bout a given line, the resulting solid is a solid of revolving a region bounded by a given function about on axis, follow the steps below:

1. Sketch the area and determine the axis of revolution (this determines the variable of integration)
2. Sketch the cross -section (disk, washer, shell) and determine the appropriate formula.
3. Determine the boundaries of the solid
4. Set up the definite integral, and integrate.

1. Finding volume of a solid of revolution using a disk method

The area bounded by the curve $y = f(x)$ $a \le x \le b$ and x-axis rotated about the x-axis. The area divided into n strips perpendicular to x- axis . Each strip when revolved resulting a disk (Right Circular Cylinder)

With radius $r_i = y_i$ & height equal $\Delta x = \frac{b-a}{n}$

$\Delta v_i =$ volume of each disk $\pi(raduis)^2 \times height$

$$\Delta v_i = \pi(r_i)^2 \times \Delta x = \pi(y_i)^2.\Delta x$$
$$V_T = \sum_{i=1}^{n} \Delta v_i = \sum_{i=1}^{n} \pi(y_i)^2.\Delta x$$
When $n \to \infty$ $\qquad \Delta x \to 0$
$$V = \lim_{\Delta x \to 0} V_T = \lim_{\Delta x \to 0} \Delta v_i = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi(y_i)^2.\Delta x$$

$$\int dv = \int \pi y^2.dx$$

$$\therefore \quad V = \int_a^b \pi y^2 dx = \pi \int_a^b (f(x))^2\, dx$$

$$\boldsymbol{V = \int_a^b \pi y^2 dx} \quad \Longrightarrow \text{ Axis of rotation is x-axis}$$

$$\boldsymbol{V = \int_c^d \pi x^2 dy} \quad \Longrightarrow \text{ Axis of rotation is y-axis}$$

Volume by disk method

$V = \int_c^d A\,(y)dy = \int_c^d \pi\,[R\,(y)]^2\,dy$ about y-axis

$V = \int_a^b A\,(x)dx = \int_a^b \pi\,[R\,(x)]^2\,dx$ about x-axis

$y_1 = f(x) \geq y_2 = f(x)$

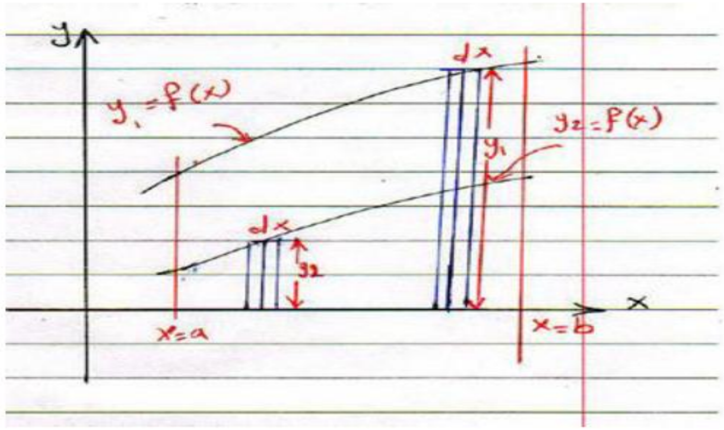
$a \leq x \leq b$

$\therefore\ V_1 = \int_a^b \pi {y_1}^2\,dx$

$V_2 = \int_a^b \pi {y_2}^2\,dx$

$V = V_1 - V_2$

$= \int_a^b \pi({y_1}^2 - {y_2}^2)dx$



<u>Case 1</u>

About the y-axis

$V_1 = \int_0^d \pi b^2\,dy$

$V_2 = \int_c^d \pi x^2\,dy$

$V_3 = \int_0^c \pi a^2\,dy$