Department of Mining Engineering
-2nd-Class
College of Petroleum and Mining Engineering
University of Mosul

Mathematics III

Lecture 3

INFINITE SEQUENCES AND SERIES (PART III)

Dr. Hudhaifa HAMZAH

Taylor and Maclaurin Series

This section shows how functions that are infinitely differentiable generate power series called Taylor series. In many cases, these series can provide useful polynomial approximations of the generating functions.

Series Representations

We know from Theorem 19 that within its interval of convergence the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval I, can it be expressed as a power series on I? And if it can, what will its coefficients be?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n(x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^2 + \cdots, \end{aligned}$$

DEFINITIONS Taylor Series, Maclaurin Series

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$ is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f is**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

EXAMPLE 1 Finding a Taylor Series

Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

Solution We need to find $f(2), f'(2), f''(2), \ldots$. Taking derivatives we get

$$
\begin{aligned}
f(x) &= x^{-1}, & f(2) &= 2^{-1} = \frac{1}{2}, \\
f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{2^2}, \\
f''(x) &= 2!x^{-3}, & \frac{f''(2)}{2!} &= 2^{-3} = \frac{1}{2^3}, \\
f'''(x) &= -3!x^{-4}, & \frac{f'''(2)}{3!} &= -\frac{1}{2^4}, \\
&\vdots & &\vdots \\
f^{(n)}(x) &= (-1)^n n! x^{-(n+1)}, & \frac{f^{(n)}(2)}{n!} &= \frac{(-1)^n}{2^{n+1}}.
\end{aligned}
$$

The Taylor series is

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

This is a geometric series with first term $1/2$ and ratio $r = -(x-2)/2$. It converges absolutely for $|x-2| < 2$ and its sum is

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

In this example the Taylor series generated by $f(x) = 1/x$ at $a = 2$ converges to $1/x$ for $|x-2| < 2$ or $0 < x < 4$. ■

Finding Taylor Series

In Exercises 21–28, find the Taylor series generated by f at $x = a$.

21. $f(x) = x^3 - 2x + 4, \quad a = 2$

22. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

23. $f(x) = x^4 + x^2 + 1, \quad a = -2$

24. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

25. $f(x) = 1/x^2, \quad a = 1$

26. $f(x) = x/(1 - x), \quad a = 0$

27. $f(x) = e^x, \quad a = 2$

28. $f(x) = 2^x, \quad a = 1$

Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercises 9–20.

9. e^{-x}

10. $e^{x/2}$

11. $\frac{1}{1 + x}$

12. $\frac{1}{1 - x}$

13. $\sin 3x$

14. $\sin \frac{x}{2}$